***Space in General Equilibrium****

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**Introduction**

How do households distribute themselves in a spatial dimension? Do they distribute themselves efficiently? What determines land use patterns? Standard intermediate microeconomic theory is ill-equipped to answer these questions because households and others using land care about the location, as well as the quantity, of land that they consume. As a result, some of the standard assumptions used in our models lead to predictions that are inconsistent with observed behavior. For example, suppose that households like to consume land and a composite consumption good (a bundle of everything that is not land). A key assumption in standard microeconomic theory is that

* We would like to thank William Alonso and Masa Fujita for divine inspiration. Helpful comments were provided by an anonymous referee and Richard Arnott. The authors retain responsibility for any errors.

preferences are strictly convex, which implies under symmetry of preferences that households prefer owning an acre of land and a unit of composite good to owning two acres of land and no composite good or to owning two units of composite good and no land, all else being equal. However, if households care about the location of the land they consume, then "land in the city" and "land in the suburbs" are essentially two different goods. In this case, convex preferences and symmetry imply that households prefer owning one acre of land in the city and one acre of land in the suburbs to owning two acres of land in the city and none in the suburbs and to owning two acres of land in the suburbs and none in the city. In general, households will want to diversify their landholdings. This is inconsistent with observed behavior.

To answer the questions we posed, we turn to the Alonso (1964) model, and rely on Berliant and Fujita (1992) for an analysis of it. This model is a straightforward extension of standard microeconomic theory to urban economics that includes land as a commodity while at the same time incorporating differences between land at different locations in a natural way. In this model, a finite number of identical households live in a long, narrow (one dimensional) city. They like to consume a composite consumption good and land. In particular, households consume parcels or intervals of land. They simultaneously choose how much composite good and land to consume and the location of their parcel. Households commute from the land they consume to an exogenously determined location, the city center or central business district, in order to receive their endowment of composite good. Commuting is costly, so they care about the location of their parcel of land because the cost of commuting between it and the central business district varies with the distance between the two. The Alonso model is distinguished from other models of urban economics by the following two features: (1) The use of a finite number of households (two in this article) instead of a continuum, and (2) the assumption that households like intervals of land in one dimension.

We begin with a brief review of the tools and definitions used in standard microeconomic theory. Next, we introduce the Alonso model. We follow this introduction with an extension of the standard tools and definitions for this model. We then provide a specific example of the model to illustrate how two identical households divide up the available land in a long, narrow city. We conclude with comments on extensions of the basic Alonso model.

**A Brief Review Of Intermediate Microeconomic Theory**

We begin with an exchange economy populated by two households, A and B, who like to consume two goods, 1 and 2. Household *i*'s consumption of good *j* is $x_j^i$. Note that we use superscripts to identify households and subscripts to identify goods. Households' preferences over different consumption bundles, or different combinations of goods 1 and 2, are represented by utility functions. The utility function for household *i* is $U^i : R^2 \to R$, where $U^i(x_1^i, x_2^i)$ is the level of utility household *i* enjoys when it consumes the bundle $(x_1^i, x_2^i)$. Preferences are *convex* if, for all possible consumption bundles $(x_1^i, x_2^i)$ and $(\hat{x}_1^i, \hat{x}_2^i)$, $U^i(x_1^i, x_2^i) = U^i(\hat{x}_1^i, \hat{x}_2^i)$ implies that for all $\alpha \in (0,1)$, $U^i(\alpha x_1^i + (1-\alpha)\hat{x}_1^i, \alpha x_2^i + (1-\alpha)\hat{x}_2^i) \geq U^i(x_1^i, x_2^i)$. In other words, consuming a linear combination of two bundles that both generate the same level of utility does not diminish utility. An *indifference curve* is a collection of consumption bundles that generate the same level of utility for a household. Thus, a household is indifferent between all the consumption bundles that make up an indifference curve. Figure 1 illustrates an indifference curve, $IC^i_{\overline{U}}$, where $IC^i_{\overline{U}} = \{(x^i_1, x^i_2) \mid U^i(x^i_1, x^i_2) = \overline{U}\}$ is the set of bundles that generate utility level $\overline{U}$ for household *i*. Household *i*'s *marginal utility of good j* is $MU^i_j : R^2 \to R$, where $MU^i_j(x^i_1, x^i_2)$ is the additional utility household *i* would get if it consumed an additional unit of good *j*, given that it is consuming the bundle $(x^i_1, x^i_2)$ and maintaining utility level $\overline{U}$. The marginal utility of good *j* is the derivative of household *i*'s utility function with respect to good *j*,

$$MU^i_j(x^i_1, x^i_2) = \frac{\partial U^i(x^i_1, x^i_2)}{\partial x^i_j}.$$

Household *i*'s *marginal rate of substitution of good 1 for good 2* is $MRS^i_{1,2} : R^2 \to R$, where

$$MRS^i_{1,2}(x^i_1, x^i_2) = \frac{MU^i_1}{MU^i_2}$$

identifies how much of good 2 household *i* is willing to give up in order to get one more unit of good 1, given that it is consuming the bundle $(x^i_1, x^i_2)$. Figure 1 shows that $MRS^i_{1,2}(\hat{x}^i_1, \hat{x}^i_2)$ is the slope of household *i*'s indifference curve at the bundle $(\hat{x}^i_1, \hat{x}^i_2)$. Household *i*'s endowment of good *j* is $\omega^i_j$. An *allocation* is a list of consumption bundles for each household: $(x^A_1, x^A_2, x^B_1, x^B_2)$. An allocation is *feasible* if there is material balance in both goods, $x^A_1 + x^B_1 = \omega^A_1 + \omega^B_1$ and $x^A_2 + x^B_2 = \omega^A_2 + \omega^B_2$. The set of feasible allocations are those contained in the standard Edgeworth box, illustrated in Figure 2. The width of the box is the total quantity of good 1 available in the economy, $\omega^A_1 + \omega^B_1$. The height of the box is the total quantity of good 2 available in the economy, $\omega^A_2 + \omega^B_2$.

Household A's origin is the point (0,0), and household B's origin is the point $(\omega_1^A + \omega_1^B, \omega_2^A + \omega_2^B)$. A feasible allocation is *Pareto optimal*, or is *efficient*, if there is no other feasible allocation that keeps every household at least as well off and makes some household better off. We can use marginal rates of substitution to characterize the set of Pareto optimal allocations: If the feasible allocation $(x_1^A, x_2^A, x_1^B, x_2^B)$ is Pareto optimal, then $MRS_{1,2}^A(x_1^A, x_2^A) = MRS_{1,2}^B(x_1^B, x_2^B)$, and if preferences are convex, then $MRS_{1,2}^A(x_1^A, x_2^A) = MRS_{1,2}^B(x_1^B, x_2^B)$ implies that $(x_1^A, x_2^A, x_1^B, x_2^B)$ is Pareto optimal, provided that it is feasible. (Please note that we are skipping some technical assumptions and details here.) Loosely speaking, the set of Pareto optimal allocations, or the *contract curve*, is the set of allocations in the Edgeworth box at which households' indifference curves are tangent to each other. The intuition, illustrated in Figure 2, is that if the marginal rates of substitution for the two households are unequal at an allocation, such as $(x_1^{'A}, x_2^{'A}, x_1^{'B}, x_2^{'B})$, then there are unexhausted gains from trade and the allocation is not efficient. Moreover, if the marginal rates of substitution at an allocation such as $(x_1^A, x_2^A, x_1^B, x_2^B)$ are equal, then the set of allocations that would make one household better off is disjoint from the set of allocations that would make the other household better off, so there is no way to make one household better off without making the other household worse off. Note that finding and characterizing the set of Pareto optimal allocations – the set of "best" allocations – is a normative exercise that says nothing about the quantities of goods each household will actually consume.

