

Explaining the Paradoxical Persistence of Momentum Traders in International Financial Markets

Extended knowledge feedback traders, as first stated by Shiller (1990a), continue to exist in the market even though rational investors who predict their behavior must consistently outsmart them. This paper addresses the question of why positive feedback traders with extended knowledge persist in international securities markets. Several different reasons have been suggested as for why positive feedback trading persists or more generally, why technical analysis persists. Singleton (1987), Hussman (1992), Brown & Jennings (1989) and Grundy & McNichols (1989) attempt to explain technical analysis as optimal dynamic signal extracting behavior under information heterogeneity and noisy prices. Benninga & Blume (1985) and Black & Perold (1992) show that dynamic hedging strategies are optimal under certain classes of utility functions depending on the content of the strategy. This suggests that positive feedback trading can be optimal behavior for people who have specific types of utility functions. De Long, Shieifer, Summers & Waldmann (1990b) suggest that positive feedback traders' learning from past mistakes might be limited because each bubble episode might look different and there are often long intervals between episodes. They also point out that there are always new investors entering the market who have never experienced past episodes. As a general principle that applies to any type of irrational investors, De Long, Shieifer, Summers & Waldmann (1988, 1990(a)) point out that in the presence of an equity premium, irrational investors can earn a higher expected return if they are on average more bullish than rational investors.

This paper suggests quite a different reason for the persistence of positive feedback traders. It shows that positive feedback traders with extended knowledge actually lose hardly anything to rational investors in spite of their irrationality. Specifically, it hypothesizes and demonstrates that the expected required rate of entrance of positive feedback traders to keep the wealth ratio of positive feedback traders to rational investors constant over time converges to 0 as the memory horizon of positive feedback traders gets longer. The source of this result is that positive feedback traders' irrational forecasting strategy turns out to be a very good one whether

they know this fact or not. Positive feedback traders form their expectation of the next period's return by averaging past periods' returns. If the return series is stationary and ergodic for the mean, this means they are using a consistent estimator of the unconditional expected return as their forecast of the next period's return. As memory horizons get longer, the size of the sample that they use grows and the law of large numbers dictates that this sample mean converges to the population mean in probability unless there are uniformly positive correlations among samples.

Positive feedback traders with long memories introduce negative serial correlation of returns over long horizon. So the law of large numbers works better than under market efficiency and as a result, the variance of their forecast of the next period's return around the true unconditional expected return converges to 0. In an environment where the only source of the deviation of the true conditional expected return from the true unconditional expected return is the deviation of positive feedback traders' forecast from the true unconditional expected return, this means that the true conditional expected return of every period also converges to the true unconditional expected return in probability. As a result, expectations of rational investors and positive feedback traders of the next period's return get so close to each other that they become indistinguishable from each other. In other words, there is hardly any difference in beliefs that rational investors can take advantage of. Therefore, positive feedback traders with extended knowledge earn about the same return as rational investors.

This paper then explains a seemingly paradoxical fact that arises in the presence of positive feedback traders with extended knowledge. Two metrics for market efficiency are (i) whether there are predictable disturbance components in returns that rational investors can take advantage of and (ii) whether prices are equal to fundamental values. If we rule out (rational bubbles, the first metric is about the price equation and the second one is about its solution. Under market efficiency, there is no predictable disturbance component in returns and prices are always equal to fundamental values. The fact that the conditional expected return of every period converges to the unconditional expected return in probability suggests that positive feedback traders with extended knowledge make the market almost efficient in terms of the first metric. This also implies near-uncorrelatedness of returns. However, even if there is hardly any predictable disturbance component in the next period's return, prices deviate widely from fundamentals in the presence of positive feedback traders with extended knowledge, suggesting that the market is far from efficient in terms of the second metric. In other words, the price equation is very close to the one in the efficient market but its solution (the price) is very different from the one in the efficient market (the fundamental value). Shiller (1984, 1990b) and

Summers (1986) have noted that the two metrics of market efficiency can be very different, but this paper shows more specifically how this can happen.

The source of this seemingly paradoxical result is the extreme persistence of the positive feedback traders' forecast. When the serial correlations of the return series get negligible, we can regard the return series as nearly a white noise. Positive feedback traders' forecast is a moving average of this white noise returns and therefore it follows approximately an MA process. When the memory horizon is very long, the order of this MA process is very high and as a result, the forecast is extremely persistent, although its variance is very small.

The stock price is determined by the expectation of the infinitely extended future dividend stream, the current forecast of positive feedback traders and the expectation of the infinitely extended future forecast stream of positive feedback traders. Although the variance of the forecast is extremely small, the combination of the extreme persistence of the forecast and the summation of the infinitely extended expected future forecast stream makes an infinitesimal change in the forecast have a significant effect on the price. This paper shows that the variance of this deviation eventually converges to 0 as the memory horizon goes to infinity but its decaying speed is much slower than other quantities such as the variance of feedback traders' forecast or the expected required rate of entrance or the autocorrelation coefficients of return. Besides the decaying speed, its decaying pattern is also different from other quantities in that the variance of the deviation first increases and then decreases, as the memory horizon gets longer. On the other hand, as the deviation of prices from fundamentals is determined by the extremely persistent forecast, the deviation of prices from fundamentals is also extremely persistent.

Empirical evidence has recently challenged the notion of market efficiency in the stock market. Shiller (1981), Mankiw, Romer & Shapiro (1985) and West (1988) show that stock prices are too volatile to be the present discounted value, using a constant discount rate, of expected dividends. Fama & French (1988a) and Poterba & Summers (1988) show that stock prices have a tendency to revert to the mean in the long run. Cutler, Poterba & Summers (1988) document that stock price movements are often not associated with any news. Fama & French (1988b) and Campbell & Shiller (1988) show that lagged dividend yields have predictive power on long run stock returns. This paper focuses first on the long run mean reversion evidence of Fama & French (1988a) and Poterba & Summers (1988) that show negative autocorrelation in returns over long horizons. Two competing theories have been suggested to explain this pattern, transitory components in prices and time varying required returns. Foterba & Summers (1988) point out the problems of the time varying required returns theory that it needs too much

variation in required returns to fit variance ratio point estimates and that it leads to negative serial correlation even in the short run under plausible assumptions.¹

Based on this observation they pursue the idea of transitory components in prices further in Cutler, Poterba & Summers (1990). They construct a model of asset price dynamics in the presence of three types of traders: rational traders who invest on the basis of rational expectations of future returns, fundamentals traders who invest on the basis of the difference of prices from perceived fundamentals and feedback traders who invest on the basis of past returns. They show with this model that positive feedback traders with extended knowledge can in principle generate positive autocorrelation in returns over short horizons and negative autocorrelation in returns over long horizons by getting numerical solutions of the autocorrelation coefficients of returns in the presence of positive feedback traders with a memory horizon of 3 periods. In their model they suppress dividends by considering a futures market and instead introduce fundamentals traders in addition to rational traders to make prices well defined. This paper develops this idea further and constructs a more manageable model of stock price dynamics in the presence of feedback traders based on the work of De Long, Shieifer, Summers & Waldmann (1990a).

The model of this paper does not suppress dividends and assumes only rational traders and feedback traders. The main difference of this model from the model of Cutler, Poterba & Summers (1990) is that it can be solved for the price analytically for an arbitrary memory horizon of feedback traders, which makes it possible to calibrate or estimate parameter values such as the wealth ratio of positive feedback traders to rational investors and the memory horizon of positive feedback traders.

The potential role of very long run technical analysis in explaining the historical stock price movement is well demonstrated in Barsky & De Long (1990). They show that the historical stock price movement can be well approximated by assuming that investors form their expectations of future dividend stream by extrapolating past dividend growth trend with an infinitely long memory.² This results in a wide deviation of prices from ex post fundamental

¹ Under AR(1) required return process, the short run serial correlation of returns can be positive or 0 if the covariance of required return innovation and dividend news is positive, which is not consistent with empirical evidences. Even if we assume a positive covariance between required return innovations and dividend news, positive(O) serial correlation of returns in the short run implies positive(O) serial correlation of returns also in the long run as the sign of serial correlation is the same for all lags in this specification. By the way, this is essentially the same way as negative feedback traders generate positive serial correlation of returns at all lags in the context of transitory components theory.

² Specifically, they use geometrically decaying weights which give one-fourth of weight to the past of less than 10 years and three-fourths of weight to the past less than 45 years.

values but they do not regard the investors as necessarily irrational since they implicitly assume that the true dividend process is not known.

In this paper I make an explicit assumption on the dividend process and assume rational investors know the true dividend process while positive feedback traders form expectations of the next period's return by extrapolating past periods' average return. This paper implements both calibration and estimation to determine parameter values of the model. I first calibrate them by comparing variance ratios from simulations of the model and the variance ratio point estimates of Poterba & Summers (1988). This comparison shows that the wealth ratio of positive feedback traders to rational investors is around 1-1.5 and the memory horizon of positive feedback traders is around 10 ~ 40 years under the model specification of this paper. I then estimate them using the annual S&P stock price and dividend data that Shillerf (1981) uses. When there are positive feedback traders with extended knowledge, stock returns show significant long run mean reversion that can be detected by long horizon variance ratio tests while individual autocorrelation coefficients of returns become indistinguishable from the autocorrelation coefficients in efficient markets at all lags. When the variance ratios are well approximated by the parameter values, the simulation results are also consistent with other often cited properties such as excess volatility, price movement without news and the predictive power of lagged dividend yields on long run stock returns.

