# Hedonism vs. Nihilism: No Arbitrage and Tests of Urban Economic Models*

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**Abstract**

We present two notions of "no arbitrage" in urban economic models and show that there is no model satisfying both. The standard hedonic housing model of urban economics and its generalizations are consistent with the first of these, but inconsistent with the second. We present a model consistent with the second notion of "no arbitrage" and a continuum of models consistent with neither notion that are observationally equivalent to the standard model, even if the utility function of consumers is known. Only one of these is the standard model. Thus, the available tests of the standard model cannot provide much evidence of its validity. Finally, we examine nonlinear price systems consistent with the second notion of "no arbitrage" and their welfare consequences. JEL classifications: R21, R13, D41, C21. Keywords: Monocentric City, Hedonic Models, No Arbitrage.

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# 1 Introduction

The equilibrium conditions for standard urban economic models require that identical households receive identical levels of utility. This requirement leads to the well-known result that if we take a parcel of land and move it farther from the city center, its price must go down by exactly the commuting cost (actually, the present discounted value of commuting cost over the infinite time horizon). In other words, the commuting cost is exactly capitalized into land prices. The relationship is known as the "Muth-Mills condition," after the seminal contributions by Muth (1969) and Mills (1972).[1] The Muth-Mills model implies that at any given location, all land parcel sizes are identical and trade at the same price. Thus, the model appears to imply that there are no arbitrage possibilities at equilibrium: each location has identical parcels with the same price.

In this paper, we show that the standard Muth-Mills model is not, in fact, consistent with a natural no-arbitrage condition. Since we attempt to cover large classes of urban economic models, we do not use assumptions on their primitives, such as utility functions, but rather we assume that general implications of equilibrium in the models hold. Our reduced form assumptions are the ones usually considered to have empirical content.[2] First, we assume that there are at least two locations and two possible parcel sizes. Second, we assume that in each location, there are no arbitrage possibilities in equilibrium, i.e., the unit price of land is identical for any quantity of land at a given distance from the city center. Finally, we assume that the Muth-Mills condition holds. From these simple assumptions, we obtain a contradiction: the Muth-Mills condition and the no-arbitrage condition cannot hold simultaneously. After providing examples, we will make the assumptions and result precise in the subsequent sections of this paper.

The source of this contradiction is the conflicting requirements of the Muth-Mills model and the no-arbitrage condition of standard hedonic models. The following example motivates our result.

[1] As there are at least two versions of the Muth-Mills condition in the literature, we will discuss their relationship in detail at the end of section 2.

[2] Examples of work testing urban models include Mills (1969), Kau and Sirmans (1979), Yinger (1979), McMillen (1990), and McMillen (2003). Of particular importance to us is Coulson (1991), which is the origination of the ideas we present. In particular, see pp. 300-301 of the Coulson paper. We noticed that conditions [3] and [4] seemed to be in contradiction, although they are derived from the same model.

**Example 1** *Suppose that the present discounted value of commuting cost is \$2000 per mile (this assumption is in rough agreement with the empirical estimate of Coulson (1991, p. 304)). Suppose that a nice 2 acre parcel sells for \$100,000 at distance 1 mile from the central business district (CBD). So if this parcel were available 2 miles from the CBD, it would sell for \$98,000. Now consider a 3 acre parcel, and suppose that it would sell for \$150,000 at distance 1 mile from the CBD if it were available. Suppose that it actually sells for \$148,000 and is available 2 miles from the CBD (in equilibrium). Now let's calculate the per unit price of land, assuming that land is priced proportionately as required by the no-arbitrage condition. If a 2 acre parcel were given land quantity 2, then the per unit price of land at distance 1 mile from the CBD is \$50,000. If the 3 acre parcel were given land quantity 3, then its unit price is also \$50,000. Now let's look at the unit price of land at distance 2 miles from the CBD. The per unit price of the 2 acre parcel is \$49,000. The per unit price of the 3 acre parcel is \$49,333.33. This contradicts the no-arbitrage condition because the unit price is lower for smaller land parcels.*[3]

To understand the meaning of this result, it is essential to review 4 fundamental facts in the application of general equilibrium theory to urban economics:[4]

[3] This result also applies when we change the example from land to housing. Change the words "acre parcel" to "bedroom house" and all of the same results go through. The Muth-Mills condition then implies that the price of housing falls by \$2,000 with each mile from the CBD. No arbitrage implies that the unit price of housing for a 2 bedroom house is \$49,000 at distance 2 miles from the CBD, while the unit price of housing for a 3 bedroom house is \$49,333.33. Thus, arbitrage opportunities are present: a developer can make a profit by buying 2 bedroom houses and converting them into 3 bedroom houses.

