Bifurcations in regional migration dynamics[*]

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Abstract

We study regional migration dynamics in a general framework and provide a foundation for the commonly used two-stage equilibrium. We also examine the robustness of bifurcations when parameters change along arbitrary smooth (C^1) paths. We show that bifurcations with crossing equilibrium loci are rare, which include, for example, the popular pitchfork bifurcation.

Keywords: Bifurcation; Genericity analysis; Migration dynamics; Two-stage equilibrium
JEL classification: R12; R23; F12

1 Introduction

Economic activities are not distributed uniformly in space. Nonagricultural production often concentrates in a few regions, resulting in a core-periphery pattern. Why do producers agglomerate? There are various reasons, for example, the economies of localization and urbanization, natural advantages, the provision of public goods, and imperfect competition. In this paper, we investigate the migration dynamics of firms and workers among regions. Particularly, we study the dynamics concerning the following question: "How does one region come to dominate the others and become a core of manufacturing?"

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Initiated by Abdel-Rahman (1988), Abdel-Rahman and Fujita (1990), and Krugman (1991a, 1993a, b), the Dixit-Stiglitz (1977) model has become the standard tool in the new economic geography. In this model, firms that produce differentiated products with increasing returns to scale technologies compete monopolistically. There are two types of pecuniary externalities that generate the forces sustaining production agglomeration. They result in the positive feedback that comes from firms locating near each other. First, manufacturing production will concentrate where there is a large market with many workers consuming manufactured goods. Second, workers will move to where the production concentrates because the manufactured goods are cheaper there. Employing this model of manufacturing production, Krugman (1991), and Fujita, Krugman and Venables (1999) explain the emergence of the core-periphery pattern with the dynamics of a "pitchfork bifurcation":[1] They consider an economy consisting of two regions with equal resources. There are equal populations of immobile farmers in both regions who produce a homogeneous agricultural good, and also a population of mobile manufacturing workers that migrate between regions. Workers move to the region where they have a higher utility level. The transportation of manufactured goods across regions bears a cost while that of the agricultural good does not. With other parameters fixed, when the transportation cost is high, the symmetric equilibrium, where both regions have the same manufacturing populations, is the only equilibrium and it is stable. When the transportation cost is moderate, two other stable equilibria emerge; when this happens, one of the two regions attracts all the manufacturing production. When the transportation cost is low, the symmetric equilibrium becomes unstable and the only stable equilibria are the two core-periphery equilibria. Note that the core-periphery equilibria are boundary equilibria where one of the regions has no workers.

Regional divergence can occur when the symmetric equilibrium becomes unstable. However, the pitchfork bifurcation does not depict a robust picture of the dynamics involved. A recent paper by Anas and Li (2002) illustrates this point. They study a more general model and examine changes in other parameters as well. They show that depending on the values of the fixed parameters, the equilibrium diagram displays different patterns (not only the pitchfork bifurcation). In some cases, there are no bifurcations at all. The following example illustrates that the pitchfork bifurcation disappears under a small perturbation of parameter values. Consider the following

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[1] It is also called the "tomahawk bifurcation" with modifications of the boundary equilibria.

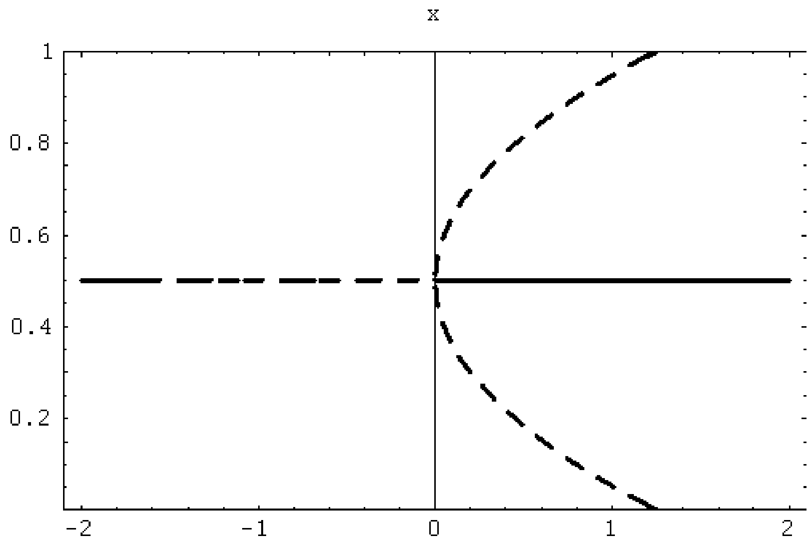

Figure 1: $a = 5,\ -2 < b < 2,\ c = 0$.

dynamical system:

$$\dot{x} = a\,(x - 0.5)^3 - b\,(x - 0.5) \tag{1}$$

where $x, a, b \in \Re$. When a is held fixed, except at $a = 0$, every equilibrium diagram obtained by varying b contains a pitchfork bifurcation (Figure 1 shows the case of $a = 5$).[2] Consider the following system ($c \in \Re$):

$$\dot{x} = a\,(x - 0.5)^3 - b\,(x - 0.5) + c. \tag{2}$$

It includes (1) as a special case at $c = 0$. However, if we perturb c slightly, the resulting equilibrium diagram does not contain a pitchfork bifurcation, even though the general contour is preserved (see Figures 2 and 3 for $c = 0.001, -0.001$ respectively). After adding a parameter c, the pitchfork bifurcation almost never occurs. Moreover, if we fix a and vary b in the original system (1), there is no pitchfork bifurcation either (see Figure 4). This raises the following question: "what kind of dynamic behavior is typical given enough parameters?"

In spite of the illustrative cases presented in previous work, there are methodological issues to be resolved concerning the study of dynamics: (i) The choice of parameters under investigation affects the dynamic behavior of a system. It is possible that when we study other parameters, which may not be present in the current

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[2] All figures are computation results. The solid and dashed lines indicate stable and unstable equilibria respectively.