This paper then investigates consequences of positive feedback traders with extended knowledge for the equity premium and the time varying volatility pattern. When the variance ratios are well approximated by the parameter values, the simulation shows that the stock returns exhibit around 2% ~ 4% of extra equity premium compared to the efficient market level. This extra equity premium must be due to something other than an increase in perceived risk because the effect of perceived risk on prices is suppressed in the model. I illustrate that this is the effect from market inefficiency. I show that market inefficiency is an extra source of the equity premium in addition to perceived risk and that it works through the covariance of the current price and the next period's excess return.

Explaining why there is a large equity premium is not enough to resolve the equity premium puzzle which is raised in the context of CCAPM. To resolve the equity premium puzzle, one needs an additional explanation on how a large equity premium can coexist with a small covariance of aggregate consumption growth and stock returns. One line of explanation for the equity premium puzzle is based on the presence of heterogeneous agents such as stockholders and nonstockholders or rational investors and noise traders, suggesting that aggregate consumption does not represent the consumption of rational agents who participate in

the international securities markets. Mankiw & Zeides (1991) show that the covariance of consumption growth and stock returns is over five times as great for stockholders as for nonstockholders. De Long, Shieifer, Summers & Waldmann (1990a) point out that since noise traders are bullish when the true expected return is low and bearish when the true expected return is high, the covariance of consumption growth and stock returns is smaller for noise traders than for rational investors who satisfy the Euler equations.

This paper formalizes the intuition of De Long, Shieifer, Summers & Waldmann (1990a) and first shows that the presence of irrational investors of any type who behave differently from rational investors always drives a wedge between the equity premium and the covariance of aggregate consumption growth and stock returns because only rational investors satisfy the Euler equations. Then by analyzing the case where the stock positions of rational investors and irrational investors sum to 0, I demonstrate that the extra equity premium of 2% ~ 4% that positive feedback traders generate in the simulation can come about without any increase in the covariance of aggregate consumption growth and stock returns. I show that in this case irrational investors change the equity premium from the efficient market level through the effect from market inefficiency while they do not change the covariance of aggregate consumption growth and stock returns from the efficient market level because the stock positions of different agents always sum to the fixed supply of stocks and the covariance of aggregate consumption growth and stock returns is determined only by the aggregate stock position. When the variance ratios are well approximated by the parameter values, the volatility pattern in the simulation also approximates well the one that is documented in the literature such as Black (1976), French, Schwert & Stambaugh (1987) and Schwert (1989) which show that changes in the volatility of stock returns are negatively related to changes in the stock price level, i.e. volatility increases during a recession, and volatility is persistent in a way that even suggests nonstationarity.

This paper shows that when the conditional variance of rate of returns is constant along the fundamentals path, any transitory component in prices increases the conditional variance when prices fall below fundamentals and decreases it when prices rise above fundamentals. Moreover, the persistence of return volatility is actually identical to the persistence of the transitory component. As positive feedback traders with extended knowledge make prices deviate widely and persistently from fundamentals, the increase of volatility when stock prices fall can be dramatic and the volatility shows an extreme persistence.

This paper shows that the deviation follows approximately another MA process of a very high order when the memory horizon is very long. This extreme persistence of the deviation is the source of the extreme persistence of volatility. Positive feedback traders with extended

knowledge make the international stock market a seemingly paradoxical place where their infinitesimal forecast error causes prices to deviate widely from fundamentals and they hardly lose to rational investors although they are not rational. This paper is organized as follows. Section 1 presents the model of stock price dynamics in the presence of feedback traders. Section 2 presents the calibration and estimation results for the parameter values of the model and also presents other simulation results. Section 3 shows how irrational investors can drive a wedge between the equity premium and the covariance of aggregate consumption growth and stock returns. Section 4 discusses the effect of transitory components in prices on volatility patterns. Section 5 describes the second model with simplified notations. Section 6 analyzes the required rate of entrance problem. Section 7 analyzes the deviation of prices from fundamentals. Section 8 concludes.

1. The Model of Stock Price Dynamics in the Presence of Feedback Traders

The infinitely extended overlapping generations model with two-lived-period agents specification creates a short investment horizon of rational investors and make available periods very long in the model. As discussed in De Long (1990a), the short investment horizon of rational investors combined with risk aversion makes it possible for the price to deviate from the fundamental value. On the other hand, numerous periods are needed in this paper to analyze price dynamics in the presence of feedback traders with extended knowledge. An infinite period model is a natural choice that simplifies algebra. The numeraire is consumption. Each agent earns exogenous labor income and invests all of it when young. The agent sells all the investments and consumes the proceeds when old. There are two assets in the economy: bonds and stocks. Bonds pay a fixed interest r and are in infinitely elastic supply with the price fixed at 1. Stocks pay a risky dividend d , in period t and are in fixed supply S . Assuming that log dividends follow a random walk with drift,

$$\ln d_t = \delta + \ln d_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d. } N(0, \sigma_\varepsilon^2).$$

Under this specification the conditional expectation and the conditional variance of the one period ahead dividend are $E_t(d_{t+1}) \approx (1 + \delta + \sigma_\varepsilon^2/2)d_t$ and $\text{Var}_t(d_{t+1}) \approx \sigma_\varepsilon^2 d_t^2$, respectively. The expected growth rate of dividends $\delta + \sigma_\varepsilon^2/2$ is assumed to be strictly smaller than the interest rate r . The ex dividend price of stocks in period t is denoted p_t . There are two types of agents: rational investors who have rational expectations and feedback traders who extrapolate past trends to form expectations. Aggregate labor income of young rational investors and feedback traders are W^R and W^F , respectively in all periods. Each agent maximizes a mean-

variance approximation to a constant relative risk aversion expected utility function.

$$E_t(U) = E_t(C_{t+1}) - (\gamma/2W)\text{Var}_t(C_{t+1})$$

where C_{t+1} is the agent's consumption in period $t+1$ and γ is the index of local relative risk aversion. The budget constraint of each agent is

$$C_{t+1} = (W - X_t p_t)(1+r) + X_t(p_{t+1} + d_{t+1})$$

where X_t is the quantity of the stocks that the agent holds in period t . For rational investors,

$$\begin{aligned} E_t(C_{t+1}) &= W^R(1+r) + X_t^R [E_t(p_{t+1}) + E_t(d_{t+1}) - (1+r)p_t] \\ \text{Var}_t(C_{t+1}) &= (X_t^R)^2 \text{Var}_t(p_{t+1} + d_{t+1}) \end{aligned}$$

Maximizing expected utility with respect to X_t^R yields rational investors' demand for stocks in period t .

$$X_t^R = [E_t(p_{t+1}) + E_t(d_{t+1}) - (1+r)p_t] W^R / \gamma \text{Var}_t(p_{t+1} + d_{t+1})$$

Feedback traders perceive the variance correctly but form their expectation of excess return by extrapolating past k periods' excess returns. Specifically, feedback traders perceive $E_t(C_{t+1})$ and $\text{Var}_t(C_{t+1})$ as

$$\begin{aligned} E_t^F(C_{t+1}) &= W^F(1+r) + X_t^F \{ \beta_1 [p_{t+1} + d_{t+1} - (1+r)p_{t+2}] + \dots + \beta_k [p_{t+k} + d_{t+k} - (1+r)p_{t+k+1}] \} \\ \text{Var}_t^F(C_{t+1}) &= (X_t^F)^2 \text{Var}_t(p_{t+1} + d_{t+1}) \end{aligned}$$

Maximizing expected utility with respect to X_t^F yields feedback traders' demand for stocks in period t

$$\begin{aligned} X_t^F &= \{ \beta_1 [p_{t+1} + d_{t+1} - (1+r)p_{t+2}] + \dots + \beta_k [p_{t+k} + d_{t+k} - (1+r)p_{t+k+1}] \} W^F \\ &\quad / \gamma \text{Var}_t(p_{t+1} + d_{t+1}) \end{aligned}$$

Unlike the noise traders of De Long, et al. (1990a), the feedback traders in this model do not present an additional source of risk and therefore any uncertainty in feedback traders' behavior is due to the fundamental risk. The market clearing condition in period t is $X_t^R + X_t^F = S$. I interpret S as the supply of stocks net of the holdings of passive investors who implement a buy and hold strategy and assume $S = 0$. I divide the market clearing condition by W^K and define $\mu = W^F/W^R$. Then the stock price is determined as

$$\begin{aligned} P_t &= (1+r)^{-1} \{ E_t(p_{t+1}) + E_t(d_{t+1}) \\ &\quad + \mu \beta_1 [p_{t+1} + d_{t+1} - (1+r)p_{t+2}] + \dots + \mu \beta_k [p_{t+k} + d_{t+k} - (1+r)p_{t+k+1}] \} \end{aligned} \quad (1)$$