To find the actual distribution of goods across households, we use the concept of competitive equilibrium, a positive concept. Let good 2 be the numeraire, so $p_2 = 1$. A *competitive equilibrium* is a feasible allocation $(x_1^{*A}, x_2^{*A}, x_1^{*B}, x_2^{*B})$ and a price $p_1^*$ such that

1. Household A maximizes its utility $U^A(x_1^A, x_2^A)$ subject to its budget constraint,

   $p_1^* x_1^A + x_2^A \le p_1^* \omega_1^A + \omega_2^A$, and

2. Household B maximizes its utility $U^B(x_1^B, x_2^B)$ subject to its budget constraint,

   $p_1^* x_1^B + x_2^B \le p_1^* \omega_1^B + \omega_2^B$,

at $x_1^A = x_1^{*A}, x_2^A = x_2^{*A}, x_1^B = x_1^{*B}, x_2^B = x_2^{*B}$. An *equilibrium allocation* is an allocation $(x_1^{*A}, x_2^{*A}, x_1^{*B}, x_2^{*B})$ such that there exists a price $p_1^*$ that makes $(x_1^{*A}, x_2^{*A}, x_1^{*B}, x_2^{*B})$ and $p_1^*$ an equilibrium. Competitive equilibrium is a positive concept that helps understand how resources will be allocated using the price mechanism in a decentralized setting with no coordination between the households. Skipping further technicalities and assuming that households exhaust their budgets, the conditions equivalent to equilibrium are that each household's marginal rate of substitution is equal to the price ratio, so for *i* = A, B,

$$MRS_{1,2}^i(x_1^{*i}, x_2^{*i}) = \frac{p_1^*}{1},$$

and that markets for both goods clear, so for $j = 1,2$,

$$x_j^{*A} + x_j^{*B} = \omega_j^A + \omega_j^B.$$

Figure 1 shows that if the first condition is not satisfied for some household *i*, then there exists an affordable consumption bundle that makes that household better off, so household *i* is not maximizing its utility subject to its budget. The budget line is the set of bundles $(x_1^i, x_2^i)$ such that $p_1^* x_1^i + x_2^i = p_1^* \omega_1^i + \omega_2^i$. The slope of the budget line is $-p_1^*$.

The bundle $(\hat{x}_1^i, \hat{x}_2^i)$ is affordable, but $MRS^i(\hat{x}_1^i, \hat{x}_2^i) \neq p_1^*$. The bundle $(x_1^{*i}, x_2^{*i})$, for which $MRS_{1,2}^i(x_1^{*i}, x_2^{*i}) = p_1^*$, is also affordable and generates more utility than $(\hat{x}_1^i, \hat{x}_2^i)$.

The welfare theorems provide the connection between equilibrium allocations, a positive idea, and Pareto optimal allocations, a normative idea, in this simple model. The First Welfare Theorem states that under certain conditions every equilibrium allocation is Pareto optimal. The Second Welfare Theorem states that if preferences are convex and it is possible to redistribute endowments, then under certain technical assumptions every Pareto optimal allocation is an equilibrium allocation for some set of endowments.

For a more thorough discussion of the topics reviewed in this section, see, for example, Varian (1993).

**The Alonso Model**

The Alonso model adds space to the basic framework we just described. Our two households, A and B, now live in a long, narrow city of length *l*. The interval from 0 to *l* describes the length of the city. See Figure 3. Households consume a composite good and land. The quantity of composite good consumed by household *i* is $z^i$, the location of the driveway or the front of the lot occupied by household *i* is $x^i \in [0, l)$, and the quantity of land consumed or length of the lot occupied by household *i* is $s^i$. Thus, household *i* owns the interval $[x^i, x^i + s^i)$. There are *C* units of composite good available in the economy. Households must commute to the city center, located at the origin, in order to pick up their endowment of composite good. In doing so, households incur a cost of *t* units of composite good per unit distance they travel, measured from the front of their lot. Households A and B have the same utility function, $U(s, z)$, where *U* is increasing in both of its arguments and land is a normal good. Recall that a good is *normal* if households consume more of it when their income increases, all else being equal. Let $U^i = U(s^i, z^i)$. Since both households have the same utility function, we can say how well off one household is relative to the other. An *allocation* is a list specifying quantities of land consumed, quantities of composite good consumed, and driveway locations for both households: $(s^A, s^B, z^A, z^B, x^A, x^B)$. Thus, $s^A, s^B, z^A, z^B > 0$ and $x^A, x^B \in [0, l)$. An allocation is *feasible* if material balance is satisfied for both the composite good and land, namely

$$z^A + z^B + tx^A + tx^B = C,$$

$$[x^A, x^A + s^A) \cap [x^B, x^B + s^B) = \emptyset,$$

and

$$[x^A, x^A + s^A) \cup [x^B, x^B + s^B) = [0, l).$$

**Pareto Optimal Allocations And The Contract Curve**

A feasible allocation is a *Pareto optimum*, or is *efficient*, if there is no other feasible allocation that keeps every household at least as well off and makes some household better off. What do Pareto optima look like? Consider a feasible allocation, $(s^A, s^B, z^A, z^B, x^A, x^B)$, where $x^A = 0$ and $s^A > s^B$. In this case, household A lives closest to the city center and occupies a lot that is larger than the lot occupied by household B. Household A's commuting cost is zero, household B's commuting cost is $tx^B = ts^A$, and $z^A + z^B + tx^B = C$. This allocation is not a Pareto optimum because there exists another feasible allocation that makes at least one household better off without making the other household worse off. For example, consider a feasible allocation $(\hat{s}^A, \hat{s}^B, \hat{z}^A, \hat{z}^B, \hat{x}^A, \hat{x}^B)$, where households switch positions but consume the quantity of land given in the original assignment, so $\hat{s}^A = s^A$, and $\hat{s}^B = s^B$. Household B consumes the same quantity of composite good, so $\hat{z}^B = z^B$, and household B lives closest to the city center, so $\hat{x}^B = 0$ and $\hat{x}^A = s^B$. Please refer to Figure 4, which shows how the two allocations are related. Household B's commuting cost is now zero and household A's commuting cost is now $t\hat{x}^A = ts^B$. Since household A lives on a larger lot than household B, $t\hat{x}^A = ts^B < ts^A = tx^B$. This allocation is feasible:

$$\hat{z}^A = C - t\hat{x}^A - \hat{z}^B = C - ts^B - z^B > C - ts^A - z^B.$$

Thus, household B is just as well off as it was with bundle $(s^B, z^B)$, and household A is better off because it is consuming the same amount of land and more composite good. Generalizing this intuition leads to the following proposition.

**Proposition 1 (Berliant and Fujita 1992):** If $(s^{\circ A}, s^{\circ B}, z^{\circ A}, z^{\circ B}, x^{\circ A}, x^{\circ B})$ is a Pareto optimum, then

1. $x^{\circ A} < x^{\circ B}$ exactly when $s^{\circ A} < s^{\circ B}$, so household A lives closest to the city center if and only if it occupies a smaller lot,
2. $x^{\circ A} < x^{\circ B}$ implies that $U^A \leq U^B$, so if household A lives closest to the city center, then household B is at least as well off as household A, and
3. $U^A < U^B$ implies that $x^{\circ A} < x^{\circ B}$, so if household B is strictly better off than household A, then household A lives closest to the city center.

Pareto optima fall into one of two categories: (1) those in which household A lives closest to the city center and (2) those in which household B lives closest to the city center. The contract curve is the union of these two sets of Pareto optima. We can portray the contract curve using an Edgeworth box modified to account for the amount of composite good used up in commuting, where the modification will depend on which household lives closest to the city center.