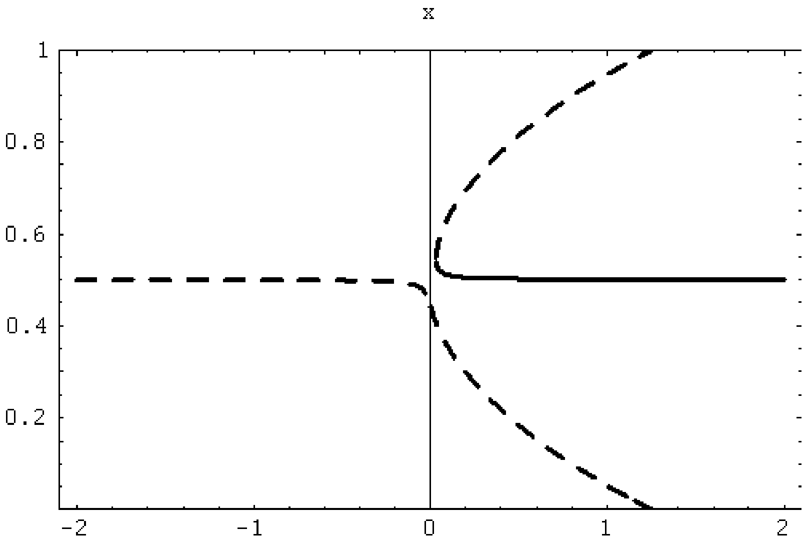

Figure 2: $a = 5$, $-2 < b < 2$, $c = 0.001$.

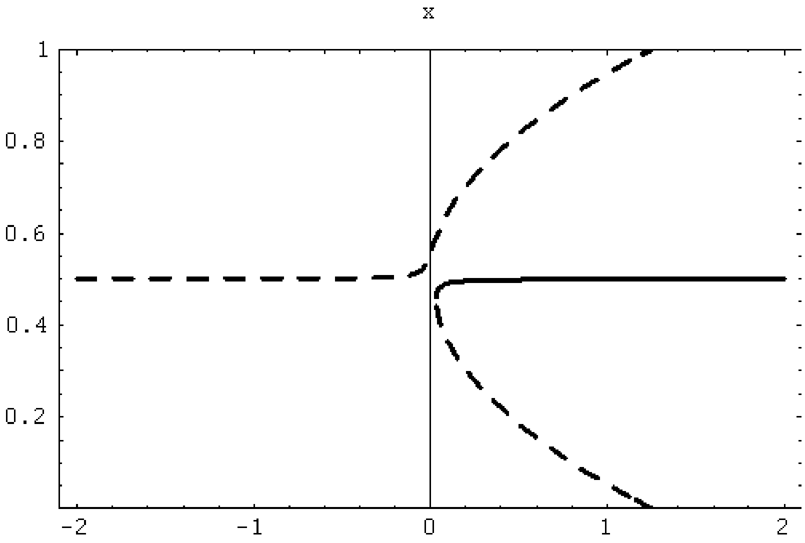

Figure 3: $a = 5$, $-2 < b < 2$, $c = -0.001$.

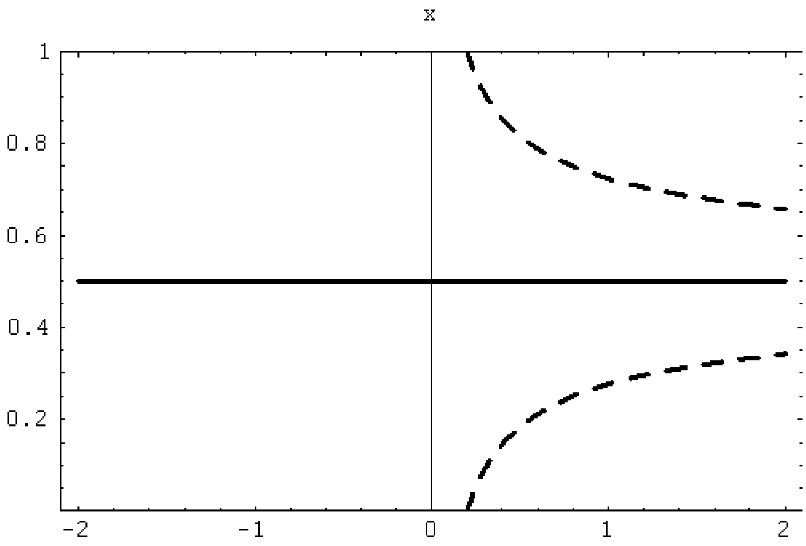

Figure 4: $-2 < a < 2$, $b = 0.05$, $c = 0$.

models, the economy displays other types of bifurcations (such as the saddle-node or transcritical bifurcations which can be generated by a one-dimensional system). (ii) It is interesting to study the effects of one parameter at a time, especially if the parameter is a dominant force changing the economy (say, transportation cost). Yet, parameters do not change one at a time in the real world. A more plausible case is that all parameters change simultaneously resulting in a path in the parameter space. (iii) Many types of equilibrium patterns (bifurcation or not) can be generated depending on the choice of parameter paths. Given enough parameters, can we distinguish the typical (i.e., generic) case that always occurs from those that occasionally or almost never occur?

We present a general approach to address these issues and, as a result, obtain a broader picture of the dynamic equilibrium of regional migration. We study equilibrium diagrams along general parameter paths in a regular parameter space where the system's Jacobian matrix with respect to endogenous variables and parameters has full rank at every equilibrium for all parameter values. It is sufficient to focus on smooth (C^1) paths since a continuous path can be approximated arbitrarily close by a smooth one (see for example, Theorem 2.6 in Hirsch 1976, ch.2). We are interested in bifurcations with crossing equilibrium loci; we call them split bifurcations. They include the pitchfork bifurcation. When the parameter space is regular, (i) there is a generic (open and dense) set of parameter paths without split bifurcations in the space of smooth paths, and (ii) in the case of two regions, the set of paths without bifurcations is open (hence bifurcations are not generic). We also demonstrate that the above mentioned model of regional migration can be embedded in a regular parameter space.

In addition, we provide a sufficient condition for the commonly used two-stage equilibrium, which deals with migration equilibrium at the first stage and suppresses the balance of commodity markets at the second stage.

This paper is organized as follows. Section 2 provides a general dynamic model of regional migration based on a static economy. Section 3 defines split bifurcations and studies its genericity in a regular parameter space. Section 4 introduces the two-region economy with monopolistic competition and provides an example of a regular parameterization. Section 5 concludes.

2 The Dynamics of Regional Migration

The following is a general framework of migration dynamics. Suppose the economy has n goods, m regions, and a given population of mobile agents. Let $q \in \left\{ q \in \Re^n_+ \,|\, \sum_{i=1}^n q_i = 1 \right\}$ denote the list of prices, and $s \in \left\{ s \in (0,1)^m \,|\, \sum_{j=1}^m s_j = 1 \right\}$ denote the list of population shares of mobile agents in all regions. We are interested in the interior equilibria and exclude in definition the boundary equilibria where $s_j = 0$ for some j. Let $\bar{f}_i(q,s)$ denote the excess demand function of good i. The equilibrium conditions for the commodity markets are $\left\{ \bar{f}_i(q,s) = 0 \right\}_{i=1}^n$. In addition to material balance in commodity markets, equations $\left\{ \bar{g}_j(q,s) = 0 \right\}_{j=1}^{m-1}$ are the migration balance conditions, which require mobile agents to have equal utility levels in all regions.[3] For example, $\bar{g}_j$ can be the utility difference between region j and region m. Assume all $\bar{f}_i$ and $\bar{g}_j$ are C^1.