A particular solution of (1) is

$$\begin{aligned} P_t &= -\mu \beta_1 p_{t-2} - \mu \beta_1 p_{t-3} - \dots - \mu \beta_1 p_{t-k+1} \\ &+ [1 + \delta + \sigma_\varepsilon^2/2 + (1+r)^{-1} \mu \beta_1 + (1+r)^{-2} \mu \beta_2 + \dots + (1+r)^{-2} \mu \beta_2] d_t / (r - \delta - \sigma_\varepsilon^2/2) \\ &+ [(1+r)^{-1} \mu \beta_1 + (1+r)^{-2} \mu \beta_2 + \dots + (1+r)^{-k} \mu \beta_k] d_{t-1} \\ &+ [(1+r)^{-1} \mu \beta_2 + \dots + (1+r)^{-(k-1)} \mu \beta_k] d_{t-2} \end{aligned}$$

$$+ \quad \quad \quad [(1+r)^{-1}\mu\beta_k] d_{t-k} \quad (2)$$

The conditional variance $\text{Var}_t(p_{t+1}+d_{t+1})$ is proportional to d_t^2 .

$$\begin{aligned} \text{Var}_t(p_{t+1}+d_{t+1}) = \\ [1 + r (1+r)^{-1}\mu\beta_1 + (1+r)^{-2}\mu\beta_2 + \dots + (1+r)^{-k}\mu\beta_k]^2 \sigma_\varepsilon^2 d_t^2 / (r - \delta + \sigma_\varepsilon^2/2)^2 \end{aligned} \quad (3)$$

Adding any rational bubble term B_t to p_t also yields a solution to the equation (1) as long as $\{B_t\}$ is a stochastic process that satisfies $E_t(B_{t+1}) = (1+r)B_t$. An example of rational bubble terms is the deterministic rational bubble $B_t = A(1+r)^t$ where A is an arbitrary constant. I ignore rational bubble terms in this paper.³ As a baseline case, I examine the properties of this model under market efficiency. Absent feedback traders ($\mu = 0$), the price is equal to the fundamental value.

$$p_t^* = (1+\delta + \sigma_\varepsilon^2/2) d_t / (r - \delta + \sigma_\varepsilon^2/2) \quad (4)$$

This fundamental value is the non-bubble solution to the equation

$$r = E_t[(p_{t+1} + d_{t+1} - p_t) / p_t] \quad (5)$$

which implies that the unconditional expectation of the rate of return is r under market efficiency. This absence of equity premium in spite of risk aversion and fundamental risk is the result of my simplifying assumption $S = 0$. If we denote the risky rate of return as R_{t+1} , the arbitrage equation (5) can be rewritten as

$$R_{t+1} = r + \eta_{t+1}, \quad E_t(\eta_{t+1}) = 0$$

where the innovation η_{t+1} reflects the change in expectations between period t and period $t+1$. This implies that conditioning on any variable x in the information set of period t , the expectation of the innovation η_{t+1} is 0.

$$E(\eta_{t+1} | x) = E[E_t(\eta_{t+1}) | x] = 0$$

The innovation η_{t+1} is thus orthogonal to any variable x in the information set of period t and since $E(\eta_{t+1}) = 0$, this implies the lack of correlation between η_{t+1} and x .

$$\text{Cov}(\eta_{t+1}, x) = E(\eta_{t+1}x) = E[E(\eta_{t+1} | x) x] = 0 \quad (6)$$

As the information set of period t contains all the lagged values of η_{t+1} , the innovation series $\{\eta_t\}$ and the rate of return series $\{R_{t+1}\}$ should be serially uncorrelated under market efficiency.⁴ In addition, my specification of the dividend process makes conditional variances of

³ Other examples would be the stochastic rational bubble of Blanchard & Watson (1982) where $B_{t+1} = [(1+r)/\varphi]B_t + v_{t+1}$ with probability φ , v_{t+1} with probability φ , v_{t+1} with probability $1-\varphi$, where $E_t(v_{t+1}) = 0$ and the intrinsic rational bubble of Froot & Obstfeld (1991) $B_t = A d_t^\lambda$, where A is an arbitrary constant and λ is the positive root of the quadratic equation $\sigma_\varepsilon^2 \lambda^2/2 + \delta \lambda - r = 0$.

⁴ Under my specification of the dividend process, η_t 's are independent.

the rate of return independent of t under market efficiency.

$$\text{Var}_t(R_{t+1}) = (1 + r - \delta - \sigma_e^2/2)^2 \sigma_e^2 \quad (7)$$

As $E_t(R_{t+1})$ is a constant, unconditional variances of the rate of return are also independent of t under market efficiency.

$$\text{Var}(R_{t+1}) = E[\text{Var}_t(R_{t+1})] + \text{Var}[E_t(R_{t+1})] = (1 + r - \delta - \sigma_e^2/2)^2 \sigma_e^2 \quad (8)$$

The model makes rate of returns serially uncorrelated and variance stationary in the absence of feedback traders, therefore the variance of any q period rate of return is q times the variance of 1 period rate under market efficiency. Moreover, there is no conditional heteroskedasticity in rate of returns in the absence of feedback traders. This means the volatility of rate of returns is constant over time under market efficiency.

2. Calibration and Estimation Results for the Model

I implement both calibration and estimation to determine the parameter values of the model: the wealth ratio of feedback traders to rational investors and the memory horizon of positive feedback traders k . In both calibration and estimation I assume that positive feedback traders' expectation of excess return is simply the average of past k periods' excess returns, which means $\beta_1 = \beta_2 = \dots = \beta_k = 1/k$. Using geometrically decaying weights in this model has the effect of shortening the memory horizon without any additional effect, so I use the average for simplicity. I take the values of δ and σ_e^2 from Froot & Obstfeld (1991) which estimate them as $\delta = 0.011$ per annum and $\sigma_e^2 = 0.015$ per annum. This gives annual expected growth rate of dividends $\delta + \sigma_e^2/2 = 0.019$, which is 1.9%. I set annual interest rate r at 6.5%, which makes annual dividend yields 4.5% and the price-dividend ratio 22 under market efficiency. I first present the calibration strategy and results. I calibrate the values of μ and k by comparing annual return standard deviations and variance ratios from simulations and their point estimates in Poterba & Summers (1988).

Calculating the variance ratio from Poterba & Summers (1988) which uses 12-month returns as the base yields

$$\text{VR}(q) = 12\text{Var}[R_t(q)]/q\text{Var}[R_t(12)]$$

where $R_t(q) = R_t + R_{t-1} + \dots + R_{t-q+1}$ is the q month returns. I assume 1 month as 1 period in the model and generate 70560 observations of rate of returns by stacking 7 samples of size 10080. This sample size reduces the theoretical standard error of variance ratios under market efficiency to less than 0.05 for up to 8 years of return horizon even with non-overlapping multi-period returns.⁵ I calculate annual return standard deviations and variance ratios for $q = 1, 24, 36, 48$,

⁵ The asymptotic standard deviation of variance ratio statistics $\text{VR}(q) = \text{Var}[R_t(q)]/q\text{Var}(R_t)$ under market efficiency with non-overlapping q period return, is $[2(q-1)/N]^{1/2}$, where N is the sample size. As variance ratios

60, 72, 84, 96 with different values for μ and k . By comparing these results with the point estimates of Poterba & Summers (1988), I calibrate the parameters μ and k .⁶ The annual return standard deviation and variance ratio point estimates of Poterba & Summers (1988) are as follows. The variance ratio decreases as the return horizon gets longer and at 96 month return horizon it is around 0.3 - 0.9, showing negative serial correlation in returns over long horizons. The annual return standard deviation is around 0.18 ~ 0.30.

Variance Ratios for U.S. Monthly Data, 1926-1985

Returns (equal weighted, value weighted, real or excess)	Annual Return Standard Deviation	Return Horizon (months)							
		1	24	36	48	60	72	84	96
Equal Weighted Real Returns	0.296	0.809	0.963	0.835	0.745	0.642	0.522	0.400	0.353
Equal Weighted Excess Returns	0.296	0.785	1.010	0.925	0.878	0.786	0.649	0.487	0.425
Value Weighted Real Returns	0.206	0.797	0.973	0.873	0.747	0.667	0.610	0.565	0.575
Value Weighted Excess Returns	0.207	0.764	1.036	0.989	0.917	0.855	0.781	0.689	0.677

are scale-free, this standard error does not depend on the length of 1 period and it is increasing with q and decreasing with N . When the observations are 70560 monthly returns, converting them to yearly returns reduces the sample size to 5880. With 5880 yearly returns, the standard error of 8 year variance ratio is $[2(8-1)/5880]^{1/2} = 0.049$.

⁶ I run the variance ratio simulation also with 3 days as 1 period in the model, generating 705600 observations of rate of returns. This is not completed as of the time of writing but the results so far for $\mu = 1$ and 1.5 are the same as with one month as one period in the model.