When household A lives closest to the city center, we modify the box as illustrated in Figure 5. Household A's origin is the lower left corner of the box. The bottom of the box is simply the line segment $[0,l]$. The height of the left side of the box is the total amount of composite good available, *C*. The top of the box is a line that identifies how much composite good remains to be divided between households A and B after household B commutes to the city center. The distance household B commutes is equal to the length of household A's lot, $s^A$. Thus, the top of the box is the line $C - ts^A$. Finally, household B's origin is the point $(l, C - tl)$, where $C - tl$ is the amount of composite good that remains if $s^A = l$ and household B commutes the entire length of the city. Since the top of the box has "rotated down" by $ts^A$, household B's indifference curves must also be "rotated down" the same way. Household A's indifference curves are unchanged. A point in this modified Edgeworth box identifies a feasible allocation in which household A lives closest to the city center, so $x^A$ is at the city center, 0. The quantity of land consumed by household A, $s^A$, is measured by the horizontal distance from the origin. The driveway of household B, $x^B$, is at $0 + s^A$. The quantity of land consumed by household B, $s^B$, is $l - s^A$. The quantity of composite good consumed by household A, $z_A$, is the vertical distance from the bottom of the box, and the quantity of composite good consumed by household B is the vertical distance from the modified top of the box, $C - ts^A$.

Suppose instead that household B lives closest to the city center. Figure 6 illustrates how the required modification of the Edgeworth box is the mirror image of that just described. Household B's origin is now the point $(l, C)$. The top of the box is simply a horizontal line segment of width *l*. The height of the right side of the box is the total amount of composite good available, *C*. The bottom of the box is a line that identifies how much composite good remains to be divided between households A and B after household A commutes to the city center. The distance household A commutes is equal to the length of household B's lot, $s^B$. Thus, the bottom of the box is the line $ts^B$. Finally, household A's origin is the point $(0, tl)$, where $tl$ is the amount of composite good that remains if $s^B = l$ and household A commutes the entire length of the city. In this case, the bottom of the box has "rotated up" by $ts^B$, so household A's indifference curves must also be "rotated up" the same way. Household B's indifference curves are unchanged. A point in this modified Edgeworth box identifies an allocation in which household B lives closest to the city center and is the same as described in the previous case except that $x^B$ is at the city center and $x^A$ is at $0 + s^B$.

The set of tangencies, or the contract curve, will characterize interior Pareto optima, just as in the standard Edgeworth box. Again, it is useful to characterize the contract curve in terms of marginal rates of substitution. Let $MRS^i : R_+^2 \to R$ be household *i*'s *marginal rate of substitution for land in terms of composite good*, where

$$MRS^i(s^i, z^i) = \frac{MU_s^i}{MU_z^i}$$

identifies how much composite good household $i$ is willing to give up for an additional piece of land, given that it is already consuming the bundle $(s^i, z^i)$. When household A lives closest to the city center, a feasible allocation $(s^A, s^B, z^A, z^B, x^A, x^B)$ is on the contract curve if $MRS^A(s^A, z^A) = MRS^B(s^B, z^B) + t$ and $s^A < s^B$. We add $t$ to $MRS^B(s^B, z^B)$ because we rotate household B's indifference curves down, just as we rotated the top of the box down to account for composite good used in commuting. Alternatively, when household B lives closest to the city center, a feasible allocation $(s^A, s^B, z^A, z^B, x^A, x^B)$ is on the contract curve if $MRS^B(s^B, z^B) = MRS^A(s^A, z^A) + t$ and $s^B < s^A$. As a result, the contract curve may be disconnected, as in Figure 7. Since we know that the household living closest to the city center can enjoy a level of utility no higher than the household living farther away from the city center, we know that both households are equally well off at the ends of both pieces of the contract curve and, in fact, enjoy the same level of utility. For example, in Figure 7, let $h$ identify the allocation $(s^A, s^B, z^A, z^B, x^A, x^B)$, which is the endpoint of the contract curve when household A lives closest to the city center, and let $k$ identify the allocation $(\hat{s}^A, \hat{s}^B, \hat{z}^A, \hat{z}^B, \hat{x}^A, \hat{x}^B)$, which is the endpoint of the contract curve when household B lives closest to the city center. Then $U^A(s^A, z^A) = U^A(\hat{s}^A, \hat{z}^A) = U^B(\hat{s}^B, \hat{z}^B) = U^B(s^B, z^B)$.

The standard argument from microeconomic theory explaining why the contract curve characterizes Pareto optima applies here. Given a feasible allocation on the curve, there is no alternative allocation in the modified Edgeworth box that makes one household better off without harming the other household. Similarly, with convex preferences, interior Pareto optima occur at tangency points, on the contract curve.

The intuition for these first order conditions is new. Consider an allocation $(s^A, s^B, z^A, z^B, x^A, x^B)$ where $(s^A < s^B)$ and $MRS^A(s^A, z^A) > MRS^B(s^B, z^B) + t$. Then we could expand household A's parcel a little, shrink household B's parcel a little to keep the allocation feasible, and transfer some composite good from household A to B, while covering the additional commuting cost. Such an operation could make both households better off, contradicting the optimality of the original allocation. Similarly, if $MRS^A(s^A, z^A) < MRS^B(s^B, z^B) + t$, then we could shrink household A's parcel, expand household B's parcel, save on commuting cost, and transfer composite good to household A, making both households better off. Thus, at any efficient allocation with $s^A < s^B$, $MRS_A(s_A, z_A) = MRS_B(s_B, z_B) + t$.

The final intuition comes from Figure 5 and calculus. To account for commuting cost, household B's indifference curves are rotated down by $C - ts^A$. As we have defined marginal rates of substitution to be non-negative, the condition characterizing tangencies in the figure is actually

$$-MRS^A(s^A, z^A) = -MRS^B(s^B, z^B) - t.$$

The right hand side follows from the application of calculus to the rotation. The first order condition follows immediately from this equation.

**Equilibrium**

To explore equilibrium, we need to identify who is endowed with the land and composite good available in the economy. Let $\omega^i$ be household *i*'s endowment of composite good. We assume that households are endowed with all the composite good available in the economy, so $\omega^A + \omega^B = C$. For simplicity, households are not endowed with land. Rather, we introduce an absentee landlord who is endowed with all the land in the economy. The landlord does not like to consume land in the city but does like to consume the composite good, so $U^L(s^L, z^L) = z^L$.

Adding the landlord to the model necessitates adjusting our definitions of an allocation and of feasibility. Now, an allocation is a list $(s^A, s^B, z^A, z^B, x^A, x^B, z^L)$, which is feasible if

$$z^A + z^B + z^L + tx^A + tx^B = C,$$

$$[x^A, x^A + s^A) \cap [x^B, x^B + s^B) = \emptyset,$$

and

$$[x^A, x^A + s^A) \cup [x^B, x^B + s^B) = [0, l).$$

The definition of a Pareto optimum is the same.

Let the composite consumption good be the numeraire. Let *P* be the land price function, where $P(V)$ is the price of parcel $V \subset [0, l)$. We would like the price of land to satisfy "no arbitrage," so we would like it to be additive across land parcels. In other words, if *V* and *W* are two parcels of land such that $V \cap W = \emptyset$, then $P(V) + P(W) = P(V \cup W)$. Let $p : [0, l) \to R_+$ be a density function. A land price function *P* such that $P(V) = \int_V p(x)dx$ satisfies this criterion.

If $(s^A, s^B, z^A, z^B, x^A, x^B, z^L)$ is a feasible allocation and $p$ is a price density, then household A pays $\int_{x^A}^{x^A + s^A} p(x)dx$ for parcel $[x^A, x^A + s^A)$ and household B pays $\int_{x^B}^{x^B + s^B} p(x)dx$ for parcel $[x^B, x^B + s^B)$. Since

$$[x^A, x^A + s^A) \cap [x^B, x^B + s^B) = \emptyset$$

and

$$[x^A, x^A + s^A) \cup [x^B, x^B + s^B) = [0, l),$$

the landlord receives

$$\int_{x^A}^{x^A+s^A} p(x)dx + \int_{x^B}^{x^B+s^B} p(x)dx = \int_0^l p(x)dx$$

units of composite good from the sale of land, so $z^L = \int_0^l p(x)dx$.