This framework applies to a general class of models, which include exchange economies and production economies with perfect or imperfect competition, as long as the firms' profit-maximizing output levels can be determined, given prices. Hence the equilibrium in the commodity markets can be reduced to requiring the excess

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[3] $m-1$ conditions suffice to equalize utility levels in m regions. To see this, consider the case of $m = 1$, there is no migration balance condition since no one can move. When $m = 2$, there is one condition which equalizes the utility levels in both regions. For each additional region, one more condition is imposed.

demand functions to equal zero. The models can include economies of localization, economies of urbanization, public goods, etc.

Note that q has only $n-1$ degrees of freedom and that s has $m-1$ degrees of freedom. We eliminate redundant variables in the following. Let $p \in S^{n-1} = \left\{p \in \Re^{n-1}_{++} | \sum_{i=1}^{n} p_i < 1\right\}$, $\lambda \in S^{m-1} = \left\{\lambda \in (0,1)^{m-1} | \sum_{j=1}^{m} \lambda_j < 1\right\}$. Define f_i for $i = 1, ..., n-1$ and g_i for $i = 1, ..., m-1$ as follows.

$$\begin{aligned} f_i(p,\lambda) &= \bar{f}_i\left(p_1, ..., p_{n-1}, 1 - \sum_{i=1}^{n-1} p_i, \lambda_1, ..., \lambda_{m-1}, 1 - \sum_{j=1}^{m-1} \lambda_j\right), \\ g_j(p,\lambda) &= \bar{g}_j\left(p_1, ..., p_{n-1}, 1 - \sum_{i=1}^{n-1} p_i, \lambda_1, ..., \lambda_{m-1}, 1 - \sum_{j=1}^{m-1} \lambda_j\right). \end{aligned}$$

Note that $\bar{f}_n$ is eliminated because of Walras' law. Let $f = (f_1, ..., f_{n-1})$, $g = (g_1, ..., g_{m-1})$, and $F = (f, g) : S^{n-1} \times S^{m-1} \to \Re^{n+m-2}$. We call this system without migration dynamics a *static economy.* A *static equilibrium* of this economy is a pair $(p, \lambda) \in S^{n-1} \times S^{m-1}$ such that $F(p, \lambda) = 0$.

Migration dynamics are usually derived from the static equilibrium. It is common practice in the literature to define a two-stage equilibrium (see, for example, Fujita et al. 1999, ch. 5): At the first stage, mobile agents choose a region to reside; at the second stage, commodity markets reach an equilibrium given the current population distribution. That is, material balance condition $f(p, \lambda) = 0$ obtains at the second stage, and migration balance $g(p(\lambda), \lambda) = 0$ obtains at the first, given that $p(\lambda)$ is a commodity equilibrium for population λ. The adjustment to equilibrium prices in commodity markets is assumed to take very little or no time in order to focus on the migration dynamics. The second stage is thus suppressed; the commodity markets are assumed to be always in equilibrium. However, this two-stage approach is valid only if there is a unique equilibrium at the second stage for any population distribution. Otherwise, a selection from multiple equilibria needs to be specified. The following assumption guarantees a unique equilibrium at the second stage.

Assumption A. *For any* $\lambda \in S^{m-1}$*, (i)* $f(p)$ *is a vector field on* S^{n-1}*, (ii)* f *is bounded below, and (iii) if* $p_k \in S^{n-1}$ and $p_k \to \partial S$ *(the boundary of* S*), then* $\|f(p_n)\| \to \infty$*. And (iv)* $|-D_p f(p,\lambda)| > 0$ *for all* (p, λ) *such that* $f(p, \lambda) = 0$.

Condition (i) is satisfied since $f(p) \in \Re^{n-1}$, the tangent space of S^{n-1}. Since the total endowment is bounded, condition (ii) is satisfied. In addition, utility nonsatiation guarantees condition (iii). All the three conditions above are satisfied by any excess demand function with nonsatiation. In addition, condition (iv) requires that the Jacobian matrix of the excess supply functions with respect to prices have full rank at every equilibrium. By A.i, ii, iii, the Index Theorem[4] applies to $f(.,\lambda)$ (Mas-Colell 1985, 5.6.1). Consequently, A.iv implies $f(p,\lambda)=0$ has a unique solution $p(\lambda)$ for every λ (Kehoe 1985, 1998).

Given a population distribution, the economy reaches the equilibrium price vector $p(\lambda)$ instantaneously. Knowing the utility levels determined by the price equilibrium, workers migrate to regions that offer higher utility levels. Migration conforms to the following dynamics: (we switch back to s temporarily for the convenience of the notation)

$$\dot{s}_j = \bar{h}_j(s) \text{ for } i = 1, ..., m,$$

where $s \in (0,1)^m$ and $\sum_{j=1}^m \dot{s}_j = \sum_{j=1}^m \bar{h}_j(s) = 0$. Since the total population is fixed, we define

$$h_j(\lambda) = \bar{h}_j\left(\lambda_1, ..., \lambda_{m-1}, 1 - \sum_{j=1}^{m-1} \lambda_j\right) \text{ for } j = 1, ..., m-1.$$

Taking $\bar{h}_m$ as redundant, the dynamical system is reduced to

$$\dot{\lambda}_j = h_j(\lambda) \text{ for } j = 1, ..., m-1 \tag{3}$$

where $\lambda \in S^{m-1}$. Note that $h(\lambda)$ is a vector field. We call this system a *dynamic economy.* A *dynamic equilibrium* of this economy is a list of population shares $\lambda \in S^{m-1}$ such that $h(\lambda) = 0$. We are particularly interested in the dynamics of C^1 vector fields h such that (i) their equilibria coincide with the static equilibria; i.e., $h(\lambda) = 0$ if and only if $g(p(\lambda), \lambda) = 0$; and (ii) their Jacobian matrices preserve the signs of $g(p(\lambda), \lambda)$; i.e., $D_\lambda h(\lambda) \lesseqqgtr 0$ if and only if $D_\lambda g(p(\lambda), \lambda) \lesseqqgtr 0$. (We need h to be C^1 for the existence and uniqueness of a solution to h; see Perko 2001, 2.2.) We rule out strange dynamics that alter the nature of the original static economy. The following are some examples of desired dynamics.