[4] It is important to relate our work to the general literature on hedonic pricing, such as Rosen (1974) or Mas-Colell (1975) and the many papers that followed them. The link is that location is one of many characteristics of a differentiated commodity called housing. The first and most obvious difference is that there is no location or commuting cost in the general hedonic model. Thus, there is no analog of the Muth-Mills condition in the general model, so a no arbitrage condition that applies across locations (or one dimension of differentiated goods) is lacking. Second, there are usually indivisibilities in the general literature, so complete repackaging of commodities is not allowed. No arbitrage at a given location or for a given type of good will not hold in general because the good cannot be completely repackaged when some characteristics are fixed or indivisible. On the other hand, in urban hedonic models, if the locational or differentiated good is land at a particular location, then repackaging is clearly possible since the quantity of land purchased is not fixed. If the locational good is housing differentiated by location, then there is the potential for a no repackaging assumption or an indivisibility. In urban economic theory, it has simply been

- No arbitrage or arbitrage possibilities are always part of the definition of an equilibrium concept; in particular, they are embedded in the notion of a price system.

- All bundles, including bundles not purchased in equilibrium, must be priced in order to discuss budget sets, demand, supply, or equilibrium.

- Feasibility constrains neither demand nor supply, so no arbitrage is defined without reference to what allocations are feasible.

- The use of duality theory, for instance the bid rent approach to equilibrium, implicitly assumes that pricing is proportional, in the sense that the value of a bundle is proportional to quantities.

The key to our result is that the hedonic price function must be well defined, *independent of whether these land parcels are actually bought in equilibrium.* At equilibrium in standard urban models, land parcels are generally identical at a given location, and land parcels generally differ across locations. However, *hedonic models require that all land parcels be priced at all locations, even when they are not actually present in equilibrium.* It does not matter that each neighborhood consists of identical lots in the Muth-Mills city; in a hedonic model, the household optimization problem is not well defined unless agents can compute the value of all bundles at all locations. And empirically, neighboring land parcels are often quite different while the same size land parcels are found at very different locations. No arbitrage requires that the hedonic price function be proportional to quantity for land parcels in the same neighborhood. The Muth-Mills model implies that parcels that are identical in all ways except location must differ in price by commuting cost multiplied by the difference in distance from the CBD. A price schedule in which prices vary across locations independently is inconsistent with no arbitrage at each location.

After stating and proving this result, we examine equilibrium price systems that satisfy one of the two conditions, as well as some that satisfy neither.

traditionally assumed that the housing good is divisible, and this is the kind of model we analyze. Moreover, for our results below, we only require that two types (or packages) of houses be conceivable; we do not require that arbitrary linear combinations of a housing good be possible. In empirical urban economics, linearity of prices at the micro level has been used to identify submarkets in a small geographical region. So an assumption of no repackaging of commodities would seem not to be empirically relevant.

There is a continuum of price systems (or models) that yield the same equilibrium allocation (the one generated by the standard monocentric city model) and price the bundles purchased in equilibrium the same, but that disagree for bundles not purchased in equilibrium. Thus, all of these models are observationally equivalent, so the "tests" of the standard model cannot distinguish it from the rest of these models, even if the consumer wealth and utility functions are known.

A simple example based on quasi-linear utility will illustrate this point. How bundles not purchased in equilibrium are priced is crucial for assessment, for instance, since in the real world heterogeneous houses in the same neighborhood can be found. We shall use standard notation, where $h$ is the quantity of the locational commodity (either land or housing), $z$ is the quantity of composite consumption good, and $u(h, z)$ is a utility function. Distance from the CBD is indexed by $x$ and $y$, while the cost per mile for commuting to the CBD is given by $t$.

**Example 2** *Take $u(h, z) = z + \ln(h)$, where the consumers are all identical and of measure $N$. For computational ease, consider a linear city so that the land available at each location is 1. Calculations using first order conditions yield an equilibrium rent density of $p(x) = Ae^{-xt}$, a standard exponential density, where $A = \frac{Nt}{1-e^{-\overline{x}t}}$ and $\overline{x}$ is the extent of the city. Moreover, equilibrium housing consumption is $h(x) = A^{-1}e^{xt}$, so $x(h) = h^{-1}(\cdot) = \ln(Ah)/t$ Define price systems $\pi(h, y) \equiv h \cdot p(x(h)) + t \cdot [x(h) - y] = 1 + \ln(Ah) - t \cdot y$, and $\pi^{\alpha}(h, y) \equiv \alpha hAe^{-yt} + (1 - \alpha)[1 + \ln(Ah) - t \cdot y]$. The price systems $p$, $\pi$, and $\pi^{\alpha}$ are all equilibrium prices. The equilibrium allocations are the same for all of these equilibrium prices, though only $p$ is derived from the standard monocentric model and values bundles proportional to quantity at each fixed location. It does not satisfy the Muth-Mills condition. To see this, fix $\overline{h}$ arbitrarily (it could, for instance, be $\overline{h} = h(x)$ for some fixed $x$). $\overline{h} \cdot p(x) - \overline{h} \cdot p(y) = \overline{h} \cdot (Ae^{-xt} - Ae^{-yt}) = \overline{h}A \cdot (e^{-xt} - e^{-yt}) \neq t \cdot [y - x]$. The price system $\pi$ satisfies the Muth-Mills condition but does not value bundles proportional to quantities at a fixed location. For any $\alpha \in (0, 1)$, $\pi^{\alpha}$ neither values bundles at a fixed location proportional to quantities nor satisfies the Muth-Mills condition.*