Now we are ready to define a competitive equilibrium. Composite consumption good is again the numeraire. A *competitive equilibrium* is a feasible allocation $(s^{*A}, s^{*B}, z^{*A}, z^{*B}, x^{*A}, x^{*B}, z^{*L})$ and a price density $p^*$ such that

1. $(s^{*A}, z^{*A})$ maximizes household A's utility subject to its budget constraint,

   $z^A + \int_{x^A}^{x^A+s^A} p^*(x)dx + tx^A \le \omega^A$, and

2. $(s^{*B}, z^{*B})$ maximizes household B's utility subject to its budget constraint

   $z^B + \int_{x^B}^{x^B+s^B} p^*(x)dx + tx^B \le \omega^B$.

Since utility functions are increasing in both arguments, budget constraints will hold with equality. Substituting for $z^A$ and $z^B$ in the respective households' maximization problems, the first order conditions for maximizing utility over driveway locations and land consumption are, for *i* = A,B,

$$MRS^i(s^i, z^i) = p(x^i + s^i)$$

and

$$p(x^i + s^i) = p(x^i) - t.$$

The intuition for these expressions is as follows. Adding land at the back of a parcel causes no change in commuting costs, so at a household optimum, the willingness to pay for an additional unit of land at the back is equal to its cost. Notice that this condition is analogous to the first order condition for equilibrium in the standard Edgeworth box. Adding a unit of land at the front of a parcel reduces commuting cost by t, so at a household optimum, adding a unit at the front must cost t more than adding a unit to the back. Otherwise, the household would choose a different parcel at a household optimum.

Since there are three agents in this economy, we cannot illustrate equilibrium allocations using an Edgeworth box. We can, however, illustrate equilibrium price densities. Suppose that $(s^{*A}, s^{*B}, z^{*A}, z^{*B}, x^{*A}, x^{*B}, z^{*L})$ is an equilibrium allocation with household A living closest to the city center. Let $p^A = MRS^A(s^{*A}, z^{*A})$ and let $p^B = MRS^B(s^{*B}, z^{*B})$. We might guess that a price density $p$, such that

$$p(x) = \begin{cases} p^A, 0 \le x < x^{*B}, \\ p^B, x^{*B} \le x < l, \end{cases}$$

is an equilibrium price density. See Panel A of Figure 8, where the horizontal axis is location and the vertical axis is price. However, since $MRS^A > MRS^B$, household A will want to expand its lot and consume part of household B's parcel. So, this price density cannot be an equilibrium price density.

What we need to do is to construct a price density so that household A cannot increase its utility by expanding its parcel, so it has no incentive to do so. Let $\overline{U}^A = U^A(s^{*A}, z^{*A})$ and let $z(\overline{U}^A, s)$ be the amount of composite good that household A must consume to enjoy that same utility level, $\overline{U}^A$, given that it is consuming $s$ units of land, where $s$ may not be equal to $s^{*A}$. Let $\hat{s}$ be such that $MRS^A(\hat{s}, z(\overline{U}^A, \hat{s})) = p^B$. See Panel B of Figure 8. Then household A will be indifferent between the parcel $[x^{*A}, x^{*A} + s^{*A})$ and a slightly larger parcel if the price density is adjusted so that

$p(x) = MRS^A(x, z(\overline{U}^A, x))$ for $s^{*A} \le x < \hat{s}$. In addition, household B will have no desire to expand its parcel in towards the city center. Thus, $p^*$ such that

$$p^*(x) = \begin{cases} p^A, & 0 \le x < s^{*A}, \\ MRS^A(x, z(\overline{U}^A, x)), & s^{*A} \le x < \hat{s}, \\ p^B, & \hat{s} \le x \le l, \end{cases}$$

is an equilibrium price density. Alternatively, let $\overline{U}^B = U(s^{*B}, z^{*B})$ and let $z(\overline{U}^B, s)$ be the quantity of composite good household B must consume to enjoy utility level $\overline{U}^B$, given that it is consuming $s$ units of land. Let $\tilde{s}$ be such that $MRS^B(\tilde{s}, z(\overline{U}^B, \tilde{s})) = p^A$. Then $p^*$ such that

$$p^*(x) = \begin{cases} p^A, & 0 \le x < \tilde{s}, \\ MRS^B(x, z(\overline{U}^B, x)), & \tilde{s} \le x \le l, \end{cases}$$

is also an equilibrium price density. See Panel C of Figure 8. Indeed, any price density that "falls between" the two is also an equilibrium price density, so there is a continuum of equilibria. The landlord likes the last one best, since that one generates the highest rent collection.

**Welfare Properties Of Equilibrium Allocations**

Two features of this model are that equilibrium allocations are Pareto optima, so there is a valid First Welfare Theorem (Berliant and Fujita 1992), and for every Pareto optimal allocation, there exists a land price density and endowments such that the Pareto optimal allocation is an equilibrium allocation with respect to that land price density and endowments, so there is also a valid Second Welfare Theorem (Berliant and Fujita 1992).

Intuitively, this can be seen from the first order conditions. If household A is located closer to the city center, then the first order conditions for equilibrium imply

$$MRS^B(s^B, z^B) = p(x^B + s^B) = p(l),$$

$$MRS^A(s^A, z^A) = p(x^A + s^A) = p(s^A) = p(x^B),$$

and

$$p(x^B + s^B) = p(x^B) - t.$$

So, $MRS^B(s^B, z^B) = MRS^A(s^A, z^A) - t$, and this is the first order condition for Pareto optima. The argument also works in the opposite direction. An important implication is that Proposition 1 applies to any equilibrium allocation, so the predictions stated in Proposition 1 are potentially testable. Moreover, in equilibrium, the consumer with higher income must naturally end up with higher utility. Thus, in equilibrium, the order of households from the city center outward is increasing in income, utility, and parcel size.

**An Example**

Consider a linear city of length $l = 8$. Let household A be endowed with 58 units of composite good and let household B be endowed with 154 units of composite good, so $C = 212$. The transportation cost of commuting per unit of distance is $t = 8$. Suppose that household A's utility function is $U^A(s^A, z^A) = z^A - (16 - s^A)^2$ and household B's utility function is $U^B(s^B, z^B) = z^B - (16 - s^B)^2$. For household A, the marginal utility of land at a bundle $(s^A, z^A)$ is $2(16 - s^A)$, while the marginal utility of composite good at a bundle $(s^A, z^A)$ is always 1. Similarly, household B's marginal utility of land at a bundle

$(s^B, z^B)$ is $2(16 - s^B)$, while the marginal utility of composite good at a bundle $(s^B, z^B)$ is always 1. (For those who know calculus, recall that the marginal utility of land is the derivative of the utility function with respect to land, and similarly for the composite good.) The marginal rate of substitution for land in terms of composite good at a bundle is the ratio of the marginal utilities at that bundle, so for household A, $MRS^A(s^A, z^A) = 2(16 - s^A)$, and for household B, $MRS^B(s^B, z^B) = 2(16 - s^B)$.

What does the set of Pareto optima look like in this model? First, we will identify the part of the contract curve such that household A lives closest to the city center. We can find this part of the curve by setting $MRS^A(s^A, z^A) = MRS^B(s^B, z^B) + t$, which implies that $2(16 - s^A) = 2(16 - s^B) + t$. Since $t = 8$ and $s^B = 8 - s^A$, we can see that $2(16 - s^A) = 2(16 - (8 - s^A)) + 8$. Solving for $s^A$, we find that $s^{\circ A} = 2$. It follows that $s^{\circ B} = 8 - 2 = 6$. When household A lives closest to the city center, its utility level must be less than or equal to household B's utility level, which implies that $z^{\circ A} - (16 - 2)^2 \leq z^{\circ B} - (16 - 6)^2$. We know that

$$z^{\circ B} = C - ts^{\circ A} - z^A = 212 - 8 \cdot 2 - z^{\circ A} = 196 - z^{\circ A}.$$

Substituting for $z^{\circ B}$, it follows that $z^{\circ A} \leq 146$. Thus, when household A lives closest to the city center, the set of Pareto optima is the union of the two line segments [(0,0), (2,0)] and [(2,0), (2, 146)]. See Figure 9. Since the two households have identical utility functions, we know by symmetry that when household B lives closest to the city center $s^{\circ B} = 2$, $s^{\circ A} = 6$, and $z^{\circ B} \leq 146$. When household B lives closest to the city center, the contract curve is the union of the two line segments [(6,66), (6, 212)] and [(6, 212), (8, 212)]. See Figure 9.

