On the Function of Two Variables of Urban Population Density

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Abstract

This paper is a pure theoretical attempt to explain tendencies revealed in the last decade urban researches, such as decentralization of population toward the cities suburbs, formation of polycentric cities, by a process of alteration of city traditional monocentricity, in time, due to the growth of population. It is shown that the process of alteration of monocentricity is governed by a function of two variables of urban population density. The function is developed following the observation that urban population density is dependent, in the same time, both on distance-falling with the distance from the city center- and on time-growing with the increase of population size in cities area. The city model based on the function of two variables also predicts other interesting urban phenomena, such as cities ''collision''.

Key words: urban population density, population growth, monocentric city model, polycentricity, decentralization of population.

JEL Classification: O18, R11, R14

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1. Introduction

An historical review over the urban spatial structure, that could be trace back to the von Thunen [24] land uses, continuing with the work of Alonso [1], Muth [18] and Mills [17], reveals that originally, urban settlements are basically monocentric. The monocentric city model has been documented and analyzed in a large body of literature dedicated to urban economics. More details can be found in [3, 6].

Central in monocentric city model is the function that relates population density with the distance from the city center. First formulated in the work of Clark [7], the function of population density predicts the exponential fall of population density with the distance from the center of the city. The dependence of population density on distance that Clark's function predicted, was questioned in many empirical studies [2, 4, 5, 8, and 11] along the time, and is still dominant in modeling urban spatial structure.

Last decades empirical researches in urban spatial structure revealed tendencies, such as decentralization of population toward the cities suburbs, formation of polycentric cities, which cannot be depicted in the monocentric city model. Empirical analyses in formation of cities dense population and employment subcenters [9, 13, 14, 15, and 16] that could form polycentric cities [5, 10] strongly suggested that cities are crossing a period of transition to a city model characterized by the above mentioned phenomena. The alteration of the city monocentricity occurs with the last decades accelerated growth of cities. Since the growth of cities is influenced mostly by population growth, the main factor that alters the city monocentricity seems to be the increase of population in the city limits, which imply a growth of urban population density. Evidences in urb an population growth can be found in [22]. Thus, the evolution of urban population density in time must be an ''ingredient'' as important as distance is, in the analysis of urban spatial patterns.

The population growth theory has a long history, starting with the seminal work of Malthus [12], on exponential growth of population. The Malthusian model assumes an indefinitely increase of population, which is unrealistic and represent the main weakness of the model. To eliminate the weakness, Verhulst [23] dev eloped a density dependent population growth model, well known in the literature as logistic model. The density dependency that appears in the logistic model stopped the population increasing indefinitely. Logistic model is widely applied at urban level to evaluate the population growth. It also is an important tool in appreciating the evolution of urban population density.

To summarize, the urban population density is dependent on two variables:

- distance, according to the Clark's negative expon ential function;
- time, according to the logistic model of population growth.

Thus, on the one hand, urban population density falls exponentially with the distance from the city center, and on the other hand, it grows, in time, due to the increasing of population size in urban limits.

It is shown in this paper that the two dependencies the function of population density is performing at the urban level can be synthesized in a unique function of two variables for urban density of population.

The city model based on the function of two variables is capable to predict and satisfactory explain tendencies of population decentralization toward the cities suburbs, formation of alternative cities centers and enlargement of the cities limits. The model also predicts other urban interesting phenomena, such as cities ‘’collision’’.

The paper is structured as follows. Sections 2 and 3 constitute preliminaries reviews of the existing literature in the urban spatial structure and population growth theory. The review is centered on the part of the literature that will be of interest in the next sections. It also pointed out the dependencies of urban population density function on distance from the city center, and on time respectively. Section 4 is designed to synthesized the two dependencies urban population density is perform, in a unique function of two variables of urban population density.

The predictions the city model based on the two variables function can made are presented and phenomenon related to evolution of urban spatial structure are explained in section 5.