Example 1. (i) $\bar{h}_j$ is region j population share times the difference between own utility and the average utility of all mobile agents:

$$\bar{h}_j = s_j \left[v_j(q,s) - \sum_{i=1}^m s_i v_i(q,s) \right],$$

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[4] Note that since we deal with $p \in S^{n-1}$ instead of a simplex or a ball, our index is defined as $sign(|-D_p f|)$. The theorem says $\sum_{p \in E} \text{sign}(|-D_p f|) = 1$ where $E = \{p \in S^{n-1} | f(p,\lambda) = 0\}$.

where $v_j(q, s)$ is the utility level in region j when the population distribution is s and the associated equilibrium price vector is q, i.e., $f(q_1, ..., q_n, s_1, ..., s_m) = 0$. This is the "replicator dynamics" from evolutionary games (Weibull 1995, Fujita et al. 1999).

(ii) $\bar{h}_j$ is region j utility minus the average utility of all regions (Tabuchi 1986, Zeng 2002):

$$\bar{h}_j = v_j(q, s) - \frac{1}{m}\sum_{i=1}^{m} v_i(q, s).$$

▮

We need further considerations on the boundary equilibria (where $s_j = 0$ for some j) to use these dynamics. According to the replicator dynamics, if a population type dies out, it cannot replicate anymore and its population share will remain zero. This is not appropriate in the context of regional migration since regions do not really die out. People will move to an empty region if they have a higher potential utility there. (Zeng 2002 specifies in detail the boundary equilibrium for Example 1.ii.) However, to define a reasonable notion of boundary equilibrium, discontinuous dynamics are needed. Hence the differentiable approach has no power on the boundary. We study C^1 dynamical systems in an open domain and focus on interior equilibria. (This does not harm our argument, since the pitchfork bifurcation is interior.)

3 The Genericity of Bifurcations

Bifurcations occur where the dynamic behavior changes qualitatively as the vector field h changes. This is formally defined below.

Definition 2. (Perko 2001, 4.1) Let $C^1(E)$ denote the set of all C^1 maps from E to $\Re^n$ where E is an open subset of $\Re^n$, and $\|.\|_1$ denote the C^1 norm.[5] A vector field $h \in C^1(E)$ is *structurally stable* if there is an $\varepsilon > 0$ such that for all $g \in C^1(E)$ with $\|h - g\|_1 < \varepsilon$, h and g are topologically equivalent on E (i.e., there is an E to E homeomorphism that maps all trajectories of $\dot{\lambda} = h(\lambda)$ onto trajectories of $\dot{\lambda} = g(\lambda)$ and preserves their orientation by time.) If h is not structurally stable, it belongs to the *bifurcation set* of $C^1(E)$.

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[5] For $f \in C^1(E)$, $\|f\|_1 = \sup_{x\in E} \|f(x)\| + \sup_{x\in E} \|Df(x)\|$. Without confusion, $\|.\|$ denotes the Euclidean norm and the matrix norm separately.

In application, a vector field changes as parameters change. We embed h into a finite dimensional parameter space Θ; $h : S^{m-1} \times \Theta \rightarrow \Re^{m-1}$ where $\Theta \subset \Re^l$ is an open subset of $\Re^l$. The changes in parameters result in a multidimensional path in Θ. We focus on smooth paths. A *path* of parameter change in Θ is a C^1 map $\eta : [0,1] \rightarrow \Theta$. The space of paths is endowed with the C^1 norm. Note that each $\eta(t)$, $t \in [0,1]$, on the path determines a vector field $h(\lambda, \eta(t))$. The parameter value t is a *bifurcation value* if the vector field $h(\lambda, \eta(t))$ is not structurally stable. Define $E(\eta) = \{(\lambda, t) \in S^{m-1} \times [0,1] \, | h(\lambda, \eta(t)) = 0\}$ as the *equilibrium diagram* of h along path η. This is the slice of the equilibrium set taken along path η. Each element $(\lambda, t) \in E(\eta)$ is an equilibrium point of h for the parameter value $\eta(t) \in \Theta$.

A segment on the equilibrium diagram is called an equilibrium locus. An equilibrium diagram has a split bifurcation where there are crossing equilibrium loci. This is defined formally as follows: An *equilibrium locus* from an equilibrium point $(\lambda, t) \in E(\eta)$ is the image of a continuous map $e : [0,1] \rightarrow E \times [0,1]$ such that $e(0) = (\lambda, t)$ and $e(z) \in E(\eta)$ for $z \in [0,1]$. Path η has a *split bifurcation* at $\hat{t}$ if there is $\left(\hat{\lambda}, \hat{t}\right) \in E(\eta)$ such that there are more than two distinct equilibrium loci from $\left(\hat{\lambda}, \hat{t}\right)$. It is necessarily a bifurcation since the number of equilibria changes in the neighborhood of $\hat{t}$. A pitchfork bifurcation is indeed a split bifurcation: the three spikes and the handle meet at a point.

A parameter space Θ is *regular* for f if $D_{(\lambda,\theta)} h(\lambda, \theta)$ has full rank whenever $h(\lambda, \theta) = 0$. Our results rely on the full rank of the Jacobian matrix. We want to study generic properties of paths. A class of paths are *generic* if they form an open and dense (in the C^1 norm) set in the space of smooth paths in Θ .

Proposition 3. *In a regular parameter space Θ, there is an open and dense set of η without split bifurcations.*

Proof. See the Appendix. ∎

Thus, generic paths do not have split bifurcations. How about the occurrence of bifurcations? This relates to the notion of a regular economy. An economy θ is *regular* if 0 is a regular value of $h(., \theta)$. If θ is not a regular economy, then $h(., \theta)$ is not structurally stable (Rosser 1991, ch.2): In this case, one of the eigenvalues of $D_\lambda h\left(\hat{\lambda}, \theta\right)$ is 0. Since regular economies are of full Lebesgue measure in Θ (Debreu 1970, 1976), we can perturb θ into a regular one so that 0 is not an eigenvalue anymore. This results in a qualitative change in the dynamics. The vector fields of regular economies, however, can be stable or unstable. This is because we only have the full rank of the Jacobian matrix (with respect to endogenous variables), while structural stability is determined by finer properties such as its eigenvalues.