Next, we will construct an equilibrium price density for this model. Since both the First and Second Welfare Theorems are valid for this model, the only candidates for equilibrium allocations are the Pareto optimal allocations. Since household A has less endowed wealth than household B, in the end it will have lower utility than household B. It follows from Proposition 1 that household A lives closest to the city center. We want to construct a land price density such that neither household can increase its utility by deviating from an allocation such that $x^{*A} = 0$, $s^{*A} = 2$, $x^{*B} = 2$, and $s^{*B} = 6$. On the margin, the price that household A pays for land should be equal to its marginal willingness to pay for land, which implies that

$$p^*(2) = MRS^A = 2(16 - s^{*A}) = 2(16 - 2) = 28.$$

Similarly, on the margin, the price that household B pays for land should be equal to its marginal willingness to pay for land, so

$$p^*(8) = MRS^B = 2(16 - s^{*B}) = 2(16 - 6) = 20.$$

Notice that the difference between the two is equal to the cost of commuting per unit distance. Next, we need to price land at $x > 2$ such that household A cannot make itself better off by increasing the size of its lot, while keeping its lot front at the origin. In other words, if household A consumes more land, it should have to give up so much composite good that it does not increase its utility. This quantity is revealed by $MRS^A$, so $p^*(x) \geq 2(16 - x)$. Notice that when $x = 6$, $2(16 - x) = 20$, and for $x > 6$, $2(16 - x) < 20$. If we set $p^*(x) = 20$ for $6 < x \leq 8$, then the condition that $p^*(x) \geq 2(16 - x)$ is satisfied, as is the condition that $p^*(8) = 20$. So, an equilibrium price density is $p^*$ such that

$$p^*(x) = \begin{cases} 28, & 0 \le x < 2, \\ 2(16-x), & 2 \le x < 6, \\ 20, & 6 \le x \le 8 \end{cases}.$$

This price density is illustrated in Figure 10.

Finally, we will construct the equilibrium allocation associated with this land price density. Since the equilibrium land price density constructed above supports an allocation such that $x^{*A} = 0$, $s^{*A} = 2$, $x^{*B} = 2$, and $s^{*B} = 6$, we just need to figure out how much composite good everyone consumes.

Household A's expenditure on land is

$$\int_{x^{*A}}^{x^{*A}+s^{*A}} p^*(x)dx = \int_0^2 p^*(x)dx = \int_0^2 28dx = 28 \cdot 2 = 56,$$

which is the area labeled A in Figure 10. Household A's utility is increasing in land and composite good, so it exhausts its budget and consumes

$$z^{*A} = \omega^A - tx^{*A} - \int_{x^{*A}}^{x^{*A}+s^{*A}} p^*(x)dx = 58 - 56 = 2.$$

Household B's expenditure on land, the area labeled B in Figure 10, is

$$\int_{x^{*B}}^{x^{*B}+s^{*B}} p(x)dx = \int_2^8 p^*(x)dx = \int_2^6 2(16-x)dx + \int_6^8 20dx = 96 + 40 = 136,$$

and its expenditure on commuting is $tx^{*B} = 8 \cdot 2 = 16$. Since household B also spends all of its income,

$$z^{*B} = \omega^B - tx^{*B} - \int_{x^{*B}}^{x^{*B}+s^{*B}} p^*(x)dx = 154 - 16 - 136 = 2.$$

Finally, the absentee landlord collects both households' expenditures on land in composite good, so $z^{*L} = 56 + 136 = 192$. So, an equilibrium allocation associated with the price density $p^*$ is $(s^{*A}, s^{*B}, z^{*A}, z^{*B}, x^{*A}, x^{*B}, z^{*L}) = (2,6,2,2,0,2,192)$. Alternative equilibria have the same allocation of land, but the landlord collects more rent from household B.

**Conclusion**

We have presented in this essay the simplest model of how two households distribute themselves across space and shown that they do so efficiently. Using the welfare theorems and Proposition 1, we can characterize the relationship between household income, the size of the lot the household occupies, and the location of the lot. In equilibrium, the household with the larger endowment of composite commodity will ultimately enjoy a higher level of utility, and so will live further away from the city center and on a larger lot than the household with the smaller endowment of composite commodity. This was, in fact, the outcome of our specific example. The Alonso model can also accommodate $n$ identical households, where $n$ is finite and greater than two. Under certain conditions, Fujita (1989) can be used to show that the model can also accommodate $n$ heterogeneous households.

An important implication of this model is that equilibrium allocations are efficient, so there is no market failure. Thus, under the assumption of perfect competition, the free market yields efficiency, as it does in similar analyses of non-locational goods.

**References**

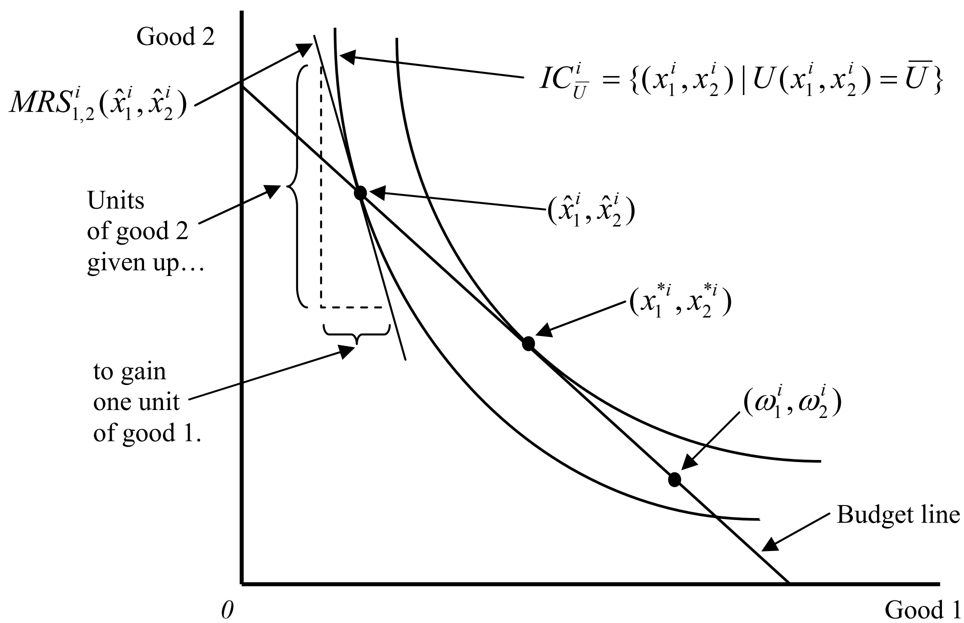

Alonso, W. (1964) *Location and Land Rent: Toward a General Theory of Land Rent*. Cambridge, MA, Harvard University Press.

Berliant, M. and Fujita, M. (1992) Alonso's Discrete Population Model of Land Use: Efficient Allocations and Competitive Equilibria. *International Economic Review* 33 (3), 535-566.

Fujita, M. (1989) *Urban Economic Theory: Land Use and City Size*. Cambridge, Cambridge University Press.