The bottom line is that there is no price system or model that satisfies both the no-arbitrage and Muth-Mills conditions, and there is a continuum of models that are observationally equivalent to the standard model. There is one such model consistent with each of the conditions, and a continuum of models consistent with neither.

In response, many have proposed (at least implicitly) nonlinear pricing to avoid the paradox. The following example satisfies the Muth-Mills condition but does not price bundles proportional to quantity, and features equilibrium allocations that are inefficient.

**Example 3** *Proportional pricing of land after accounting for the discount in the total price of a parcel due to its distance from the CBD amounts to a 2 part tariff. The general form of this price structure is* $P(h,x) = K - t \cdot x + p(x) \cdot h$*, where* $K$ *is a constant. An important special case, which incorporates the idea that once commuting cost is accounted for, there is no essential difference in vacant land in different locations, is* $P(h,x) = K - t \cdot x + p \cdot h$*. Running this through the consumer budget constraint, it's obvious that commuting cost cancels, namely that the budget at every location is* $w = z + p \cdot h + K$*. If (for instance) all consumers are identical and utility is strictly quasi-concave, the solution to the consumer problem is independent of location, so the consumers at all locations choose the same amount of composite good to consume and the same amount of land to consume. This contradicts both the prediction of the monocentric city model, that land consumption increases with distance from the CBD, and efficiency, that requires the same. Allowing the more general form of a two part tariff given above, or nonlinear pricing of this sort more generally, will only yield more equilibria inconsistent with the monocentric city model, many of which are inefficient.*

We begin with the notation and assumptions in Section 2. Section 3 simply makes Example 1 formal and general. Section 4 discusses the continuum of models that are observationally equivalent to the standard monocentric city model. Section 5 discusses the nonlinear pricing of land. Section 6 contains our conclusions.

## 2 The Notation and Assumptions

Our assumptions are not on primitives. This is done both to reduce the amount of notation as well as to cover classes of models that generate the same or similar testable implications. The following basic framework is typical of most urban models.

**Notation 1** *We assume that location is represented by a variable* $x \in X \subseteq \Re_+$*, the set of all possible locations. We assume that the CBD to which every* *consumer must commute is represented by* $0 \in X$. *Multiple subcenters would cause no problems for us, but we defer the discussion of extensions to Section 6. The locational good (land or housing) is represented by* $h \in H \subseteq \Re_+$. *We define* $G$ *to be an abstract set of other locational characteristics, and let* $g \in G$ *be a generic element of the set. (The set* $G$ *need not be a subset of a Euclidean space.)*

**Remark 1** *For example, suppose that there were certain characteristics that vary with location in equilibrium, such as neighborhoods, schools and local public goods in general. These need not be exogenous variables in the structural model. The value of* $g$ *could be exogenous (for example, if we took schools associated with a location as exogenous)* ***or*** *it could be the equilibrium value of some endogenous variables, like neighborhood income.*

**Notation 2** *The hedonic pricing function takes as its domain housing purchase, location, and the values of these characteristics at the selected location. In other words, we write the hedonic pricing function as* $P(h, x, g)$, *where* $P : H \times X \times G \to \Re$. *Its value is the total price of the specified house, not the unit price.*

**Remark 2** *If there are no locational characteristics other than location itself, we can simply let* $G$ *be a singleton (constant across location). In fact, if one interprets* $h$ *as land consumption, one could also include in* $g$ *other housing characteristics, such as the number of bedrooms. Our analysis is robust to all of these variations and interpretations. In general, we will assume that* $P$ *is the equilibrium hedonic price function, since we shall be examining conditions that characterize equilibrium in urban models. This formulation of the hedonic price function is simple but general.*

**Notation 3** *We assume that commuting cost is given by* $t \cdot x$ *for a person living at* $x \in X$, *where* $t > 0$. *This is also standard in the literature.*