Fortunately, in the case of two regions, we can show that paths with bifurcations is not generic: When there are two regions, the dynamical system is one-dimensional. So, an equilibrium is hyperbolic[6] if and only if $D_\lambda h \neq 0$. Moreover, a regular economy has a finite number of equilibria (Debreu 1970). This means θ is not a bifurcation value of h if and only if it is a regular economy (Perko 2001, 4.1, Theorem 3). When $m = 2$, the set of paths without bifurcations is open in Θ (Kung 2002, Proposition 6).

Therefore, paths with bifurcations are not generic, since the complement set is open. The equilibrium diagrams of regular paths are generic. In the proof of Proposition 3, we show that for a regular η, $E(\eta)$ is a C^1 curve in a neighborhood of any $\left(\hat{\lambda}, \hat{t}\right) \in E(\eta)$. Thus, a generic equilibrium diagram contains open segments of (nonintersecting) smooth curves (it is a one-dimensional differentiable manifold).[7] When $m = 2$, this allows only the saddle-node bifurcations (Figures 2 and 3) or no bifurcations (Figure 4).

4 Two-Region Migration with Monopolistic Competition and Increasing Returns

In this section, we study the migration of mobile workers between two regions (à la Fujita et al. 1999). We present an example of a regular parameterization for the dynamic economy. More precisely, we construct a regular parameter space for the static economy and show that it is regular for the dynamic economy as well (this is not generally true for $m > 2$). Therefore, results in Section 2 apply to this model.

First, we introduce the model. There are two regions in the economy which are denoted by $i \in \{1, 2\}$. There are two types of commodities: a homogeneous agricultural good and differentiated manufactured goods. There is a continuum of manufactured goods of size $n \in \Re_+$, which is determined endogenously. Each manufactured good is

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[6] An equilibrium λ of vector field h is *hyperbolic* if none of the eigenvalues of $D_\lambda h(\lambda, \theta)$ has zero real part.

[7] It is known that "all bifurcations of one-parameter families at an equilibrium with a zero eigenvalue can be perturbed to saddle-node bifurcations" (Guckenheimer and Holmes 1997, p. 149).

denoted by $j \in [0, n]$. Let $p_i^A \in \Re_{++}$ denote the local price of the agricultural good, and let $p_i(j)$, where $p_i : [0, n] \to \Re_{++}$ is a measurable function, denote the local price of each manufactured good j in region i. There are two types of consumers: immobile farmers of population L_i^A in region $i \in \{1, 2\}$, and mobile workers of population L^M who migrate between regions. Each of them is endowed with one unit of labor.

Let $A \in \Re_+$ denote the quantities of the agricultural good, and let $m(j)$, where $m : [0, n] \to \Re_+$ is a measurable function, denote the quantity of manufactured good j. All consumers have the same utility function

$$u(m, A) = M^{\mu} A^{1-\mu}$$

where $M = \left[\int_0^n m(j)^{\rho} dj\right]^{\frac{1}{\rho}}$ and $0 < \mu, \rho < 1$. A consumer in region i with income Y solves the following problem.

$$\begin{array}{c} \underset{A,\, m(j) \in \Re_+}{Max} \;\; u(m, A), \\ s.t.\; p_i^A A + \int_0^n p_i(j)\, m(j)\, dj = Y. \end{array} \tag{4}$$

The demand functions are

$$\begin{array}{c} \hat{A}_i(Y) = (1-\mu) Y / p_i^A, \\ \hat{m}_i(j, Y) = \mu Y G_i^{\frac{\rho}{1-\rho}} / p_i(j)^{\frac{1}{1-\rho}}, \end{array}$$

where $G_i = \left[\int_0^n p_i(j)^{\frac{\rho}{\rho-1}} dj\right]^{\frac{\rho-1}{\rho}}$ is the manufacturing price index.

The agricultural good is produced with labor by farmers with a one-to-one input-output ratio. The transportation of the agricultural good bears no cost. Thus, the equilibrium agricultural price is the same in both regions by no arbitrage; let $p_1^A = p_2^A = p^A$. Farmers retain all the revenue; they have income p^A.

Manufactured goods are produced by firms that employ mobile workers. Labor is the only input required. All firms have the same inverse production function

$$l = F + cq$$

where $F, c > 0$ are the fixed and the marginal input requirements. This means l units of labor are required for q units of output. The production technology exhibits increasing returns to scale due to the fixed costs. There is free entry into the market. Because of increasing returns to scale, each j-good is produced by and is the only product of an operating firm. Operating firms choose locations and engage in Chamberlinian monopolistic competition. Each firm chooses a location and charges a uniform free on board (f.o.b.) price for its product. Firms make decisions simultaneously. Let $w_i \in \Re_{++}$ denote the wage rate in region i. Suppose a firm locates in region i, charges price p, pays wage w_i, and sells output $q(p)$, where $q : \Re_{++} \to \Re$ is the demand of consumers. Its profit is

$$\pi_i(p) = pq(p) - w_i\left[F + cq(p)\right].$$

A firm in region i solves the following problem.

$$\underset{p\in\Re_{++}}{Max}\ \pi_i(p). \tag{5}$$

Because of the assumed constant elasticity utility function and the iceberg transportation cost (detailed later), the elasticity of demand facing a firm is independent of the locations of its consumers. (This is widely known; see Fujita et al. 1999.) A monopolistically competitive firm charges a price marked up from the marginal cost. The profit-maximizing price for a firm in region i is $p_i = cw_i/\rho$. Its maximized profit is

$$\pi_i = \frac{1-\rho}{\rho}cw_i\left[q - \frac{F}{(1-\rho)c}\right].$$

The transportation cost of manufacturing goods takes the Samuelson iceberg form. If one unit of good is shipped across regions, $1/T$ unit arrives. Since firms are identical and their behavior differs only in location, we label firms and their products with their locations. This simplifies the notation to $j \in \{1, 2\}$. Let p_i^j denote the price of region j products in region i, and $\hat{m}_i^j(Y)$ denote the demand for region j products of region i consumers (to replace $p_i(j)$ and $\hat{m}_i(j, Y)$). Let n_i denote the number of firms in region i. The total number of operating firms equals the total variety of products; $n_1 + n_2 = n$. Note that $G_i = \left[n_1\left(p_i^1\right)^{\frac{\rho}{\rho-1}} + n_2\left(p_i^2\right)^{\frac{\rho}{\rho-1}}\right]^{\frac{\rho-1}{\rho}}$.