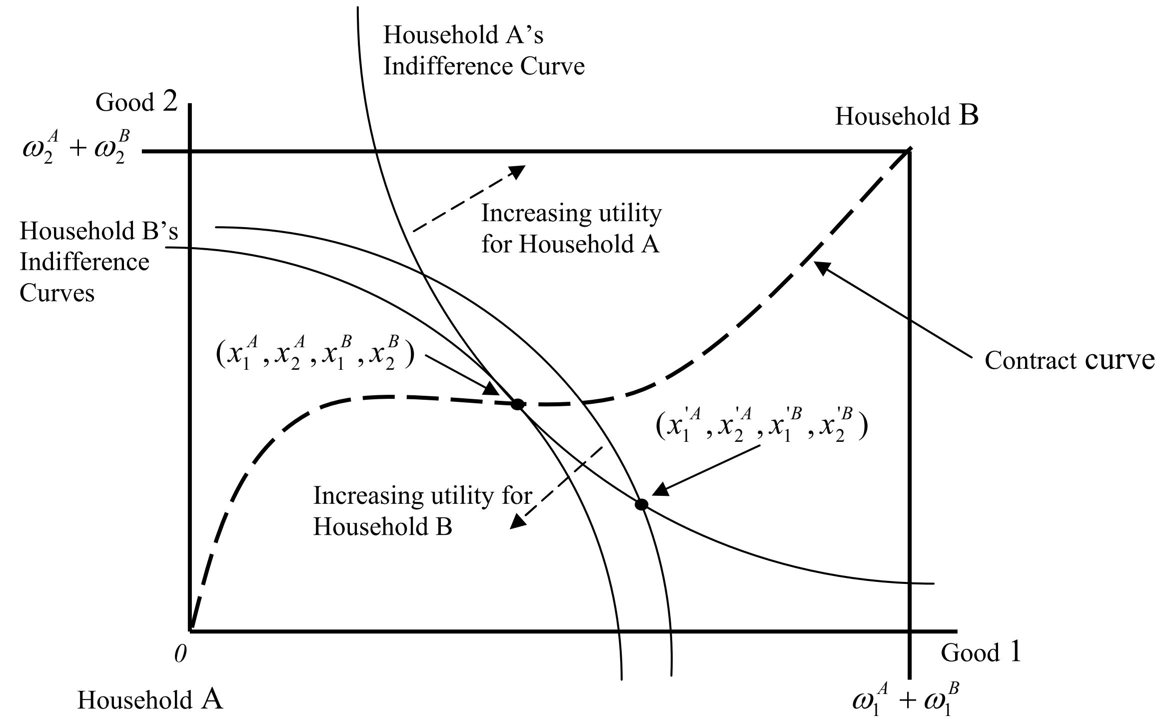
Varian, H. (1993) *Intermediate Microeconomics: A Modern Approach*. New York, W. W. Norton.

Figure 1



Good 2
$MRS^i_{1,2}(\hat{x}^i_1, \hat{x}^i_2)$
$IC^i_{\overline{U}} = \{(x^i_1, x^i_2) \mid U(x^i_1, x^i_2) = \overline{U}\}$
Units
of good 2
given up…
$(\hat{x}^i_1, \hat{x}^i_2)$
$(x^{*i}_1, x^{*i}_2)$
to gain
one unit
of good 1.
$(\omega^i_1, \omega^i_2)$
Budget line
0
Good 1

Figure 2



Household A's
Indifference Curve
Good 2
Household B
$\omega_2^A + \omega_2^B$
Increasing utility
for Household A
Household B's
Indifference
Curves
Contract curve
$(x_1^A, x_2^A, x_1^B, x_2^B)$
$(x_1^{'A}, x_2^{'A}, x_1^{'B}, x_2^{'B})$
Increasing utility for
Household B
0
Good 1
Household A
$\omega_1^A + \omega_1^B$