**Remark 3** *In hedonic models, it is necessary to have prices defined for all possible bundles of goods at each location. For example, this is common in the local public finance literature, where membership in every jurisdiction, whether it happens to be present in equilibrium or not, must be priced (see Ellickson, Grodal, Scotchmer and Zame (2000) for a recent incarnation of this feature). The reason, of course, is that without prices for some goods (or jurisdictions), the optimization problems of agents are not well-defined since they cannot compute the cost of each bundle. In our context, the implication is that* $P(h, x, g)$

*must be defined for each $x \in X$, each $h \in H$ and each $g \in G$. The theoretical models imply that all house types or lot sizes must be priced at all locations. The empirical hedonic models actually do price all house types at all locations. Of course, this is necessary in order to assess properties, one of the main applications of the model.*

**Axiom 1** *There are at least two distinct members of $X$ (called $x$ and $y$), at least two distinct members of $H$ (called $h$ and $i$), and at least one member of $G$, $g \in G$.*

It shouldn't be too controversial to assume that there are at least two locations and two parcel sizes or types of houses. Moreover, we assume that there is at least one common member of $G$ that is priced across these two locations and for these two parcel sizes. **As long as the hedonic price function $P$ is defined for the four bundles $P(h, x, g)$, $P(i, x, g)$, $P(h, y, g)$, and $P(i, y, g)$, how the hedonic price function $P$ is defined for other bundles is completely irrelevant. Without loss of generality, we take $H = \{h, i\}$, $X = \{x, y\}$, and $G = \{g\}$.** For consistency below, we will often say "$\forall g \in G$"; for our purposes, the reader can take this to mean "for the unique element of $G$".

**Axiom 2** *(No Arbitrage): At each location, the total price of land is linear in the quantity consumed. Formally, $\forall g \in G$, $\forall x \in X$, $\forall h \in H$, $P(h, x, g) = h \cdot p(x, g) + c(g)$, where $p : X \times G \rightarrow \Re$ and $c : G \rightarrow \Re$.*

The interpretation, of course, is that people can not make positive profits by simply trading in land or housing at a given location. The price function $p(x, g)$ is the type usually used in the urban economics literature. It represents the per unit price of land with characteristics $g$ located distance $x$ from the CBD. If one wants to dispense with this condition and use nonlinear pricing (for example, two part tariffs) for $h$, then as illustrated in Example 3 of the Introduction, one can kiss the first welfare theorem good-bye; equilibrium allocations might not be Pareto optimal.[5] The usual proof of the first welfare theorem relies heavily on *proportional* pricing. In the theoretical literature, it is typical that $c \equiv 0$. In the empirical literature, it is typical for $c$ to represent components of the regression other than terms and interaction terms involving $h$.

[5] Theorem 2 below shows that with nonlinear pricing, there is a continuum of equilibrium allocations for fixed endowments, and many of these are inefficient.

When $H$ is land and $c \equiv 0$, the interpretation of Axiom 2 is particularly compelling. If we consider $h$ and $i$ to be vacant lots where one is twice the size of the other, then since the parcels can be split into pieces, no arbitrage says that agents cannot make a profit by breaking the parcels up or combining them.[6] Vacant lots need not be present in equilibrium; we only require that the hedonic price function prices them. We allow the generalization $c \neq 0$ to cover empirical work.

When $H$ is housing rather than land, since we only need to use the values at two points (at each location), this is really just agreement (among agents) about the units used to measure the housing good. Such an assumption is usually justified in the literature by a production function for housing that is location-independent (often Cobb-Douglas in capital and land). This leads to a location-independent scale for measuring housing, though the **price** of housing can still be location-dependent due to the location-dependence of the price of land. In that case, lot size can either be embodied in $h$ through the production function and/or included in $g$.

One way to capture many of the empirical models in our framework is to make $h$ land and throw all of the housing characteristics into $g$. Then the houses we are comparing differ only in lot size and, once again, no arbitrage in land seems compelling.

Empirically, this is a semiparametric regression.

**Axiom 3** *(Coulson-Muth-Mills): If we take a land parcel and move it $d$ units farther from the CBD, then its price goes down by the commuting cost multiplied by $d$. Formally, $\forall g \in G$, $\forall h \in H$, $\forall x, y \in X$, $P(h, x, g) = P(h, y, g) + t \cdot [y - x]$.*

There are many remarks to be made about the last axiom. First, the intuition is clear and well-known. If we take two vacant lots or houses that are identical in every respect except distance from the CBD, then the difference in price must be equal to the commuting cost. In terms of Coulson (1991), equation [3] of that paper is the differential version of our Axiom 3. In our notation, presuming that $P$ is differentiable in $x$, Coulson's equation [3] is

$$\partial P(h, x, g)/\partial x = -t. \tag{1}$$

[6] As is standard in most of economics, we assume that consumers ignore feasibility constraints when solving their optimization problems, so they think it's possible to purchase vacant lots of any size at any location.