Variance Ratios for U.S. Annual Data, 1871-1985

Returns (real, excess)	Annual Return Standard Deviation	Return Horizon (months)						
		24	36	48	60	72	84	96
Real Returns	0.190	1.035	0.880	0.876	0.855	0.797	0.769	0.781
Excess Returns	0.189	1.047	0.922	0.929	0.913	0.856	0.821	0.833

Tables 1-1 through 1-4 show the annual return standard deviations and variance ratios from simulations for different values of μ and k . These show that the most appropriate values for μ and k that satisfy the restrictions are μ around 1 ~ 1.5 and k around 120 ~ 480 months.⁷ For comparison, I also report the annual return standard deviations and variance ratios in the presence of negative feedback traders in Table 1-5. Negative feedback traders are specified as using the average of past periods' excess return multiplied by -1 as their forecast of the next period's excess return. The price equation (1) shows that multiplying a by -1 yields the price in the presence of negative feedback traders. Table 1-5 shows that negative feedback traders make serial correlation of returns positive at all lags.

I now present the estimation strategy and results. In the model, the excess return $\pi_{t+1} = p_{t+1} + d_{t+1} - (1+r)p_t$ follows

$$\pi_{t+1} = -\mu k^{-1}(\pi_{t-1} + \pi_{t-2} + \dots + \pi_{t-k}) + \{1 + r + \mu k^{-1} r^{-1} [1 - (1+r)^{-k}]\} [\exp(\delta + \varepsilon_{t+1}) - \exp(\delta + \sigma_\varepsilon^2/2)] d_t / (r - \delta - \sigma_\varepsilon^2/2)$$

Dividing both sides by the fundamental value $p_t^* = (1 + \delta + \sigma_\varepsilon^2/2) d_t / (r - \delta - \sigma_\varepsilon^2/2)$ yields

⁷ The model does not quite explain the positive autocorrelation in returns over short horizons which is reported in Lo & MacKinlay (1988) and Poterba & Summer (1988). To the model, the serial correlations in the short run get so close to 0 that the 1 month variance ratio approaches 1 as the memory horizon of positive feedback traders gets longer. So I focus on the point estimates from 24 month variance ratio. One possible way to generate both positive autocorrelation in returns over short horizons and negative autocorrelation in returns over long horizons is to shift the memory window of positive feedback traders a little bit to the past so that they do not take account of the most recent past in forming their forecast. However, because it is not clear why they do not take account of the most recent past, I do not adopt this specification. Another possibility is that the positive serial correlations in the short run might get more significant if the investment horizon of rational investors gets longer than one period, considering the result of Cutler, Poterba & Summers (1990). Extending the investment horizon of rational investors might also decrease the magnitude of the deviation of prices from fundamentals as argued by De Long et al. (1990a). However, there are several difficulties in modeling investment horizon of many periods. It is hard to specify rational investors' object of maximization. It is hard to specify the behavior of positive feedback traders. It is difficult to make a tractable OLG structure where prices are determined every period. It is hard to endogenize the timing of liquidating positions. I do not pursue this issue further in this paper.

$$\pi_{t+1}/p_t^* = -\mu k^{-1}(\pi_{t-1}/p_t^* + \pi_{t-2}/p_t^* + \dots + \pi_{t-k}/p_t^*) \\ + \{1 + r + \mu k^{-1} r^{-1} [1 - (1+r)^{-k}]\} (1 - \delta + \sigma_\varepsilon^2/2)^{-1} [\exp(\delta + \varepsilon_{t-1}) - \exp(\delta + \sigma_\varepsilon^2/2)]$$

The error term is i.i.d. with mean 0. Using the annual S&P stock price and dividend data that Shiller (1981) uses, now extending from 1871 to 1995, I determine k by running OLS regressions

$$\pi_{t+1}/p_t^* = \beta_0 + \beta_1 \pi_{t-1}/p_t^* + \beta_2 \pi_{t-2}/p_t^* + \dots + \beta_k \pi_{t-k}/p_t^* + e_{t+1}$$

for progressively higher values of k and finding out the values of k that maximizes the adjusted R² (for degrees of freedom).⁸ Table 1-6 shows that the maximum occurs at k = 29 years. I then run a restricted least square regression at k = 29 and perform an F test on the validity of the restriction

$$H_0 : \beta_0 = 0 \text{ and } \beta_1 = \beta_2 = \dots = \beta_k$$

The F statistic has a value of 1.086 and the 5% critical value of an F(29, 64) variable is 1.645. Now μ is estimated as $-k\beta$ where β is the coefficient of the restricted least square regression. The coefficient of the restricted least square regression is - 0.113 with a standard error of 0.043. Therefore, μ is estimated as 3.277 with a standard error of 1.247. The estimation result using the adjusted R² is consistent with the calibration results. In the rest of simulations, I use the calibrated parameter values since the procedure for determining k in the estimation has no firm basis in theory.

The final part of this section presents other simulation results of the model. Table 1-7 shows the annual equity premia (%) in the same simulations as for calibration. The equity premium in the presence of positive feedback traders is positive for all parameter values and it is increasing in the wealth ratio of feedback traders to rational investors μ . When μ is around 1 ~ 1.5 and k is around 10 ~ 40 years, the equity premium is around 2% ~ 4% annually. The equity premium in the presence of negative feedback traders is negative. I run another simulation to investigate other properties of the model when μ is around 1 ~ 1.5 and k is around 10 ~ 40 years. I assume 3 days as one period in the model and generate 7200 observations of rate of returns, which corresponds to 60 years. I regard 10 periods in the model as 1 month and estimate the conditional variance of monthly rate of returns with each 10 periods' sample variance multiplied by 10. I estimate the conditional standard deviation of monthly rate of

⁸ The usual R², which does not penalize the loss of degree of freedom, is monotonically increasing in k. On the other hand, both Amemiya's prediction criterion and Ataike's information criterion are minimized at k=1, which penalize the loss of degree of freedom more heavily than the adjusted R². In principle, it is also possible to determine k by looking at the significance of the coefficients from the OLS regressions for progressively higher values of k. However, since the small sample size makes the power of test low, hardly any coefficient is significant at 5% significance level, as can be seen in the t statistics of the unrestricted OLS coefficients in Table 1-6.

returns by taking the square root of the conditional variance of the monthly rate of returns. Then I calculate the autocorrelation coefficients of monthly standard deviations. Because individual autocorrelation coefficients of the rate of return are indistinguishable from 0 at all lags, I do not report the autocorrelation coefficients of the rate of return. I report four examples of the simulations. For comparison, I first report the autocorrelation coefficients of monthly standard deviation estimated by French, Schwert & Stambaugh (1987).

Lag (month)	1	2	3	4	5	6	7	8	9	10	11	12
Coefficients	0.71	0.59	0.55	0.54	0.50	0.51	0.54	0.54	0.49	0.51	0.49	0.45

Figure 1-1 to 1-4 show the examples. They show the price and the fundamental value series, the rate of return series and the monthly standard deviation series. The parameter values and the autocorrelation coefficients of monthly standard deviation are as follows.

< $\mu = 1$ and $k = 20$ years >

Lag (month)	1	2	3	4	5	6	7	8	9	10	11	12
Coefficients	0.44	0.42	0.46	0.41	0.41	0.45	0.42	0.41	0.42	0.41	0.42	0.42

< $\mu = 1$ and $k = 30$ years >

Lag (month)	1	2	3	4	5	6	7	8	9	10	11	12
Coefficients	0.49	0.47	0.47	0.45	0.44	0.48	0.44	0.45	0.45	0.44	0.46	0.45

< $\mu = 1.5$ and $k = 20$ years >

Lag (month)	1	2	3	4	5	6	7	8	9	10	11	12
Coefficients	0.76	0.70	0.69	0.64	0.62	0.58	0.53	0.52	0.51	0.49	0.47	0.44

For comparison, I report a typical example of negative feedback traders in Figure 1-5.

< $\mu = -0.9$ and $k = 10$ years >

Lag (month)	1	2	3	4	5	6	7	8	9	10	11	12
Coefficients												

Coefficients	0.69	0.69	0.70	0.68	0.68	0.68	0.68	0.66	0.70	0.69	0.65	0.68
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Individual autocorrelation coefficients of the rate of return are indistinguishable from 0 at every lag, just as in the case of positive feedback traders.

3. Can Irrational Investors Drive a Wedge Between the Equity Premium and the Covariance of Aggregate Consumption Growth and Stock Returns?

The simulation shows that the equity premium in the presence of positive feedback traders with extended knowledge is significantly higher than the equity premium in the absence of feedback traders, which is 0 in this paper. Specifically, the equity premium is increasing with μ and it is around 2% - 4% when μ is around 1 ~ 1.5 and k is around 10 - 40 years. This extra equity premium must be due to something other than an increase in perceived risk because the effect of perceived risk on prices is suppressed in the model.⁹ I illustrate with an analytically tractable version of the model that this is the effect from market inefficiency. I show that market inefficiency is an extra source of the equity premium in addition to perceived risk and that it works through the covariance between the current price and the next period's excess return which is generally different from 0 when the market is not efficient. To resolve the Mehra- Prescott (1985) puzzle, one needs an explanation on how a large equity premium can coexist with a small covariance of aggregate consumption growth and stock returns.¹⁰

Formalizing the intuition of De Long et al. (1990a), I first show that the presence of irrational investors of any type who behave differently from rational investors always drives a wedge between the equity premium and the covariance of aggregate consumption growth and stock returns because only rational investors satisfy the Euler equations. Then by analyzing the case where the stock positions of rational investors and irrational investors sum to 0, I

⁹ Note that (3) shows that the presence of positive feedback traders increases perceived risk. Unlike the noise traders of De Long et al. (1990a) which present an additional source of risk, positive feedback traders of this paper do not present any additional source of risk. However, they increase perceived risk by amplifying fundamental risk.