Figure 3

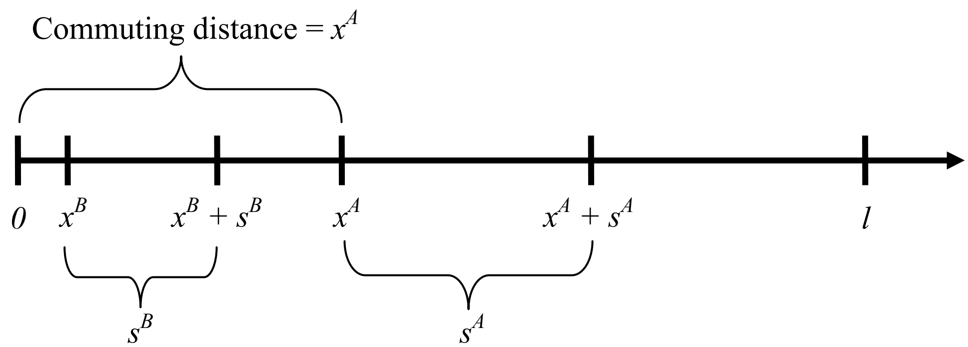


Figure 4

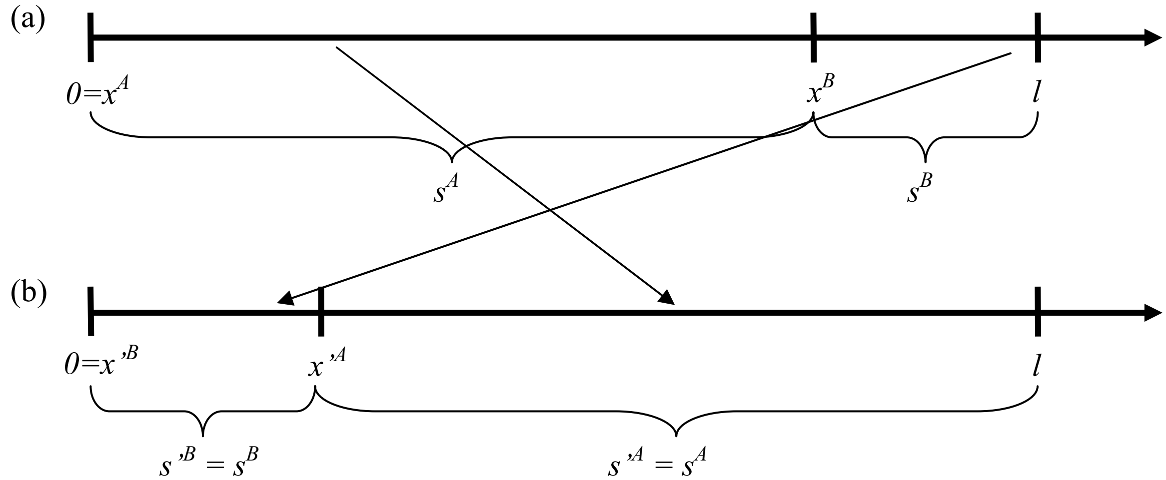


Figure 5

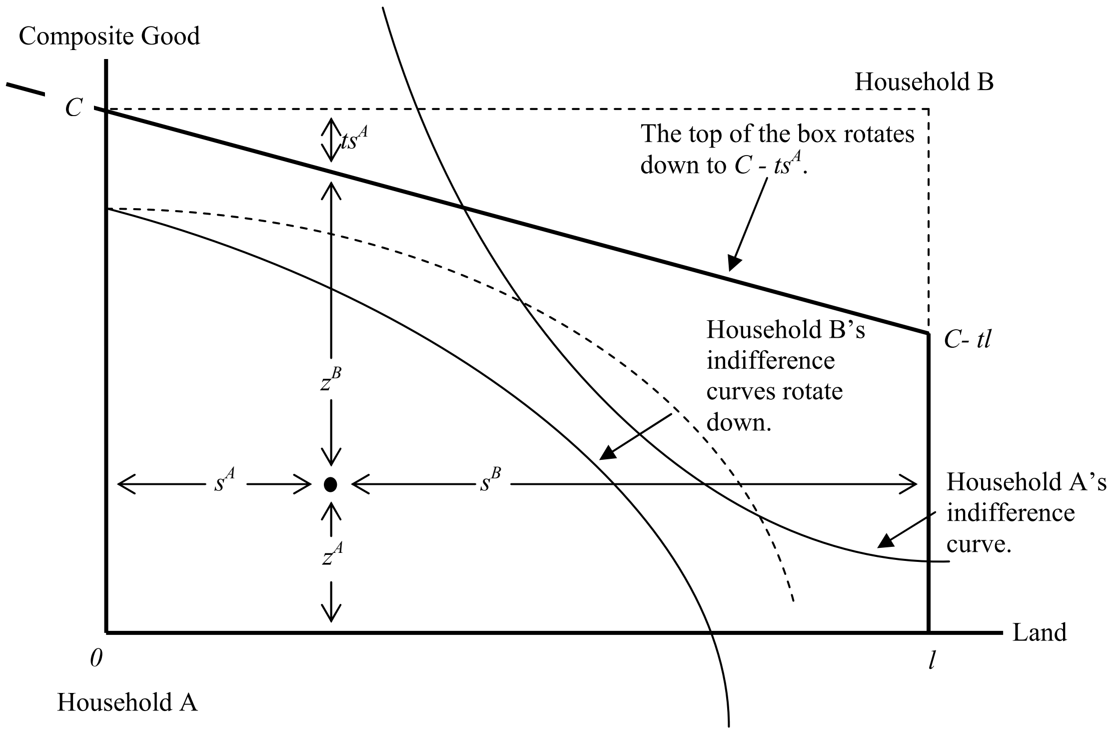

Composite Good
Household B
C
ts^A
The top of the box rotates down to C - ts^A.
C- tl
Household B's indifference curves rotate down.
z^B
s^A
s^B
z^A
Household A's indifference curve.
0
l
Land
Household A

Figure 6

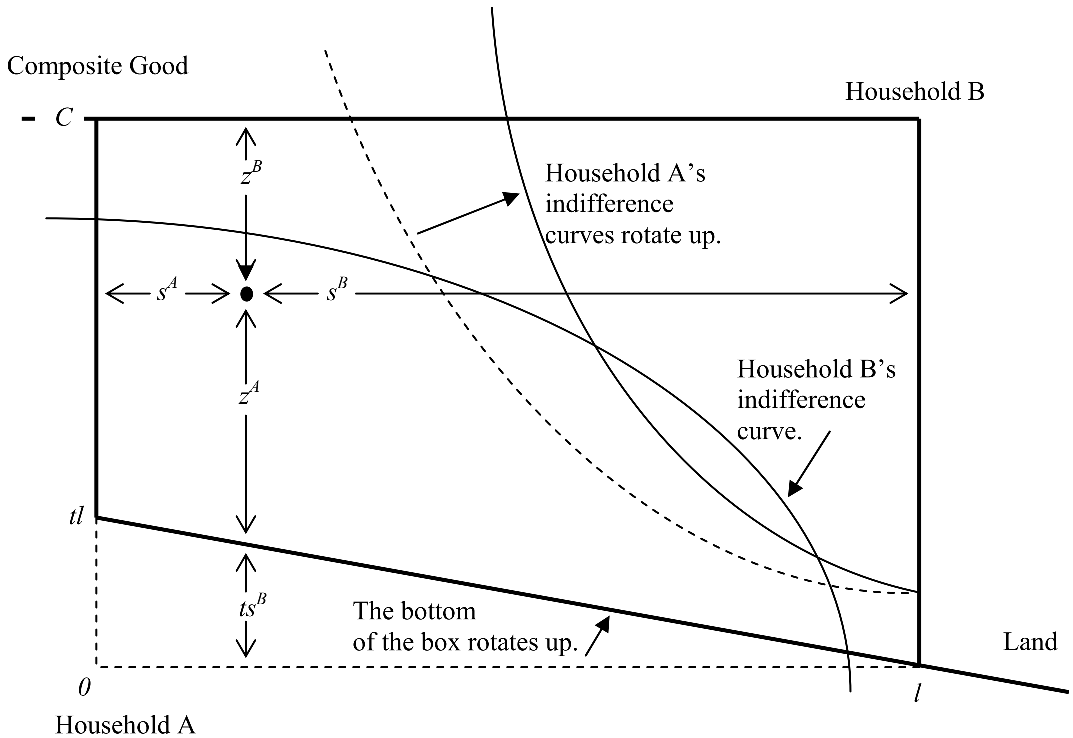

Composite Good
Household B
C
$z^B$
Household A's
indifference
curves rotate up.
$s^A$
$s^B$
Household B's
indifference
curve.
$z^A$
$tl$
$ts^B$
The bottom
of the box rotates up.
Land
0
$l$
Household A

Figure 7

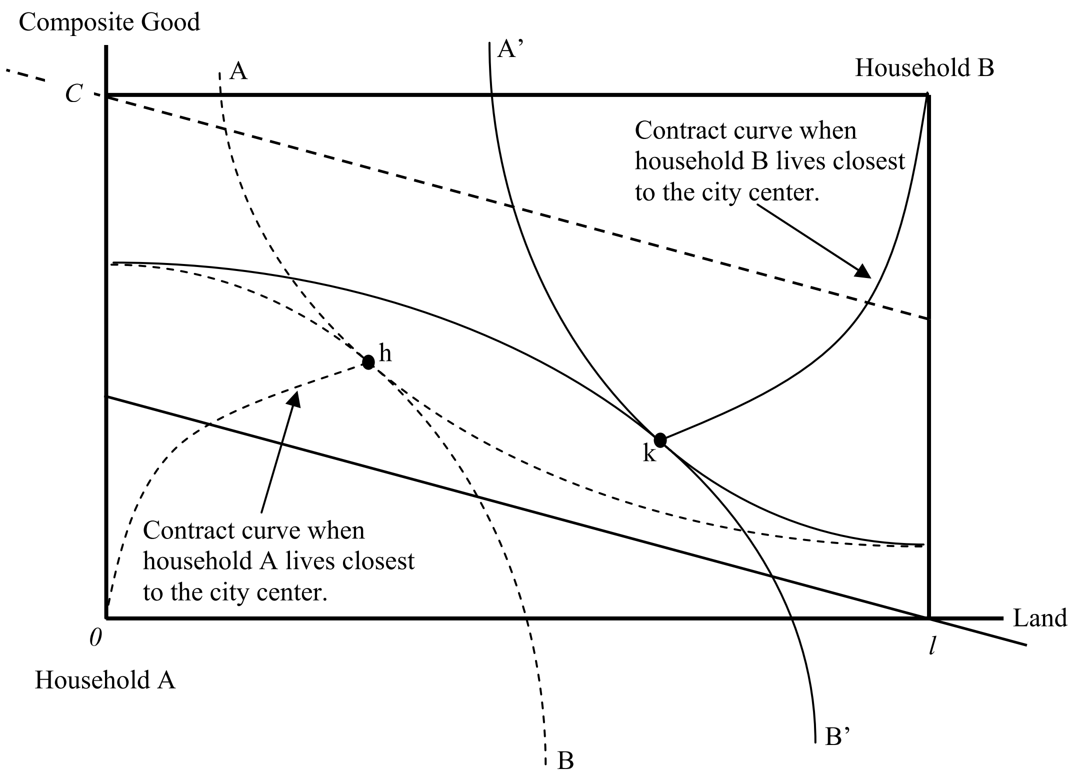

Composite Good
A'
A
Household B
C
Contract curve when
household B lives closest
to the city center.
h
k
Contract curve when
household A lives closest
to the city center.
Land
0
l
Household A
B'
B