This can be derived from Axiom 3 by letting $x$ tend to $y$ and using the definition of a derivative.

As Coulson notes, the argument for the standard Muth-Mills condition is a bit harder, since the *ceterus paribus* conditions are more complicated. That's because the equilibrium consumption of land increases with distance from the CBD under the "normal"[7] assumptions. Let $h(x)$ denote the equilibrium value of $h$ at $x$, the number of units of land or housing purchased in equilibrium at distance $x$ from the CBD. In terms of our notation, the standard differential version of the Muth-Mills condition is as follows.

$$\partial p(x,g)/\partial x \cdot h(x) = -t \tag{2}$$

It can be derived from the budget constraint and equal utility equilibrium condition of the standard model (see the usual texts Fujita (1989) or Mills and Hamilton (1994, chapter 6), or see Coulson (1991, equation [4a])), though its derivation can be a bit messy.

We prefer the discrete version since it simplifies our analysis and avoids the use of additional assumptions, such as differentiability of $P$. The discrete version is a bit more involved.

$$\forall x, y \in X,\ P(h(y), y, g) + t \cdot y \leq P(h(y), x, g) + t \cdot x \tag{3}$$

The intuition here is pretty much the same as the other intuitions we have advanced. Bear in mind that in these models, price is generally decreasing with distance from the CBD while housing consumption is generally increasing with distance from the CBD. If we look at the equilibrium housing profile across locations $h(y)$, then if we move farther from the CBD and look at the same house, the price decrease from moving farther from the CBD must not exceed the increase in commuting cost. If we move closer to the CBD and look at the same house, the price increase must be at least as large as the decrease in commuting cost. The reason this differs from the differential statement (2) above that uses only the equilibrium value of housing at a point is that in the discrete case, equilibrium housing consumption changes with a discrete change in location, and no arbitrage should prevent consumers from moving and buying the identical house in equilibrium. Evidently, we can obtain the differential version (2) by imposing Axiom 2, by using the inequalities (3), and

[7] That is, the usual assumptions plus normality of housing. Pun intended. See Fujita (1989).

by applying the definition of derivative by taking limits as $x$ approaches $y$.[8] Notice that the difference between our inequality (3) and our Axiom 3 is that Axiom 3 uses equalities, while inequality (3) uses inequalities. In fact, *Axiom 3 implies inequality (3), but not conversely.*

*Since Axiom 3 is stronger than inequality (3), why do we impose it?* The reason lies in the economic intuition. Let's begin by stating Axiom 3 in a slightly different form:

$$P(h,x,g) - P(h,y,g) = t \cdot [y - x]$$

What does this mean? *It means that there is exactly 100% capitalization of commuting cost differentials into land price differentials.* We want to emphasize that this is the easiest interpretation of this axiom. The way to see this is to suppose that equation (3) holds with strict inequality for some locations $x$ and $y$ and some land or housing quantity $h(y)$. For the purpose of illustration, let us take $h(y)$ to be the equilibrium quantity of land or housing consumed at location $y$. Then

$$P(h(y),y,g) + t \cdot y < P(h(y),x,g) + t \cdot x$$

or

$$P(h(y),x,g) - P(h(y),y,g) > t(y - x)$$

Suppose first that $x > y$. Then this says that the commuting cost differential is capitalized less than 100%. If $x < y$, then the commuting cost differential is capitalized more than 100%.

So now let's return to the crux of the issue. Assume that we have an equilibrium price for a nice monocentric city that satisfies Axiom 2 but does not satisfy Axiom 3. So there is not precisely 100% capitalization of commuting cost differentials into land price differentials. Suppose a city manager or developer or department chair comes in and offers the consumers an alternative city with 100% capitalization of commuting cost differentials into the land pricing scheme. For example, consider the quasi-linear utility case. A two part tariff (with the fixed subsidy, zero at the CBD, for each location accounting for the commuting cost, and the marginal cost at a location equal to the marginal cost at that location in the old city) would work, as in Example 3. Such a scheme satisfies Axiom 3, of course. And everyone moves out. The idea here

[8] In fact, Mills and Hamilton (1994, p. 110) requires some correction in this respect, since they refrain from using calculus in the text but nevertheless derive a discrete version of the result that uses an equality rather than inequalities.

is that there is more freedom in price systems satisfying Axiom 3 than those satisfying Axiom 2, so developers compete away rents to the consumers more readily. So Axiom 3 is essentially a no arbitrage condition for city entry. The proposal is an equilibrium for the new city, with a rent transfer to consumers.

Axiom 3 is what sets urban economic hedonic models apart from hedonic models in other fields.