A region i firm charges a f.o.b. price $p_i = cw_i/\rho$. Thus, $p_i^i = p_i$ and $p_i^j = p_jT$ for $j \neq i$ by no arbitrage. Substituting Y with w_i, we have region i workers' indirect utility

$$v_i = \mu^\mu(1-\mu)^{1-\mu}w_iG_i^{-\mu} \text{ for } i \in \{1, 2\}.$$

Workers are freely mobile. They choose a region that offers a higher utility level.

An *economy* is an 8-tuple $\left(L_1^A, L_2^A, L^M, \mu, \rho, c, F, T\right)$. Let L_i^M denote the worker population in region i, and A_i, m_i^j denote their consumptions of agricultural and manufactured goods respectively. Let A_{Ai}, m_{Ai}^j denote the consumption of farmers in regional i. Let q^j denote the output level of region j firms. An *allocation* in the economy is an 18-tuple $\left\{L_i^M, A_i, A_{Ai}, \left\{m_i^j, m_{Ai}^j\right\}_{j=1}^2, n_i, q^i\right\}_{i=1}^2$ (equal treatment of consumers and firms at the same location is implied; this is harmless since they will behave the same in equilibrium). A *feasible* allocation satisfies the following constraints:

$$L_1^M + L_2^M = L^M. \tag{6}$$

$$L_1^M m_1^1 + L_1^A m_{A1}^1 + L_2^M m_2^1 T + L_2^A m_{A2}^1 T - q^1 = 0. \tag{7}$$

$$L_1^M m_1^2 T + L_1^A m_{A1}^2 T + L_2^M m_2^2 + L_2^A m_{A2}^2 - q^2 = 0. \tag{8}$$

$$L_1^M A_1 + L_1^A A_{A1} + L_2^M A_2 + L_2^A A_{A2} = L^A. \tag{9}$$

Equation (6) balances the total worker population and the total demand for workers. Equations (7) and (8) balance the consumption of manufactured goods and their production. Equation (9) balances agricultural consumption and production.

Facing prices p^A, p_1, p_2, w_1, and w_2, the following conditions are satisfied in equilibrium. (Note that we have already imposed no-arbitrage on the transportation of goods.) The free entry of new firms drives the profit of operating firms down to zero.

$$\pi_1 = \pi_2 = 0. \tag{10}$$

Workers are freely mobile and identical, so their utility levels are equal if there are workers living in both regions.

$$v_1 = v_2, \text{ if } L_1^M, L_2^M > 0. \tag{11}$$

Note that workers' utility v_i is not defined if there are no workers in region i. To handle the boundary equilibria, we can define the potential wage as the limit of the equilibrium wage when worker population goes to zero. Then, the potential utility is derived accordingly. Having all workers in one region constitutes an equilibrium if none of them wants to move out; that is, the potential utility in the other region is not higher. As mentioned in Section 2, we focus on interior equilibria only: $L_1^M, L_2^M > 0$.

An *equilibrium* is a list of prices and a feasible allocation such that conditions (4), (5), (10), and (11) are satisfied. We simplify the system in the Appendix and reach the following definition.

Definition 4. An *equilibrium* is a list $(\lambda, w_1, w_2) \in (0,1) \times S^2$ that satisfies the following equations. (Note that $\lambda = L_1^M / L^M$, $p^A = 1 - w_1 - w_2$, and $\bar{q} = \frac{F\rho}{c(1-\rho)}$.)

$$\frac{\lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_1}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_1}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{(1-\lambda) L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_1}{\rho} T\right)^{\frac{1}{1-\rho}}} + \frac{L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_1}{\rho} T\right)^{\frac{1}{1-\rho}}} - \bar{q} = 0, \quad (12)$$

$$\frac{\lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_2}{\rho} T\right)^{\frac{1}{1-\rho}}} + \frac{L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_2}{\rho} T\right)^{\frac{1}{1-\rho}}} + \frac{(1-\lambda) L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}}}{\left(\frac{cw2}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}}}{\left(\frac{cw2}{\rho}\right)^{\frac{1}{1-\rho}}} - \bar{q} = 0. \quad (13)$$

$$w_1 - w_2 G_2^{-\mu} G_1^{\mu} = 0. \quad (14)$$

To apply the two-stage approach, we need to verify assumption A. Conditions A.i to A.iii are straightforward. Condition A.iv guarantees a unique commodity equilibrium (so that the two-stage approach is sound) and is necessary for the vector field to be C^1 when the dynamics in Example 1 are used. Since this paper studies the genericity of bifurcations rather than the existence of equilibria, we do not repeat the task but rather assume that previous work has found the parameter range where A.iv is satisfied.

It is difficult to check whether the parameters introduced so far constitute a regular parameter space. We augment the system with more parameters and show that the new parameter space is regular. Let Θ be an open subset of $\Re^3_{++}$, its elements are denoted by $\theta = (v, \gamma)$, where $v \in \Re^2_{++}$ and $\gamma \in \Re_{++}$. These parameters enter the model in the following way. (i) v parameterizes "regional fixed input": The fixed labor input of firms in region i is $F + v_i$. Note that firms' chosen prices (cw_i/ρ) are not affected by v_i, but the zero-profit output level for region i firms is now $\frac{\rho(F+v_i)}{c(1-\rho)}$. (ii) γ parameterizes "regional amenity": Workers have preferences over regions in the following way. If a worker lives in region 2, her utility function is unchanged. If she lives in region 1, her utility is factored up by $1/\gamma$. The new utility function of region 1 workers is

$$\frac{1}{\gamma} u\left(m_1^1, m_1^2, A_1\right).$$

Lemma 5. *Θ is a regular parameter space for vector fields $h(\lambda, \theta)$ defined on page 8.*

Proof. See the Appendix. ▮

Note that the augmented parameter space, with the new and original parameters combined, is also regular. The key to this construction is that these new parameters break the symmetric parameterization of the original model. Take the amenity factor γ for example. Consider a two-parameter system with T and γ (set $L_1^A = L_2^A$, $v = 0$). If $\gamma = 1$, which means the workers do not particularly prefer one region to the other, then the equilibrium diagram along the transportation cost will have a pitchfork bifurcation. If we set $\gamma > 1$ or < 1, the resulting equilibrium diagram (still, along T) will look like Figures 2 or 3. Therefore, the pitchfork bifurcation is the result of a symmetric parameterization along transportation cost. It is not robust against arbitrary perturbations in parameters.