Figure 8 Panel A

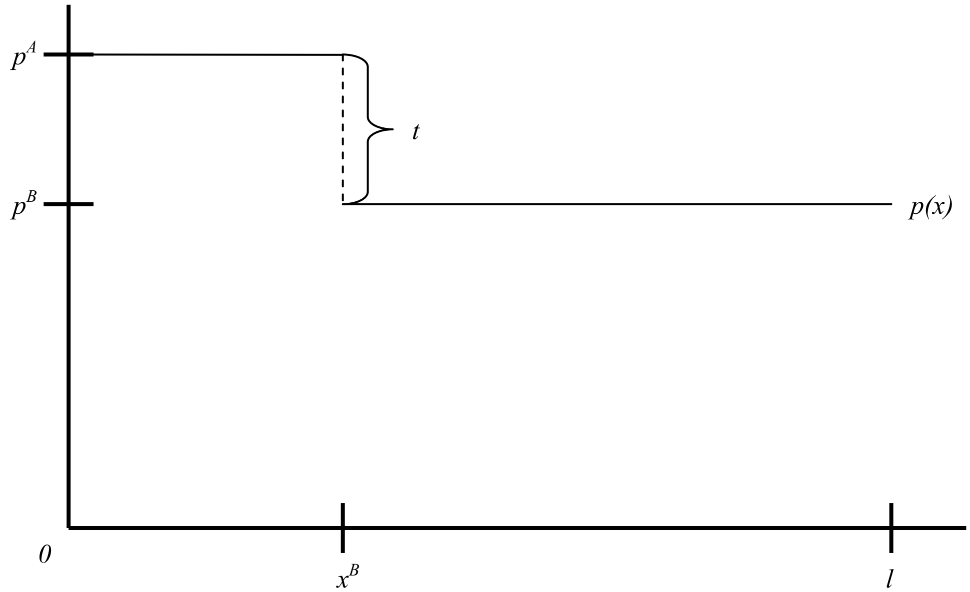


Figure 8 Panel B

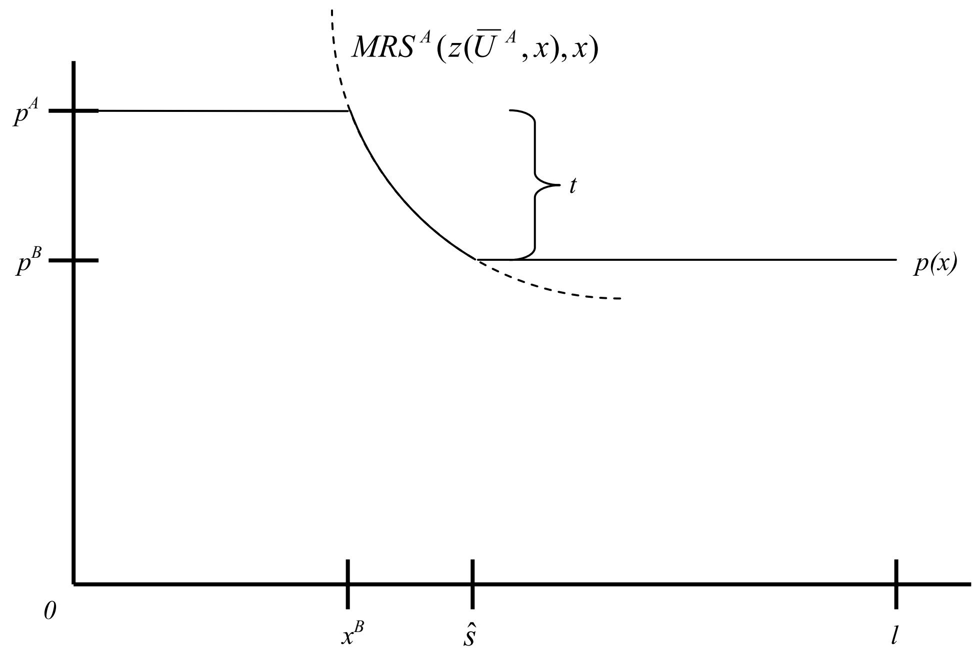


Figure 8 Panel C

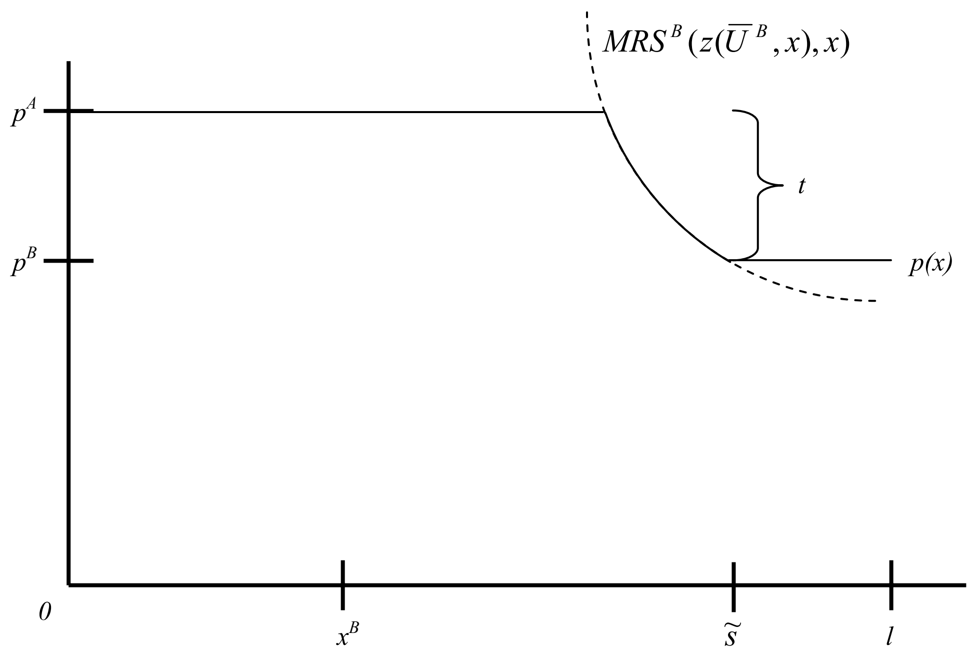


Figure 9

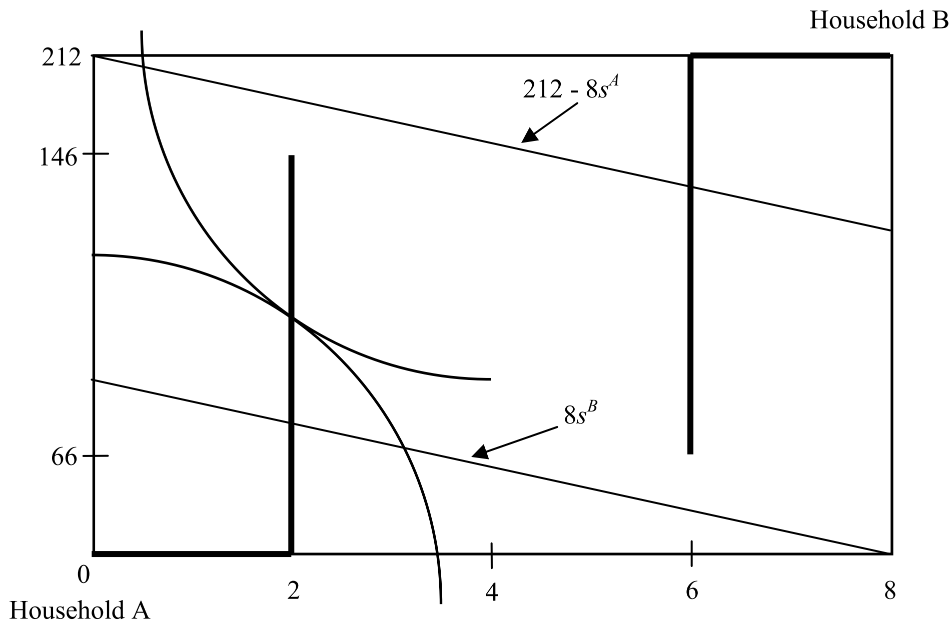


Figure 10