# 3 The Result

**Theorem 1** *Axioms* 1, 2, *and* 3 *together imply a contradiction. Thus, for any test of a hedonic housing model, at least one of the three axioms must be violated. If all three axioms are assumed (either explicitly or implicitly), then the model is internally inconsistent.*[9]

Proof: An almost trivial proof using calculations. Let the two different locations given in Axiom 1 be $x, y \in X$, let the two different housing types given in Axiom 1 be $h, i \in H$, and let the element of $G$ given in Axiom 1 be $g \in G$. Then Axiom 3 implies $P(h, x, g) = P(h, y, g) + t \cdot [y - x]$ and $P(i, x, g) = P(i, y, g) + t \cdot [y - x]$. Applying Axiom 2 to these two equations, $h \cdot p(x, g) = h \cdot p(y, g) + t \cdot [y - x]$ and $i \cdot p(x, g) = i \cdot p(y, g) + t \cdot [y - x]$. Note that the last two equations and $x \neq y$ imply that $p(x, g) \neq p(y, g)$. Combining the last two equations, one obtains that $h = i$, a contradiction. Q.E.D.

# 4 The Take Home Lesson

In this section, we will show that there are many models that are observationally equivalent to the standard monocentric city model, so empirical "tests" cannot distinguish among them. Even if the utility function and income of the identical consumers in the one type model are known to the observer, there is a continuum of models (equilibrium price systems) consistent with the observed housing purchases and rent density, only one of which is the standard model. If the income and utility function are unknown, there are huge numbers of models that are observationally equivalent to the standard model, having downward sloping rent and upward sloping land consumption. The

[9] Notice that the Axioms to not impose restrictions on what agents are in the model. Thus, Theorem 1 covers economies with any combination of consumers, producers, governments, and Martians.

Introduction provides an example based on a common version of the standard model using log linear utility.

Note that the standard urban economic theory model satisfies Axiom 2 with $c \equiv 0$. Evidently, this model does not satisfy Axiom 3. However, it is easy to construct an equilibrium price system for such a model that does satisfy Axiom 3 but not Axiom 2. For simplicity, let's ignore $g$. Suppose that the equilibrium price of housing satisfying Axiom 2 is given by $p : X \to \Re$ (defining $P$ according to Axiom 2), and that the equilibrium housing function $h : X \to \Re$ is one-to-one and onto (for example, $h(0) = 0$ and $dh/dx > k > 0$). Denote the function $h^{-1}$ by $x(h)$. Define $\pi : H \times X \to \Re$ by $P(h, y, g) = \pi(h, y) \equiv h \cdot p(x(h)) + t \cdot [x(h) - y]$. Then the price system $p$ satisfies Axiom 2 but not Axiom 3. The price system $\pi$ satisfies Axiom 3 but not Axiom 2. The two price systems agree on the equilibrium bundles $h(x)$, but no price system satisfies both Axioms. Moreover, for any $\alpha \in (0, 1)$, define the price system $P(h, x, g) = \pi^{\alpha}(h, y) \equiv \alpha hp(y) + (1 - \alpha)\pi(h, y)$. Then $\pi^{\alpha}$ is also an equilibrium price system (with the same equilibrium bundles), but satisfies neither Axiom 2 nor Axiom 3.

Next we prove formally that if $p$ is an equilibrium price system, so is $\pi$ (with the same equilibrium bundles). To see this, let $u(h, z)$ be the utility function of all of the identical consumers, where $z$ is a non-negative number representing composite good consumption. If $w$ is a positive number representing income, the budget constraint is $w \geq t \cdot x + P(h, x) + z$. Now suppose that $[h(x), z(x), p(x)]$ is an equilibrium in the sense that $w \geq t \cdot x + h(x) \cdot p(x) + z(x) \forall x$ and for all $(h', z', x')$ with $w \geq t \cdot x' + h' \cdot p(x') + z'$, $u(h(x), z(x)) \geq u(h', z')$. Suppose that $\exists (h', z')$ and $\exists x$ with $u(h', z') > u(h(x), z(x))$. Then $\forall x'$, $t \cdot x' + \pi(h', x') + z' = t \cdot x' + h' \cdot p(x(h')) + t \cdot [x(h') - x'] + z' = h' \cdot p(x(h')) + t \cdot x(h') + z' > w$. In fact, $\forall \alpha \in [0, 1]$, $P(h, x, g) = \pi^{\alpha}(h, y) \equiv \alpha hp(y) + (1 - \alpha)\pi(h, y)$ is also an equilibrium price system (this follows easily from the fact that the equilibrium allocation is the same), and $\pi^{\alpha}$ satisfies neither Axiom 2 nor Axiom 3 when $\alpha \in (0, 1)$.

# 5 Nonlinear Pricing

The consequences of putting $h$ as an argument in $p$ (namely, allowing the unit price of land or housing to depend on quantity purchased) are twofold, as we will show formally.