In section 6 a limited, unrealistic but still interesting case is briefly taken into account. Conclusions of the present paper are summarized in the last section.

2. Urban Spatial Structure- Monocentric City Model.

Extensively documented for many years, urban spatial structure constitutes an important research object in urban analysis and planning. In model the urban spatial patterns, monocentric city model still occupy a central place. Theoret ical basis for monocentric model could be found in the work of von Thunen, on land uses. The modern approach of urban analysis can be considered as started with the work of Clark, Alonso, Muth and Mills.

Central in monocentric city model is the function th at relates urban population density to distance from the city center. First formulated in the classic study undertaken by Clark, the population density function is still dominant in modeling urban spatial structure. The function has negative exponential pa tterns and its central prediction is that population density and related, land uses, house prices and employment density falls exponentially with the distance from the city center.

To summaries, the Clark’s classical function of population density can be w ritten mathematically in the form:

$$D_{(x)} = D_{(0)} e^{-\lambda x} \tag{1}$$

Where $D_{(x)}$ stands for density of population at a distance x from the city center, $D_{(0)}$ is the population density in the city center and λ represent the rate of density decrease or density gradient.

The density gradient measures the proportional decline in residential density per unit of distance.

In a graphical manner the distanc e from the city center dependence of population density is drown in the figure 1.

[Figure 1 about here]

Apart from the distance dependency, urban population density experiences also an evolution in time due to the continuous growth of population in t he cities limits.

3. Population Growth-Logistic Model

In response to the Malthusian, indefinitely growth of population model, Verhulst developed a self-limited model, known in the literature as logistic growth model. First formulated in the work of Verhu lst, and representing the basis of the model, logistic function was lately rediscovered by Pearl and Reed [19,20], and promoted to the status of predictive law of population growth in writings by Pearl [21].

The Verhulst model of population growth is widel y used to evaluate the dynamic of population at the level of the world, regions or urban settlements.

Logistic function of population growth can be written as:

$$N_{(t)} = \frac{K}{1 + \left(\frac{K - N_{(0)}}{N_{(0)}} \right) e^{-rt}} \tag{2}$$

Where $N_{(t)}$ stands for population size at a time t, $N_{(0)}$ is the initial size of population, r represent the growth rate and K stands for **carrying capacity**.

The growth rate is the difference between birth and death of individuals sit uated in the city limits.

The carrying capacity parameter in the logistic model is a measure of the available resources and represents the maximum of population that a city can sustain. When population reaches the size K, all resources are used to keep th e population at this level, and no further growth is possible. The carrying capacity value is influenced by factors of economical, political or technological nature. As a result of changes of above factors, the value of carrying capacity could change also in time.

The graph in figure 2 shows the evolution of population dynamic in the logistic model assumption.

[Figure 2 about here]

The ''S-shape'' of the graph is typical for the logistic function.

For a small size of population, comparing wi th carrying capacity the increase is exponentially. The growth of population began to slow in the neighborhood of carrying capacity and when the K size is reached, no further growth is possible.

Because of the carrying capacity appearance as a parameter, logistic function is also known in literature being density dependent. The dependency on density makes the logistic model of population growth to be a useful tool in evaluating the dynamics of population density. To refer to urban area is more convenient t o consider in relation (2), population density instead of population size. Reporting population size to the urban area in logistic function yields to:

$$D_{(t)} = \frac{K}{1 + \left(\frac{K - D_{(0)}}{D_{(0)}} \right) e^{-rt}} \tag{3}$$

where $D_{(t)}$ is the predicted population density at a time t and $D_{(0)}$ represent the initial population density. Logistic function for density, applied at urban level, shows the urban population density evolution in time.

4. The Function of Two Variables of Urban Population Density

Conclusion last two sections drown is that urban population density is dependent on the one hand on distance-from the city center-and on the other hand on time due to the population growth in urban area. The mat hematical meaning is that population density function is, in the same time, dependent on two variables, a fact that leads to the necessity of modeling urban spatial structures by a function of two variables.