¹⁰ Explanations for this equity premium puzzle can be classified into several groups. One of them focuses on market frictions that prevent rational agents from taking optimal positions in stocks such as transactions costs, borrowing constraints and short-sales constraints. The other group modifies the utility function of rational agents into a inseparable form from past consumptions or other variables such as leisure. Habit formation theory, which is making the utility function inseparable over time, tries to explain the equity premium puzzle by making the intertemporal marginal rate of substitution more volatile without changing the volatility of consumption growth. Another group of explanations is based on the presence of heterogeneous agents such as stockholders and non-stockholders or rational investors and noise traders, suggesting that aggregate consumption does not represent the consumption of rational agents who participate in the stock market. The explanation of this paper belongs to this third group.

demonstrate that the extra equity premium of 2% - 4% which positive feedback traders generate in the simulation can come about without any increase in the covariance of aggregate consumption growth and stock returns. I show that in this case irrational investors change the equity premium from the efficient market level through the effect from market inefficiency while they do not change the covariance of aggregate consumption growth and stock returns from the efficient market level. I first state the equity premium puzzle in the two-period CCAPM model. There are two periods t and $t+1$ and there is a rational representative agent who earns an exogenous wage w_t in period t . The numeraire is consumption. There are two assets in the economy: bonds and stocks. Bonds pay a fixed interest rate r and stocks pay a risky rate of return R_{t+1} . In the first period, the agent consumes C_t and invests the rest in bonds and stocks. In the second period, the agent sells all the investment and consumes the proceeds C_{t+1} . The budget constraint of the agent is

$$C_{t+1} = [1 + r + \alpha_t(R_{t+1} - r)](w_t - C_t)$$

where α_t is the fraction of savings in stocks. The agent has additively separable CRRA utility and makes an optimal consumption and investment plan by maximizing the expectation of this utility in the first period.

$$E_t(U) = (C_t^{1-\gamma}-1)/(1-\gamma) + \beta E_t[(C_{t+1}^{1-\gamma}-1)/(1-\gamma)]$$

where γ is the index of local relative risk aversion and β is the discount factor. Plugging the budget constraint into the expected utility and maximizing the expected utility with respect to first period consumption C_t and the fraction of stocks α_t yields

$$C_t^{-\gamma} = \beta E_t\{ C_{t+1}^{-\gamma} [1 + r + \alpha_t(R_{t+1} - r)] \} \quad (9)$$

$$E_t[C_{t+1}^{-\gamma} (R_{t+1} - r)] = 0 \quad (10)$$

These equations in turn yield the Euler equations for stocks and bonds.

$$\beta E_t[(C_{t+1}/C_t)^{-\gamma} (1 + R_{t+1})] = 1 \quad (11)$$

$$\beta E_t[(C_{t+1}/C_t)^{-\gamma} (1 + r)] = 1 \quad (12)$$

Approximating the term inside the bracket of (11) with a second order Taylor series around $C_{t+1}/C_t = 1$ and $R_{t+1} = 0$, and using the approximations $E_t[(C_{t+1}/C_t) - 1]^2 \approx 0$ yields¹¹

$$E_t(R_{t+1}) = (1-\beta)/\beta + \gamma E_t[(C_{t+1}/C_t) - 1] - (1/2)\gamma(\gamma+1)\text{Var}_t[(C_{t+1}/C_t) - 1] + \gamma\text{Cov}_t[(C_{t+1}/C_t) - 1, R_{t+1}] \quad (13)$$

Similarly, (12) can be approximated as

$$r = (1 - \beta)/\beta + \gamma E_t[(C_{t+1}/C_t) - 1] - (1/2)\gamma(\gamma + 1)\text{Var}_t[(C_{t+1}/C_t) - 1] \quad (14)$$

¹¹ This approximation method follows Mankiw & Zeides (1991). There are several other ways to derive essentially the same result, such as assuming joint lognormality of C_{t+1}/C_t and $1 + R_{t+1}$ or approximating (10) directly after dividing both sides by $C_t^{-\gamma}$. All the results derived from (10), (11), (12) hold both conditionally and unconditionally as (10), (11), (12) hold both conditionally and unconditionally.

Subtracting (14) from (13) gives the relationship between the equity premium and the covariance of consumption growth and stock returns.

$$E_t(R_{t+1}) - r = \gamma \text{Cov}_t[(C_{t+1}/C_t) - 1, R_{t+1}] \quad (15)$$

To see more clearly what equation (15) means, I divide the budget constraint by C_t and plug it into (15).

$$E_t(R_{t+1}) - r = \gamma \text{Cov}_t[(w_t - C_t)/C_t - 1, R_{t+1}] \quad (16)$$

This shows that the covariance of consumption growth and stock returns depends on three factors in this model. The covariance gets bigger if people save more or if they invest more in stocks or if stock returns get riskier. So what equation (15) shows is that when the equity premium gets higher, which means stocks become more attractive, rational people should either save more or invest more in stocks unless stock returns get riskier or people get more risk averse. The equity premium puzzle arises because the equity premium is much higher than the covariance of aggregate consumption growth and stock returns. Specifically, the equity premium is 0.062 and the covariance is 0.00235 in the United States for the period 1890-1979, implying a value around 26 for the index of relative risk aversion.¹²

From now on, I assume log utility. This is assuming a plausible value of 1 for the index of relative risk aversion γ . As an additively separable log utility function is homothetic and its intertemporal elasticity of substitution is 1, first period consumption is not affected by the rate of return from investment and this makes the consumption decision and the investment decision separated. Specifically, plugging the budget constraint into (9) and setting $\gamma = 1$ yields

$$C_t = w_t/(1+\beta) \quad (17)$$

Therefore, (16) is simplified to

$$E_t(R_{t+1}) - r = \beta \alpha_t \text{Var}_t(R_{t+1}) \quad (18)$$

This shows that the ratio of fast period saving to first period consumption is determined only by the discount factor and when the equity premium gets higher, rational people should invest more in stocks unless stock returns get riskier. The investment decision of a rational agent is immediate from (18).

$$\alpha_t^R = E_t(R_{t+1} - r)/\beta \text{Var}_t(R_{t+1}) \quad (19)$$

Now I introduce irrational investors. They earn the same amount of wage w_t in period t and they choose the optimal fast period consumption. The only difference of irrational investors from rational investors is that they do not choose the optimal investment allocation between

¹² The figures are from Mankiw & Zeides (1991). As a related problem, the risk free rate puzzle is about the equation (14). The equation (14) shows that the right hand side is a quadratic equation in γ that has a global maximum. The risk free rate puzzle is that for values of γ near the global maximum, a low risk free rate can imply a value of β bigger than 1.

stocks and bonds. I denote the fraction of stocks that an irrational investor holds as α_t^N . First and second period consumption of a rational agent and an irrational agent are

$$C_t^R = w_t/(1+\beta)$$

$$C_t^N = w_t/(1+\beta)$$

$$C_{t+1}^R = [1 + r + \alpha_t^R (R_{t+1} - r)] w_t \beta / (1+\beta)$$

$$C_{t+1}^N = [1 + r + \alpha_t^N (R_{t+1} - r)] w_t \beta / (1+\beta)$$

I assume that there are L^R rational investors and L^N irrational investors and denote the ratio of irrational investors to rational investors as $\mu = L^N/L^R$. First and second period aggregate consumption are

$$\begin{aligned} C_t &= C_t^R L^R + C_t^N L^N = (L^R + L^N) w_t / (1+\beta) \\ C_{t+1} &= C_{t+1}^R L^R + C_{t+1}^N L^N = (L^R + L^N) (1+r) w_t \beta / (1+\beta) \\ &\quad + (\alpha_t^R L^R + \alpha_t^N L^N) (R_{t+1} - r) w_t \beta / (1+\beta) \end{aligned}$$

Therefore, the covariance of aggregate consumption growth and stock returns is

$$\text{Cov}_t[(C_{t+1}/C_t) - 1, R_{t+1}] = \beta[(\alpha_t^R + \mu \alpha_t^N)/(1+\mu)] \text{Var}_t(R_{t+1})$$

On the other hand, the equity premium is equal to the covariance of rational investors' consumption growth and stock returns. These show that as long as there are irrational investors whose investment decision is different from that of rational investors, the equity premium is not equal to the covariance of aggregate consumption growth and stock returns. Therefore, any rational investors drive a wedge between the equity premium and the covariance of aggregate consumption growth and stock returns.