1) For given endowments, a continuum of feasible allocations are

equilibrium allocations. (In the New Urban Economics, typically the equilibrium allocation is unique for given endowments.)

2) Among these, only a few are efficient. (In the case of the New Urban Economics, few = 1.)

*By letting unit price depend on quantity, one has introduced nonlinear pricing*, and thus a nonlinear budget constraint. In general, we don't think that the literature of the New Urban Economics features properties 1 and 2, and thus does not allow the unit price to depend on quantity.

*Example 3 of the Introduction gives a price system consistent with Axiom 3 (but not Axiom 2) that results in an equilibrium allocation that is inefficient.*

**Theorem 2** *Let utility be quasi-linear, $u(h,z) = z + v(h)$, for instance $v(h) = \ln(h)$, as in Example 2. Let consumer endowments of composite good be denoted by $w$. Fix any feasible allocation $(\widehat{z}(x), \widehat{h}(x))$ such that $\widehat{z}(x) \leq w$ and such that utility is constant across locations: $u(\widehat{h}(x), \widehat{z}(x)) = u(\widehat{h}(x'), \widehat{z}(x'))$. Then there exists $p(h,x)$ (or equivalently $P(h,x) = h \cdot p(h,x)$) such that $p(h,x), (\widehat{z}(x), \widehat{h}(x))$ is an equilibrium for fixed endowments $w$.*

**Proof:** Define $p(h,x) = \frac{w - t\cdot x + v(h) - v(\widehat{h}(x)) - \widehat{z}(x)}{h}$. Then $(\widehat{z}(x), \widehat{h}(x))$ is affordable for each $x$: $\widehat{z}(x) + t\cdot x +$ $p(\widehat{h}(x), x) \cdot \widehat{h}(x) = w$. Anything better is too expensive: $z' + v(h') > \widehat{z}(x) + v(\widehat{h}(x))$ implies $z' + t \cdot x' + p(h', x') \cdot h' = z' + t \cdot x' + \frac{w - t\cdot x' + v(h') - v(\widehat{h}(x')) - \widehat{z}(x')}{h'} \cdot h' = w + z' + v(h') - [\widehat{z}(x') + v(\widehat{h}(x'))] = w + z' + v(h') - \widehat{z}(x) + v(\widehat{h}(x)) > w$.

Q.E.D.

For instance, $\widehat{h}(x)$ could decrease in $x$, inconsistent with the prediction of the standard model as well as efficiency.

# 6 Conclusions

Clearly what we have in mind is nihilism (as opposed to the hedonism of hedonic models). There is no equilibrium price system that simultaneously satisfies two natural no arbitrage conditions, Axioms 2 and 3.[10] For each condition, there is one equilibrium price system satisfying it and a continuum of price systems satisfying neither. These price systems generate the same equilibrium bundles, but price bundles not purchased in equilibrium differently.

[10] It is amusing to note that most theorists prefer Axiom 2, and most empiricists prefer Axiom 3. Since the two axioms cannot be distinguished empirically, this must somehow be connected to matters of faith.

These models (with alternate price systems) are observationally equivalent, so the tests of the standard hedonic model cannot distinguish among them. The use of one particular price system out of a continuum represents an assumption that cannot be tested.[11]

The main advantage of price systems or models satisfying Axiom 2 is that it's *possible* that every equilibrium allocation will be efficient; see Berliant, Papageorgiou and Wang (1990) for some counterexamples in small variations of the standard model. If one chooses a model and price system consistent with Axiom 3 instead of Axiom 2, then the nonlinear pricing in housing can generate more equilibria, perhaps with heterogeneous land parcels or housing in each location. This would correspond closer to the data. On the other hand, such equilibrium allocations are unlikely to be efficient, as in Example 3 of the Introduction.

Coulson (1989) presents empirical evidence that Axiom 2, linear prices for housing at a given location, is violated. Evidently, this argues for acceptance of Axiom 3 in its place.

Let us turn next to a detailed discussion of Theorem 1. One might think that a way to get around the problem we have presented is to estimate a hedonic model where commuting cost is not additively separable in price. However, this approach would deny the essential intuition of Axiom 3, that the prices of identical land parcels or houses in different locations should be the same except for the difference in commuting cost. If a hedonic model is additively separable in commuting cost, then it likely violates Axiom 2, and has the potential to generate a continuum of equilibrium allocations, at least some of which are inefficient.