5 Conclusion

It is important to study an economic system in a low dimensional parameter space, especially when the chosen parameters are the main forces changing the economy. The study of a specific type of bifurcation may provide strong insight into the dynamics of regional migrations. However, the real world has countless parameters that are neglected in a model. Since the choice of parameters affects the dynamic behavior of a system, this raises the following question: "what kind of behavior is typical given enough parameters?" It is inevitable to work on models with only a few parameters, but, at the same time, it is not desirable to study properties that are rare occurrences in the real world. We show that, in a regular parameter space, there is a generic set of smooth parameter paths without split bifurcations, and in the case of two regions, smooth paths with bifurcations are not generic. In particular, we study the migration of workers in a two-region economy where firms that produce with increasing returns to scale technologies compete monopolistically as in Fujita et al. (1999). We resent an example of a regular parameter space for this model.

Regional divergence can occur when the symmetric equilibrium becomes unstable. However, the popular pitchfork bifurcation, as a split bifurcation, is not a robust representation of the dynamics involved . Thus, a broader picture is needed to understand the full extent of the dynamic behavior of this model. In the case of two regions, the generic case contains the saddle-node bifurcations or no bifurcations .

Appendix

Simplify the system (Section 4)

First, by (4), the demand of workers for the agricultural good and manufactured goods are $A_i = (1-\mu)\, w_i / p^A$ and $m_i^j = \mu w_i G_i^{\frac{\rho}{1-\rho}} \left(p_i^j\right)^{\frac{-1}{1-\rho}}$ respectively, and the demand of farmers for the two types of goods are $A_{Ai} = (1-\mu)$ and $m_{Ai}^j = \mu p^A G_i^{\frac{\rho}{1-\rho}} / \left(p_i^j\right)^{\frac{1}{1-\rho}}$. By (5), $p_i = cw_i/\rho$. Then by (10),

$$\begin{aligned} q^1 &= q^2 = \frac{F\rho}{c\,(1-\rho)} = \bar{q}, \\ n_i &= \frac{L_i^M}{F + c\frac{F\rho}{c(1-\rho)}} = \frac{L_i^M\,(1-\rho)}{F}. \end{aligned}$$

Let $\lambda = L_1^M / L^M$; by (6) and the fact that we study interior equilibria only, $0 < \lambda < 1$. Equations (7), (8) and (9) are dependent because of Walras' law; we take the last one as redundant. Plug the above results into (7) and (8), and we have

$$\frac{\lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_1}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_1}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{(1-\lambda)\, L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_1}{\rho} T\right)^{\frac{1}{1-\rho}}} + \frac{L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_1}{\rho} T\right)^{\frac{1}{1-\rho}}} - \bar{q} = 0,$$

$$\frac{\lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_2}{\rho} T\right)^{\frac{1}{1-\rho}}} + \frac{L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_2}{\rho} T\right)^{\frac{1}{1-\rho}}} + \frac{(1-\lambda)\, L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}}}{\left(\frac{cw2}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}}}{\left(\frac{cw2}{\rho}\right)^{\frac{1}{1-\rho}}} - \bar{q} = 0.$$

Equation (11) can be replaced with

$$w_1 - w_2 G_2^{-\mu} G_1^{\mu} = 0.$$

Finally, normalizing prices to $w_1 + w_2 + p^A = 1$ and letting $p^A = 1 - w_1 - w_2$, the remaining is a system of three variables and three equations.

Proof of Proposition 3.[8] A path η is called *regular* if $D_{(\lambda,t)} h\,(\lambda, \eta\,(t))$ has full rank (which is $m-1$) whenever $h\,(\lambda, \eta\,(t)) = 0$ for $t \in [0,1]$. First, we show that a regular path does not have split bifurcations.

Suppose $D_{(\lambda,t)} h \left(\hat{\lambda}, \eta\left(\hat{t}\right)\right)$ has full rank, which is $(m-1) \times m$, then it has $m-1$ independent columns. Without loss of generality, suppose $D_{(\lambda_{-1},t)} h \left(\hat{\lambda}, \eta\left(\hat{t}\right)\right)$ has full

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[8] The proof is adopted from Lemma 2 and Proposition 4 in Kung (2002), which deals with one-dimensional parameterizations of vector fields.

rank (where λ_{-1} denotes $(\lambda_2, ..., \lambda_{m-1})$). By the implicit function theorem, $h\left(\hat{\lambda}, \eta\left(\hat{t}\right)\right) = 0$ can be locally solved by a C^1 function of λ_1. This means $E(\eta)$ is a C^1 curve in a neighborhood of $\left(\hat{\lambda}, \hat{t}\right)$. So, there can be only two equilibrium loci from $\left(\hat{\lambda}, \hat{t}\right)$.

Next, we show the set of regular paths is open and dense. Since a perturbation (in the C^1 norm) yields small changes in $h(\lambda, \eta(t))$ and its first order derivatives, openness is straightforward by the continuity of $h(\lambda, \eta(t))$ and $D_{(\lambda,t)}h(\lambda, \eta(t))$. To show density, we need the following theorem (see Guillemin and Pollack 1974, p. 68, and Mas-Colell 1985, p. 320). For a C^r map $f : M \to N$ between manifolds, $y \in N$ is a *regular value* if $Df(x)$ has full rank whenever $f(x) = y$.

Transversality Theorem. *Suppose that* $f : X \times S \to \Re^m$ *is a* C^r *map where* X, S *are* C^r *boundariless manifolds with* $r > \max\{0, \dim(X) - m\}$*; let* $f_s(x) = f(x, s)$, $f_s : X \to \Re^m$. *If* $c \in \Re^m$ *is a regular value for* f*, then except for* s *in a set of measure zero in* S*,* c *is a regular value for* f_s.

Augment the parameter space with $A = \Theta$ whose elements are $a \in A$. For a path η, we construct a map $\varphi : S^{m-1} \times [0,1] \times A \to \Re^{m-1}$,

$$\varphi(\lambda, t, a) = h(\lambda, \eta(t) + a).$$

Apparently, $D_{(\lambda,a)}\varphi = D_{(\lambda,\theta)}h(\lambda, \theta)$ whenever $\theta = \eta(t) + a$ for any t for all a close to 0. And $D_{(\lambda,\theta)}h(\lambda, \theta)$ has full rank whenever $h(\lambda, \theta) = h(\lambda, \eta(t) + a) = 0$ because Θ is regular. This means $D_{(\lambda,a)}\varphi$, and also $D_{(\lambda,t,a)}\varphi$, has full rank whenever $\varphi(\lambda, t, a) = 0$. By the Transversality Theorem, for almost all a, $D_{(\lambda,t)}\varphi(\lambda, t, a)$ has full rank whenever $\varphi(\lambda, t, a) = 0$. So, we can find $\bar{a}$ arbitrarily close to 0 such that $D_{(\lambda,t)}\varphi(\lambda, t, \bar{a})$ has full rank whenever $\varphi(\lambda, t, \bar{a}) = 0$. Then η', where $\eta'(t) = \eta(t) + \bar{a}$, is a regular path arbitrarily close to η. ▮