I now analyze the case where the stock positions of rational investors and irrational investors sum to 0. This assumption suppresses the role of the variance of stock returns and emphasizes the role of the aggregate stock position in the covariance of aggregate consumption growth and stock returns. I first translate (19) into stock prices and stock quantities. To do this, I denote the dividend, the ex dividend stock price and the stock demand of a rational investor as d_{t+1} , p_t and X_t^R , respectively. Then I can rewrite α_t^R and R_{t+1} in (19) as

$$\alpha_t^R = X_t^R p_t / (w_t - C_t),$$

$$R_{t+1} = (p_{t+1} + d_{t+1} - p_t) / p_t$$

A rational investor's demand for stock is

$$X_t^R = [E_t(p_{t+1} + d_{t+1}) - (1+r)p_t] / \beta \text{Var}_t(p_{t+1} + d_{t+1})$$

I denote the stock demand of an irrational investor as X_t^N and assume the aggregate stock supply is fixed at 0. Then the market clearing condition is $X_t^R L^R + X_t^N L^N = 0$. Multiplying this by $p_t / (w_t - C_t)$ yields

$$A_t^R L^R + \alpha_t^N L^N = 0 \quad (20)$$

Because of (20), second period aggregate consumption reduces to

$$C_{t+1} = (1+r)\beta(L^R + L^N)w_t / (1+\beta)$$

Therefore, aggregate consumption growth is simply $(1+r)\beta$ and the covariance of aggregate consumption growth and stock returns is 0 no matter how many irrational investors are in international securities markets.

$$C_{t+1}/C_t = (1+r)\beta$$

$$\text{Cov}[(C_{t+1}/C_t) - 1, R_{t+1}] = \text{Cov}_t[(C_{t+1}/C_t) - 1, R_{t+1}] = 0$$

This means the presence of irrational investors does not change the covariance of aggregate consumption growth and stock returns from the efficient market level as long as the aggregate stock supply remains fixed at 0. This is because the stock positions of different agents always sum to the fixed supply of stocks and the covariance of aggregate consumption growth and stock returns is determined only by the aggregate stock position. As I did not specify the type of irrational investors in the model, this result applies to any type of irrational investors.

I now extend the two-period model to an overlapping generations model with two-period-lived agents. I assume that the wage W_t grows at an exogenous rate g and the number of young rational investors L^R and young irrational investors L^N are constant over time. Under the assumption that the aggregate stock supply is fixed at 0, aggregate consumption growth in this overlapping generations model is simply $(1+g)$ and therefore the covariance of aggregate consumption growth and stock returns is still 0 no matter how many irrational investors are in international securities markets.

$$C_{t+1}/C_t = 1 + g$$

$$\text{Cov}[(C_{t+1}/C_t) - 1, R_{t+1}] = 0$$

On the other hand, the presence of irrational investors alters the equity premium from the efficient market level, as is shown in the simulation results. The simulation shows that positive feedback traders raise the equity premium above the efficient market level while negative feedback traders lower the equity premium below the efficient market level. To illustrate where the extra equity' premium in the simulation comes from, I specify irrational investors as feedback traders with a memory horizon of 1 period. The stock demand of a feedback trader in period t is

$$X_t^N = [p_t - 1 + d_t - 1 - (1+r)p_{t-2}](w_t - C_t) / \beta \text{Var}_t(p_{t+1} + d_{t+1})$$

For analytical tractability, I assume that dividends follow an AR (1) process

$$D_t = d + \varepsilon_t, \varepsilon_t = \rho\varepsilon_{t-1} + \eta_t, 0 < \rho < 1, \eta_t \sim \text{i.i.d.}(0, \sigma\eta^2)$$

The market clearing condition $X_t^R L^R + X_t^N L^N = 0$ yields the price equation

$$E_t[p_{t+1} + d_{t+1} - (1+r)p_t] = -\mu[p_t - 1 + d_t - 1 - (1+r)p_{t-2}]$$

Notice that the perceived risk term $\text{Var}_t(p_{t+1} + d_{t+1})$ disappears from the price equation

by the assumption that the aggregate stock supply is 0. The price without rational bubbles is

$$P_t = d/r + \rho \varepsilon_t / (1+r-\rho) + \mu [\varepsilon_t / (1+r)(1+r-\rho) + \varepsilon_{t-1} / (1+r) - p_{t-2} + d/r]$$

The price series follows an ARMA (3,1) process and I impose the stationarity condition $|\mu| < 1$.

$$P_t - \rho p_{t-1} + p_{t-2} + \rho \mu p_{t-3} = (1-\rho)(1-\mu)d/r + [\rho(1+r) + \mu] \eta_t / (1+r)(1+r-\rho) + \mu \eta_{t-1} / (1+r)$$

Denoting the excess return $[p_{t+1} + d_{t+1} - (1+r)p_t]$, the excess return series follows an AR(2) process.

$$\pi_{t+1} = -\mu \pi_t - 1 + [(1+r)2 + \mu] \eta_{t+1} / (1+r)(1+r-\rho)$$

The equity premium is

$$E(R_{t+1} - r) = E(\pi_{t+1} / p_t)$$

Approximating π_{t+1} / p_t with a second order Taylor series around $E(\pi_{t+1})$ and $E(p_t)$, and taking expectations yields

$$E(R_{t+1} - r) = E(\pi_{t+1}) [1/E(p_t) + \text{Var}(p_t)/E(p_t)^3] - \text{Cov}(\pi_{t+1}, p_t)/E(p_t)^2$$

In this expression, $E(\pi_{t+1})$ is the term that would include the perceived risk $\text{Vart}(p_{t+1} + d_{t+1})$ if I did not assume $S = 0$. The assumption $S = 0$ removes any effect of perceived risk of the equity premium and as a result, $E(\pi_{t+1}) = 0$. However, the equity premium is still generally different from 0 when the market is not efficient because of the term $\text{Cov}(\pi_{t+1}, p_t)$, which would be 0 if the market were efficient ($\mu = 0$).

$$E(R_{t+1} - r) = \mu(\mu + \rho^2) [(1+r)^2 + \mu] \sigma \eta^2 / [(1-\mu^2)(1+\mu\rho^2)(1+r)(1+r-\rho)^2 E(p_t)^2]$$

When the market is not efficient, there are predictable disturbance components in the next period's return as can be seen in the price equation and this makes $\text{Cov}(\pi_{t+1}, p_t)$ generally different from 0. The above expression shows that in the presence of positive feedback traders ($\mu > 0$), the equity premium is positive and increasing in μ . while in the presence of negative feedback traders ($\mu < 0$), the equity premium is negative as long as μ is greater than $-\rho^2$. In the limiting case where ρ approaches 1, negative feedback traders always make the equity premium negative.

The other side of market inefficiency is that prices deviate transitorily from fundamental values as can be seen in the expression of the price without rational bubbles. To see how the pattern of deviation is related to the equity premium, I denote the price p_t as the sum of the fundamental value p_t^* and the weakly stationary deviation component u_t .¹³

$$P_t = p_t^* + u_t$$

Noting that $E_t[p_{t+1}^* + d_{t+1} - (1+r)p_t^*] = 0$, the term $-\text{Cov}(\pi_{t+1}, p_t)$ can be rewritten as

¹³ In this model, u_t follows an ARMA (2,1) process.
 $U_t = -\mu u_{t-2} + \mu \eta_t / (1+r)(1+r-\rho) + \mu \eta_{t-1} / (1+r-\rho)$

$$\begin{aligned}
& -\text{Cov}(\pi_{t+1}, p_t) = -\text{Cov}[\text{Cov}[u_{t+1}-(1+r)u_t, p_t]] \\
& = -\text{Cov}[u_{t+1}-(1+r)u_t, p_t^*] - \text{Cov}[u_{t+1}-(1+r)u_t, u_t]
\end{aligned}$$

The second term can be rewritten as

$$-\text{Cov}[u_{t+1}-(1+r)u_t, u_t] = [1+r-\text{Cov}(u_{t+1}, u_t)/\text{Var}(u_t)] \text{Var}(u_t)$$

Since $\{u_t\}$ is weakly stationary, this term is always positive. Intuitively speaking, the deviation component always tends to revert to its mean, which is 0. The fast term is related to the correlation between the fundamental value and the deviation component. I define overreaction, underreaction, counterreaction of the price to news as follows.