The result can be extended in a number of ways. First, multiple city subcenters and different directions of commuting can be accommodated, as long as there are two locations from which consumers commute to the same subcenter. As Coulson (1991) notes, multiple incomes or types of consumers make no difference to Axiom 3 (or the other axioms). Notice that the construction

[11] Scotchmer (1986) is a related model. There, the hedonic price function is assumed linear and as a result is identified, but the preferences consistent with an equilibrium hedonic price function are not. As a consequence, the benefits of non-marginal changes in amenities ($g$ in our notation) are not identified. In our model, the hedonic price function itself is not identified (outside of equilibrium values), so the benefits of non-marginal changes in amenities are unknown. This follows because it is unclear how such non-marginal changes will affect prices of or willingness to pay for locational goods via $c(g)$ or $P(h, x, g)$, as the only values of these functions that are known are those for equilibrium values of $g$.

of the hedonic regression itself, linear in distance from the CBD, implies Axiom 3, and thus leads to a contradiction. If we actually *observe* two pairs of identical houses[12] in different locations, then even without the specification of the hedonic regression model, we've got a serious problem, since Axiom 3 is satisfied (this is essentially Example 1 of the introduction). The latter idea can be extended to chains of pairs of identical houses, where the locations are the same for the second element of one pair and the first element of the next. If we get a non-trivial cycle of pairs (returning to the location where the chain starts), then there is a contradiction. The proof involves stringing together all of the equations implied by the chain. Non-linear commuting cost can be accommodated easily in the axioms and proof. Finally, the locational good need not be land. One could replace all occurrences of "land" in this paper by "housing." The symbol $h$ would represent housing consumption in that case.

There is yet another level of generalization of our result that is feasible; it replaces the semiparametric regression in Axiom 2 with a non-parametric regression. Let $y > x$. Retain Axioms 1 and 3. These alone imply that $P(h, x, g) - P(h, y, g) = P(i, x, g) - P(i, y, g)$. In differential terms, this would imply that $\partial P(h, x, g)/\partial x \equiv \partial P(i, x, g)/\partial x$ (or its messy discrete alternative). Thus, $\partial^2 P(h, x, g)/\partial x \partial h = 0$ (or its messy discrete alternative). In place of Axiom 2, one can use the generalization $\partial^2 P(h, x, g)/\partial h \partial x \neq 0$ or its messy discrete equivalent. (When Axioms 2 and 3 are satisfied, $\partial^2 P(h, x, g)/\partial h \partial x = -t$.) This yields a contradiction with a new, more general condition on the hedonic price function than no arbitrage. It is essentially a single crossing condition on $P$! This avenue of research clearly deserves further attention; supermodularity could play a key role.

It should be clear that when we test urban economic models, we are actually assuming more than the classical Muth-Mills condition. We are generally using Axiom 3 (stronger than the Muth-Mills condition).

So where do we go from here? There are four obvious paths and one less obvious path. The first is to make commuting cost individual-specific. This path is likely to lead nowhere for two reasons. First, the framework presented here remains a special case, so the results still apply. Second, the empirical evidence (Deacon and Sonstelie (1985)) is that commuting cost is not heterogeneous.

The second path is to study imperfect competition models of the housing market. The hedonic regressions would have to be modified, for example

[12] A convenient example is the "Chicago bungalow."

to account for the number of firms in the industry. Again, the framework presented here will be a special case when firms have little or no market power, so this path does not seem promising either.

The third path is to allow commuting cost to depend on both distance from the CBD and on $h$. There are two problems with this idea. First, $h$ is a choice variable of each household. Does it make sense to have households choosing housing knowing that this choice affects commuting cost independent of distance from the CBD? This property seems unrealistic. Second, Axioms 2 and 3 imply that commuting cost must be proportional to $h$. So, for example, commuting cost at any location must vanish as $h$ tends to zero. Again, this seems unrealistic.

The fourth and less obvious path[13] is to allow nonlinearities of the hedonic price function $P(h, x, g)$ in both housing $h$ and in location $x$. For instance, one could use a nonparametric regression to test the hypothesis that $P$ is linear in $h$ or $x$, in other words, to see if Axioms 2 or 3 are satisfied. Thus, the data itself would speak to the issue of whether or not the no arbitrage conditions are satisfied. Of particular interest is whether or not Axiom 2 is consistent with the data, since this would be a *real* test of the standard model.[14]

The fifth path, suggested to us by several people, is to recast the entire New Urban Economics in the context of nonlinear pricing, as illustrated in Example 3 and Theorem 2. This approach implies an entirely new paradigm and set of results.

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[13] We are grateful to Nancy Wallace for this suggestion.

[14] Notice that the standard model predicts that housing or land consumption should be homogeneous at almost every location. If the standard model satisfying Axiom 2 is the null hypothesis, then under the null hypothesis, one should certainly never observe nonlinearities in the hedonic price function at a given location, both because housing is homogeneous (so only one value of $h$ should be observed) and because the pricing function is linear. Thus, it is certainly possible to reject the model.

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