The construction of the population density funct ion of two variables is not difficult taking into account the fact that evolution tendencies separately, with distance and in time, are known.

It is theorized, as section 3 shows, that in urban area population density grows according to the logistic model. The model assumes an initial density $D_{(0)}$ having the same value for the entire urban area. Although, population it is not spread equally in the urban limits. According to Clark's function the center of the city has a high density of population; density that falls exponentially with the distance from the center of the city. This fact leads to different levels of growth for population density, depending on the distance from the center of the city.

For the city center logistic function, ta ke the form:

$$D_{(t)} = \frac{K}{1 + \left(\frac{K - D_{(0)}}{D_{(0)}} \right) e^{-rt}} \tag{4}$$

For an initial population density situated at a distance x from the center of the city logistic function can be written:

$$D_{(t)} = \frac{K}{1 + \left(\frac{K - D_{(x)}}{D_{(x)}} \right) e^{-rt}} . \tag{5}$$

At the distance x from the center of the city the population density $D_{(x)}$, following the Clark's function is:

$$D_{(x)} = D_{(0)} e^{-\lambda x} \ . \tag{6}$$

Substituting relation (6) in the logistic function (5), yield to:

$$D_{(t)} = \frac{K}{1 + \left(\frac{K - D_{(0)} e^{-\lambda x}}{D_{(0)} e^{-\lambda x}} \right) e^{-rt}} \ . \tag{7}$$

The logistic function of population density growth becomes dependent on distance variable too. In relation (7) appears as v ariables both distance and time. Using the mathematical common notation for a function of two variables, relation (7) can be written:

$$D_{(x,t)} = \frac{K}{1 + \left(\frac{K - D_{(0,0)} e^{-\lambda x}}{D_{(0,0)} e^{-\lambda x}} \right) e^{-rt}} \tag{8}$$

which represent the **function of two variables of urban population density** and basically is the core of a city model that evaluate urban spatial patterns both on distance and time, a model whose predictions will be explore later in this paper.

In the above relations $D_{(x,t)}$ stands for population density at a moment t and at a distance x from the city center, $D_{(0,0)}$ is the initial population density.

The two variable function of urban population density must satisfy, in a first approximation, two obvious limit conditions:

a. taking t=0 similar with neglecting the time variable, Clark's classical function had to be recovered;

b. taking x=0 meaning that distance is neglected, the two variable function had to reduces its form to logistic function.

To investigate the validity of the two conditions applied on function (8), first is neglected the time dependency. For t=0 the function can be written:

$$D_{(x,0)} = \frac{K}{1 + \left(\frac{K - D_{(0,0)} e^{-\lambda x}}{D_{(0,0)} e^{-\lambda x}} \right) e^{\lambda x - 0}} \ . \tag{9}$$

After some simple calculations and arrangement of terms take the traditionally function of urban population density form:

$$D_{(x,0)} = D_{(0,0)} e^{-\lambda x} \ .$$

Neglecting distance in the relation (8) leads to:

$$D_{(0,t)} = \frac{K}{1+\left(\frac{K-D_{(0,0)}e^{0}}{D_{(0,0)}e^{0}}\right)e^{0-rt}}. \tag{10}$$

After simple calculus and taking into account that e^0 equals unity, the logistic function is recovered:

$$D_{(0,t)} = \frac{K}{1+\left(\frac{K-D_{(0,0)}}{D_{(0,0)}}\right)e^{-rt}}.$$

The function of two variable of urban population density (8) holds for both limit conditions.

5. The Meaning of Two Variables Urban Population Density Function

The evolution of urban spatial patterns according to the function of two variables of urban population density can easily be interpreted following the graph shown in the figure 3.