(i) overreaction:

$$\text{Cov}[p_t^*-E_{t-1}(p_t^*), u_t-E_{t-1}(u_t)] > 0$$

(ii) underreaction:

$$\begin{aligned}
& \text{Cov}[p_t^*-E_{t-1}(p_t^*), u_t-E_{t-1}(u_t)] < 0 \text{ and } \text{Cov}[p_t^*-E_{t-1}(p_t^*), p_t-E_{t-1}(p_t)] > 0 \\
& (-\text{Var}[p_t^*-E_{t-1}(p_t^*)] < \text{Cov}[p_t^*-E_{t-1}(p_t^*), u_t-E_{t-1}(u_t)] < 0)
\end{aligned}$$

(iii) counterreaction:

$$\begin{aligned}
& \text{Cov}[p_t^*-E_{t-1}(p_t^*), p_t-E_{t-1}(p_t)] < 0 \\
& (\text{Cov}[p_t^*-E_{t-1}(p_t^*), u_t-E_{t-1}(u_t)] < -\text{Var}[p_t^*-E_{t-1}(p_t^*)])
\end{aligned}$$

In this model, the fast term turns out to be

$$-\text{Cov}[u_{t+1}-(1+r)u_t, p_t^*] = \text{Cov}[p_t^*-E_{t-1}(p_t^*), u_t-E_{t-1}(u_t)] [(1+r)^2 + \mu] \rho / (1+\mu \rho^2)$$

The sign of the term $\text{Cov}[p_t^*-E_{t-1}(p_t^*), u_t-E_{t-1}(u_t)]$ corresponds to whether the price overreacts to news or under/counterreacts to news. Positive feedback traders ($\mu > 0$) make the price overreact to news and as a result, they make this term positive. Therefore, they make the equity premium unambiguously positive. Negative feedback traders ($\mu < 0$) make the price under/counterreact to news and as a result, they make this term negative. The condition that they make the equity premium negative is $-\rho^2 < \mu < 0$. This can be rewritten as

$$-\rho/(1+r)]\text{Var}[p_t^*-E_{t-1}(p_t^*)] < \text{Cov}[p_t^*-E_{t-1}(p_t^*), u_t-E_{t-1}(u_t)] < 0$$

When the price underreacts to news, $-\text{Var}[p_t^*-E_{t-1}(p_t^*)] < \text{Cov}[p_t^*-E_{t-1}(p_t^*), u_t-E_{t-1}(u_t)] < 0$ holds. Since $[\rho/(1+r)]$ is smaller than 1, the condition for negative equity premium implies the condition for underreaction to news. In the limiting case where ρ approaches 1, negative feedback traders always make the equity premium negative and this implies that they always make the price underreact to news. Noise traders of the type analyzed in De Long et al. (1990a) introduce no correlation between the deviation component and the fundamental value. Therefore they make the equity premium unambiguously positive.¹⁴

¹⁴ The equity premium in the U.S. has declined since the 1950s as reported in Blanchard (1993). He suggests two explanations for this. One is that the decline in the volatility of stock returns reduced people's

4. The Effect of Transitory Components in Prices on Volatility Patterns

The simulation results in the presence of positive feedback traders with extended knowledge also exhibit the volatility patterns that are observed in historical data. Volatility increases as prices fall and decreases as prices rise. When the price enters a deep trough, volatility increases dramatically.¹⁵ Volatility is so persistent, as is shown in the autocorrelation coefficients of the monthly standard deviation series, that some authors even suggest nonstationarity.¹⁶ The source of these volatility patterns in the simulations is the presence of transitory components in prices when the conditional variance of rate of returns is constant along the fundamentals path. Under my geometric random walk specification of the dividend process, there is no conditional heteroskedasticity in rate of returns along the fundamentals path, as is shown in (7).

$$\text{Var}_t(r_{t+1}) = (1 + r - \delta - \sigma_e^2/2)^2 \sigma_e^2$$

In the presence of feedback traders, the conditional variance of gross returns $\text{Var}_t(p_{t+1}+d_{t+1})$ is proportional to d_t^2 as is shown in (3).

$$\begin{aligned} \text{Var}_t(p_{t+1}+d_{t+1}) = \\ [1 + r + (1+r)-1\mu\beta 1 + (1+r)-2\mu\beta^2 + \dots + (1+r)-k\mu\beta k]^2 \sigma_e^2 d_t^2 / (r-\delta- \sigma_e^2/2)^2 \end{aligned}$$

The fundamental value is proportional to d_t as is shown in (4).

$$p_t^* = (1+\delta+ \sigma_e^2/2)d_t / (r-\delta- \sigma_e^2/2)$$

I denote the price in the presence of feedback traders p_t , as the quotient of the fundamental value p_t^* and a transitory component u_t , where u_t is strictly positive and follows a weakly stationary process.

$$p_t = p_t^* / u_t, u_t > 0$$

perceived risk of holding stocks. The other is that institutional investors have become increasingly important and they took advantage of the high equity premium of the past with a longer investment horizon. My interpretation of his first explanation is that a decrease in the variability of dividends reduced the fundamental risk and this reduced people's perceived risk of holding stocks. Barsky and De Long (1990) report that the year-to-year growth rate of dividends has become less volatile since the 1950s. This would probably be a result of decreasing output variability since World War in which De Long and Summers (1986) show. My interpretation of his second explanation is that if we can regard institutional investors as rational investors and individual investors as positive feedback traders or noise traders, then the growing importance of institutional investors will reduce the equity premium both through the reduced market inefficiency effect and the reduced perceived risk.

¹⁵ There are several explanations on this phenomenon so far. One is the leverage explanation of Black (1976) and Christie (1982) that argue as stock prices fall, leverage of firms increase and this increases stock price volatility. However, as Schwert (1989) shows, while leverage is significantly correlated with volatility, it explains only a small part of the movements in stock volatility. Another is the market psychology explanation of Duffie (1990) which argues that financial panics make investor confidence lower thereby making the price elasticity' of stock demand lower resulting in a self-fulfilling volatility increase. My explanation is complementary to these explanations on this matter but it explains the persistence of volatility at the same time that others do not address.

¹⁶ French, Schwert & Stambaugh (1987), Schwert (1989). However, Poterba & Summers (1986) do not support the nonstationarity hypothesis. They instead show that AR (1) fits well to the volatility series.

Then the conditional variance of rate of returns is proportional to u_t^2 .

$$\text{Var}_t(\mathbf{R}_{t+1}) = \text{Var}_t(p_{t+1} + d_{t+1})/p_t^2 = Au_t^2$$

where A is a constant. So the conditional standard deviation of rate of returns $\text{St}(\mathbf{R}_{t+1})$ is proportional to u_t .

$$S_t(\mathbf{R}_{t+1}) = A^{1/2}u_t$$

This implies that volatility increases when prices fall below fundamentals and that volatility decreases when prices rise above fundamentals. As positive feedback traders with extended knowledge generate a widely varying transitory component, the increase in volatility during a recession can be dramatic. Moreover, the autocorrelation coefficients of $\{S_t(\mathbf{R}_{t+1})\}$ series are identical to the autocorrelation coefficients of $\{u_t\}$ series at all lags as multiplying by constants does not affect correlation coefficients. If the transitory component is extremely persistent as in the presence of positive feedback traders with extended knowledge, volatility is also extremely persistent. This shows that the extreme persistence of volatility is not necessarily from any nonstationarity but can be just the result of an extreme persistence of the transitory component in the presence of positive feedback traders with extended knowledge. Negative feedback traders also generate a transitory component, and $\text{Var}_t(p_{t+1} + d_{t+1})$ and p_t^* have the same form as in the case of positive feedback traders. The transitory component gets more persistent as the memory horizon gets longer. Therefore the argument so far applies both to positive feedback traders and negative feedback traders. The only difference which is actually crucial is that as negative feedback traders smooth the price series, prices get lower than fundamentals when fundamentals get higher, contrary to the case of positive feedback traders, and therefore times of high volatility do not coincide with recessions.

5. A Second Model with Simplified Modifications

The model is an infinite period model with one period investment horizon of agents. This means agents care only about the next period's wealth in each period. This short investment horizon combined with risk aversion makes a deviation of prices from fundamentals possible. The numeraire is wealth. There is neither exogenous labor income nor consumption and all the proceeds from investment are reinvested the next period. There are two assets in the economy: bonds and stocks. Bonds pay a fixed interest r and are in infinitely elastic supply with the price fixed at 1. Stocks pay a risky dividend d , in period t and are in fixed supply S . I assume that dividends follow a random walk.

$$d_t = d_{t-1} + \varepsilon_t, \varepsilon_t \sim \text{i.i.d.}(0, \sigma_\varepsilon^2)$$

The ex dividend price of stocks in period t is denoted p_t . There are two types of agents: rational investors who have rational expectations and feedback traders who extrapolate past trends to form expectations. The wealth of young rational investors and feedback traders in period t are W_t^R and W_t^F , respectively. I assume that the wealth of feedback traders is adjusted by entry or exit each period such that the wealth ratio of feedback traders to rational investors is kept constant over time. Each agent maximizes a mean-variance approximation to a constant relative risk aversion expected utility function

$$E_t(U) = E_t(W_{t+1}) - (\gamma/2W_t)\text{Var}_t(W_{t+1})$$

where γ is the index of local relative risk aversion. The budget constraint of each agent is

$$W_{t+1} = (W_t - X_t p_t)(1+r) + X_t(p_{t+1} + d_{t+1})$$

where X_t is the quantity of the stocks that the agent holds in period t . For rational investors,

$$\begin{aligned} E_t(W_{t+1}^R) &= W_t^R (1+r) + X_t^R [E_t(p_{t+1}) + E_t(d_{t+1}) - (1+r)p_t] \\ \text{Var}_t(W_{t+1}^R) &= (X_t^R)^2 \text{Var}_t(p_{t+1} + d_{t+1}) \end{aligned}$$

Maximizing expected utility with respect to X_t^R yields rational investors' demand for stocks in period t .