Proof of Lemma 5. Parameter v changes conditions (12) and (13) to

$$\frac{\lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_1}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}}}{\left(\frac{cw_1}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{(1-\lambda) L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_1}{\rho}T\right)^{\frac{1}{1-\rho}}} + \frac{L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_1}{\rho}T\right)^{\frac{1}{1-\rho}}} - \frac{\rho(F + v_1)}{c(1-\rho)} = 0, \tag{15}$$

$$\frac{\lambda L^M \mu w_1 G_1^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_2}{\rho}T\right)^{\frac{1}{1-\rho}}} + \frac{L_1^A \mu p^A G_1^{\frac{\rho}{1-\rho}} T}{\left(\frac{cw_2}{\rho}T\right)^{\frac{1}{1-\rho}}} + \frac{(1-\lambda) L^M \mu w_2 G_2^{\frac{\rho}{1-\rho}}}{\left(\frac{cw2}{\rho}\right)^{\frac{1}{1-\rho}}} + \frac{L_2^A \mu p^A G_2^{\frac{\rho}{1-\rho}}}{\left(\frac{cw2}{\rho}\right)^{\frac{1}{1-\rho}}} - \frac{\rho(F + v_2)}{c(1-\rho)} = 0. \tag{16}$$

Note that γ does not affect consumers' demand; it plays a role in their location choices only. Condition (14) is changed to

$$\frac{w_1}{\gamma} - w_2 G_2^{-\mu} G_1^{\mu} = 0. \tag{17}$$

This is because equilibrium wage rates are adjusted according to the amenity factor. Next, we show that Θ is regular for the static economy. Let f_1, f_2, g denote the left-hand side functions of (15), (16), and (17), respectively. Define $F = (f_1, f_2, g)$, $F : \Re^3_{++} \to \Re$. We have

$$D_\theta F(p, \lambda, \theta) = \begin{pmatrix} -\frac{\rho}{c(1-\rho)} & 0 & 0 \\ 0 & -\frac{\rho}{c(1-\rho)} & 0 \\ 0 & 0 & w_1 \end{pmatrix}$$

So, $D_{(p,\lambda,\theta)}F = \left(D_{(p,\lambda)}F, D_\theta F\right)$ always has full rank at every equilibrium for all $\theta \in \Theta$.

The conditions imposed on h (page 8) ensure that Θ is a regular parameter space for $h(\lambda, \theta)$ if and only if it is regular for $g(p(\lambda), \lambda, \theta)$. The next lemma shows that the Jacobian matrix of $g(p(\lambda), \lambda)$ at λ has full rank (which is 1), if and only if that of $F(p^*, \lambda)$ at (p^*, λ), $p^* = p(\lambda)$ does. So, Θ is regular for $g(p(\lambda), \lambda, \theta)$. Note that $|-D_p f(p, \lambda, \theta)| > 0$ (i.e., A.iv) for all θ in a neighborhood of $(0, 0, 1)$ by continuity. We further restrict Θ to the range where A.iv is satisfied. Note that $D_\theta F(p, \lambda, \theta)$ has full rank regardless of the values of the original parameters in the relevant range.

Lemma 6. *When $m = 2$, for any θ, for all (p^*, λ) such that $F(p^*, \lambda) = 0$ (i.e., $p^* = p(\lambda)$) $\left|D_{(p,\lambda)}F(p^*, \lambda)\right| = 0$ if and only if $\frac{d}{d\lambda}g(p(\lambda), \lambda) = 0$.*

Proof. Since $p(\lambda)$ is derived from $f(p, \lambda) = 0$ and $|D_p f(p(\lambda), \lambda)| \neq 0$ (by assumption A.iv), by the implicit function theorem,

$$D_\lambda p(\lambda) = -\left[D_p f(p(\lambda), \lambda)\right]^{-1} D_\lambda f(p(\lambda), \lambda).$$

And

$$\begin{aligned} \frac{d}{d\lambda} g(p(\lambda), \lambda) &= D_p g(p(\lambda), \lambda) D_\lambda p(\lambda) + D_\lambda g(p(\lambda), \lambda), \\ &= -D_p g(p(\lambda), \lambda)\left[D_p f(p(\lambda), \lambda)\right]^{-1} D_\lambda f(p(\lambda), \lambda) + D_\lambda g(p(\lambda), \lambda). \end{aligned}$$

Next, we expand $\left|D_{(p,\lambda)}F\right|$ along its nth row.

$$\begin{aligned}\left|D_{(p,\lambda)}F\right| &= \left|\begin{array}{cc} D_pf & D_\lambda f \\ D_pg & D_\lambda g \end{array}\right| \\ &= D_\lambda g\left|D_pf\right| + \textstyle\sum_{k=1}^{n-1}(-1)^{n+k}D_{p_k}g\left|M_k\right|.\end{aligned}$$

where M_k is the matrix obtained from D_pf by eliminating its kth column and adding $D_\lambda f$ as the last column.

Let x be the vector that solves

$$(D_pf)\,x = D_\lambda f.$$

Then $x = [D_pf]^{-1}D_\lambda f$, and by Cramer's rule, $x_k = \left|D_pf\right|^{-1}\left|N_k\right|$ where N_k is the matrix obtained from D_pf by replacing its kth column with $D_\lambda f$. Notice that M_k and N_k differ only in the positions of their columns. N_k can be obtained from M_k by switching the last column, $D_\lambda f$, with the preceding column $n-k-1$ times. Thus, $\left|N_k\right| = (-1)^{n-k-1}\left|M_k\right|$. So,

$$\begin{aligned}\left|D_{(p,\lambda)}F\right| &= D_\lambda g\left|D_pf\right| + \textstyle\sum_{k=1}^{n}(-1)^{2n-1}D_{p_k}g\left|N_k\right| \\ &= D_\lambda g\left|D_pf\right| - \textstyle\sum_{k=1}^{n}D_{p_k}g\left|D_pf\right|x_k \\ &= D_\lambda g\left|D_pf\right| - \left|D_pf\right|D_pgx \\ &= \left|D_pf\right|\left[D_\lambda g - D_pg\left[D_pf\right]^{-1}D_\lambda f\right] \\ &= \left|D_pf\right|\frac{d}{d\lambda}g\left(p\left(\lambda\right),\lambda\right).\end{aligned}$$

▮

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