[Figure 3 about here]

The graph in figure 3 is made using values for parameters K, r, and λ appropriate to have as result a useful shape for the function of two variables of urban population density. Relatively exact values of the function parameters in different cities can be finding empirically using exact data provided by databases services, such as U.S. Census Bureau.

At the initial moment, urban population density is much less than the carrying capacity. Population density will increase exponentially. The growth of popul ation density is not happened simultaneously in urban area. Because of the negatively exponential dependency on distance, population density falls with distance from the city center. It means that the carrying capacity will not be reached simultaneously by the entire city area. The center of the city performing a higher population density will reaches first the carrying capacity. Achieving the carrying capacity, no further growth of population density is possible in the center of the city.

The possible places for further growth of density remain the immediate city peripheries, which did not yet reached the carrying capacity. The population growth in suburbs is related to the decentralization of population. Decentralization is mainly a process of accelerated growth of population density in city peripheries in contrast with the city center stagnation due to the impossibility of further growth.

When the city center reached the carrying capacity, and no further growth is possible, the only observable growth of population is in suburbs. The impossibility of further growth of density in the center of the city imposes the accelerated development of suburbs and enlargement of cities limits.

Enlargement of city limits could eventually lead to a process of ''collisio n'' of cities. Collision of cities is a process of crossing of suburbs of two neighborhood cities due to the enlargement of cities limits. Being the most frequent case, collision of a metropolitan area with a small city situated in the neighborhood is show n in figure 4.

[Figure 4 about here]

The collision of a metropolitan area with a small city situated in the neighborhood tends to alter the traditionally monocentric metropolitan structure to polycentricity, due to the existence of the smaller cit y center. After collision the existing center of the smaller city did not dissipate, it will increase in density forming a subcenter for the metropolitan area. Although, not at the size of the traditional metropolitan center, the new formed subcenter will alter the monocentric metropolis structure. The metropolitan area becomes polycentric.

Carrying Capacity is susceptible of changes in value being influenced by factors of political, economical or technological nature. Technological progress could increase the value of carrying capacity. As a result, population can grow temporarily in the center of the city. The phenomenon is followed by a process of centralization of population toward the city center, until the new value of carrying capacity is reached. Th e carrying capacity being reached no further growth is possible in the center and a new effect of decentralization of population toward the suburbs will start.

Empirical evidences for decentralization of employment density and formation of strong employme nt subcenters lead to the conclusion that the city model based on a function of two variables can be successfully applied not only for population density but, in an equal manner to other related parameters that define urban economics.

6. Unlimited Resources-A Limit Case

An interesting case to be studied is that of the unlimited resources in urban area. Although, unrealistic because it admit an indefinitely growth of population density, the unlimited resources case has theoretical significance.

In the hypothesis of unlimited resources, the carrying capacity parameter is very large comparing with population size. It means that 1/K quantity can be neglected. This operation reduces the form of the function of two variables of urban population density to:

$$D_{(x,t)} = D_{(0,0)} e^{rt-\lambda x} . \qquad (11)$$

The missing of carrying capacity parameter in the function of two variables, leads to an urban spatial structure, which will performs an indefinitely and unalt ered monocentricity.

Hypothetical cities that have a population density structure according to a function of the form (11) will not encounter phenomena of decentralization, formation of alternative center or enlargement of city limits.

7. Conclusions

The aim of this paper was to present a city model based on a two variables function of urban population density, capable to predict and explain tendencies observed in the last decades in urban spatial structure.

The two variables function that is the c ore of the model is developed following the observation that urban population density is dependent both on distance -from the city center- and on time, due to the population growth in urban area.

The city model on the assumption of a two variables function can explain phenomenon of decentralization of population toward the cities suburbs, formation of polycentric cities and enlargement of cities limits. It also predicts other interesting urban phenomenon, like cities ''collision''.

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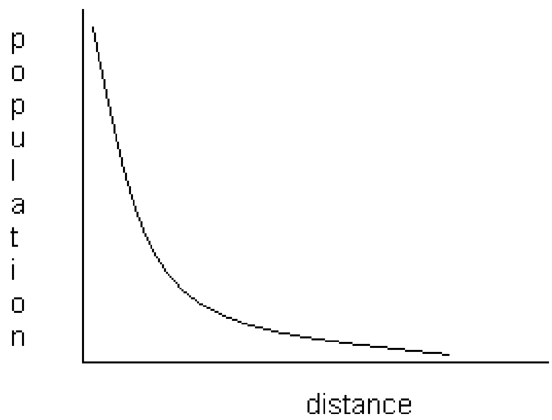
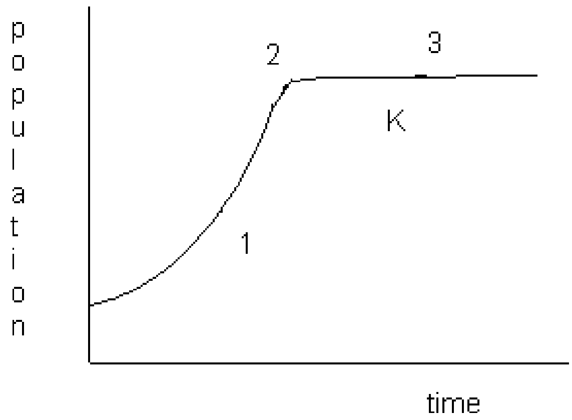


Figure 1. Exponential fall of population density with the distance from the center of the city.



Legend: 1.exponential population growth.
2.stagnation in population growth.
3.no further growth is possible.

Figure 2. Logistic function of population growth.

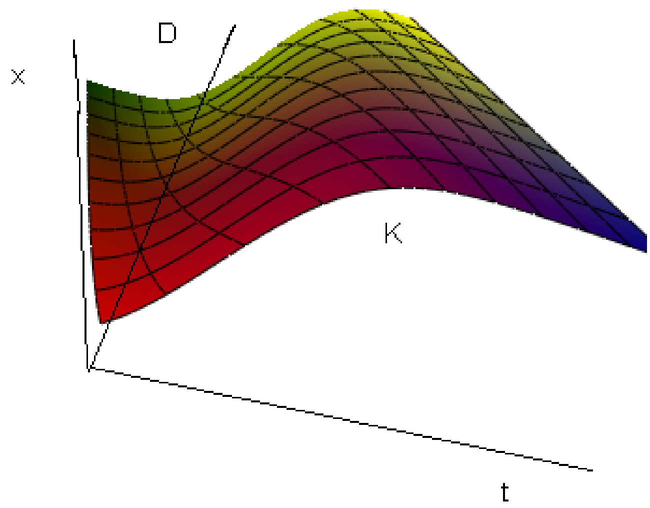


Figure 3. Evolution of urban population density
according to the function of two variables.

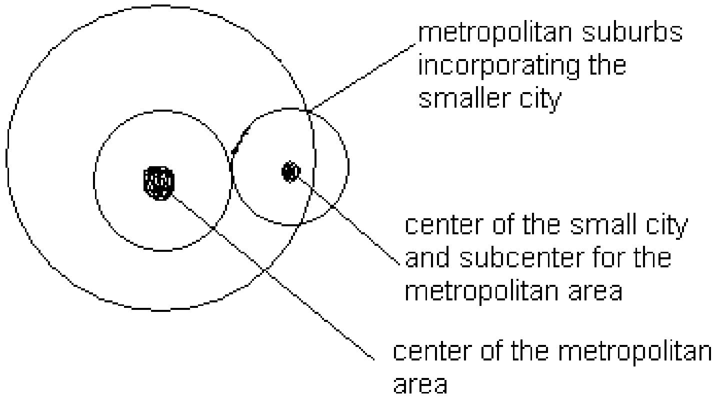


Figure 4. Collision of a metropolitan area with a small city situated in the neighborhood