$$X_t^R = E_t[p_{t+1} + d_{t+1} - (1+r)p_t] W_t^R / \gamma \text{Var}_t(p_{t+1} + d_{t+1})$$

Feedback traders perceive the variance correctly but form the expectation of excess return by extrapolating past k periods' excess returns. Specifically, feedback traders perceive

$$\begin{aligned} &E_t(W_{t+1}^F), \text{Var}_t(W_{t+1}^F) \text{ as} \\ E_t(W_{t+1}^F) &= W_t^F (1+r) + X_t^F \{ \beta_1 [p_{t-1} + d_{t-1} - (1+r)p_{t-2}] + \dots + \beta_k [p_{t-k} + d_{t-k} - (1+r)p_{t-k-1}] \} \\ \text{Var}_t(W_{t+1}^F) &= (X_t^F)^2 \text{Var}_t(p_{t+1} + d_{t+1}) \end{aligned}$$

I assume that feedback traders give equal weight to each past period's excess return that they take account of when they form their expectation of the next period's excess return, for example $\beta_1 = \beta_2 = \dots = \beta_k = 1/k$. This is for analytical tractability and this does not affect the results of this paper in any crucial way. Maximizing expected utility with respect to X_t^F yields feedback traders' demand for stock in period t

$$\begin{aligned} X_t^F &= k-1 \{ [p_{t-1} + d_{t-1} - (1+r)p_{t-2}] + \dots + [p_{t-k} + d_{t-k} - (1+r)p_{t-k-1}] \} W_t^F \\ &\quad / \gamma \text{Var}_t(p_{t+1} + d_{t+1}) \end{aligned}$$

The market clearing condition is $X_t^R + X_t^F = S$. I interpret S as the supply of stocks net of the holdings of passive investors who implement a buy and hold strategy and assume $S = 0$. This assumption removes any equity premium in efficient markets in spite of risk aversion and fundamental risk. I divide the market clearing condition by W_t^R and define $\mu = W_t^F / W_t^R$. As is assumed, μ is held constant over time by the entry or exit of feedback traders' wealth in each period. With these assumptions, the stock price equation is

$$E_t[p_{t+1} + d_{t+1} - (1+r)p_t] = -\mu_{k-1} \{ [p_{t-1} + d_{t-1} - (1+r)p_{t-2}] + \dots + [p_{t-k} + d_{t-k} - (1+r)p_{t-k-1}] \}$$

The price without rational bubbles is

$$\begin{aligned} p_t = & -\mu k^{-1}(p_{t-2} + p_{t-3} + \dots + p_{t-k-1}) \\ & + [1 + (1+r)^{-1}\mu k^{-1} + (1+r)^{-2}\mu k^{-1} + \dots + (1+r)^{-k}\mu k^{-1}] d_t / r \\ & + [(1+r)^{-1}\mu k^{-1} + (1+r)^{-2}\mu k^{-1} + \dots + (1+r)^{-k}\mu k^{-1}] d_{t-1} \\ & + [(1+r)^{-1}\mu k^{-1} + \dots + (1+r)^{-(k-1)}\mu k^{-1}] d_{t-2} \\ & + \dots \\ & + [(1+r)^{-1}\mu k^{-1}] d_{t-k} \end{aligned}$$

The conditional variance $\text{Var}_t(p_{t+1} + d_{t+1})$ is constant over time.

$$\text{Var}_t(p_{t+1} + d_{t+1}) = \{ 1 + r + \mu k^{-1} r^{-1} [1 - (1+r)^{-k}] \}^2 \sigma_e^2 / r^2$$

Before proceeding further I will show the properties of this model under market efficiency.

Absent feedback traders $\mu = 0$, the price is equal to the fundamental value.

$$P_t^* = d_t / r$$

This fundamental value is the non-bubble solution to the price equation under market efficiency

$$E_t[p_{t+1} + d_{t+1} - (1+r)p_t] = 0$$

If we denote the excess return as π_{t+1} , the price equation can be rewritten as

$$E_t(\pi_{t+1}) = 0.$$

The excess return π_{t+1} reflects the change of expectation between period t and period $t+1$. This implies that conditioning on any variable x in the information set of period t , the expectation of the excess return π_{t+1} is 0.

$$E(\pi_{t+1} | x) = E[E_t(\pi_{t+1}) | x] = 0$$

The excess return π_{t+1} is thus orthogonal to any variable x in the information set of period t and since $E(\pi_{t+1}) = 0$, this implies the uncorrelatedness of π_{t+1} and x .

$$\text{Cov}(\pi_{t+1}, x) = E(\pi_{t+1} x) = E[E(\pi_{t+1} | x) x] = 0$$

As the information set of period t contains all the lagged values of π_{t+1} , the excess return series $\{\pi_{t+1}\}$ should be serially uncorrelated under market efficiency. Under my specification of the dividend process, the conditional variance of the excess return is constant over time and in the efficient market it reduces to

$$\text{Var}_t(\pi_{t+1}) = (1+r)^2 \sigma_e^2 / r^2$$

Therefore, the unconditional variance of the excess return is also constant over time under market efficiency.

$$\text{Var}(\pi_{t+1}) = E[\text{Var}_t(\pi_{t+1})] + \text{Var}[E_t(\pi_{t+1})] = (1+r)^2 \sigma_\varepsilon^2 / r^2$$

The excess returns are serially uncorrelated and variance stationary in the absence of feedback traders, therefore the variance of the sum of k periods' excess returns is k times the variance of 1 period excess return under market efficiency.

Now I introduce new notations to simplify the expressions. I denote feedback traders' forecast of the next period's excess return in period t as Π_{t-1} and the conditional variance as Σ .

$$\begin{aligned}\Pi_{t-1} &= k^{-1}(\pi_{t-1} + \pi_{t-2} + \dots + \pi_{t-k}) \\ \Sigma &= \text{Var}_t(p_{t+1} + d_{t+1})\end{aligned}$$

A little bit of algebra shows the excess return process is

$$\begin{aligned}\Pi_{t+1} &= -\mu k^{-1}(\pi_{t-1} + \pi_{t-2} + \dots + \pi_{t-k}) \\ &+ \{1 + r + \mu k^{-1} r^{-1} [1 - (1+r)^{-k}]\} \varepsilon_{t+1} / r\end{aligned}$$

So the excess return series is an AR(k+1) process. I impose the stationarity condition. Then it is immediate that

$$E(\pi_{t+1}) = 0$$

From the definition of the feedback traders' forecast, it is also immediate that

$$E(\Pi_{t-1}) = 0$$

This shows that feedback traders in this model are the same as rational investors in terms of average bullishness. So feedback traders in this model can never earn a higher expected return than rational investors regardless of the existence of equity premium, unlike the noise traders of De Long, Shieifer, Summers & Waldmann (1988,1990a). The AR(k+1) stochastic difference equation of excess returns can be rewritten as

$$\Pi_{t+1} = -\mu \Pi_{t-1} + \{1 + r + \mu k^{-1} r^{-1} [1 - (1+r)^{-k}]\} \varepsilon_{t+1} / r$$

The next period's excess return has a predictable disturbance component that makes the conditional expected excess return diverge from the unconditional expected excess return. The source of this predictable disturbance component is the forecast of feedback traders Π_{t-1} . The conditional expected excess return is

$$E_t(\pi_{t+1}) = -\mu \Pi_{t-1}$$

The fact that the source of the conditional expected excess return's divergence from the unconditional expected excess return is the forecast of feedback traders also implies that the source of the difference, in beliefs between rational investors and feedback traders is the belief of feedback traders itself. The difference in beliefs is

$$E_t(\pi_{t+1}) - \Pi_{t-1} = -(\mu+1) \Pi_{t-1}$$

6. Analyzing the Required Rate of Entrance Problem

Rational investors' wealth evolves according to

$$W_{t+1}^R = W_t^R [1 + r + (X_t^R / W_t^R)\pi_{t+1}]$$

Feedback traders' wealth before adjustment is

$$W_{t+1}^F = W_t^F [1 + r + (X_t^F / W_t^F)] \pi_{t+1}$$

Feedback traders' wealth after adjustment is simply μW_{t+1}^R . So the required entrance of feedback traders' wealth to keep the wealth ratio constant is

$$\Delta W_{t+1}^F = \mu W_{t+1}^R - W_{t+1}^F = W_t^F [(X_t^R / W_t^R) - (X_t^F / W_t^F)] \pi_{t+1}$$

Dividing this by W_t^F , I get the required entrance rate of feedback traders' wealth to keep the wealth ratio constant. I denote this rate as q_{t+1} .¹⁷

$$q_{t+1} = \Delta W_{t+1}^F / W_t^F = [(X_t^R / W_t^R) - (X_t^F / W_t^F)] \pi_{t+1}$$

Plugging the expressions for stock demands into the aforementioned equation yields

$$q_{t+1} = [E_t(\pi_{t+1}) - \Pi_{t-1}] \pi_{t+1} / \gamma \Sigma$$

The question of this section is: what is the expectation of this rate? It is determined by the covariance of the difference in beliefs and the next period's excess return. As the difference in beliefs comes from the forecast of feedback traders and as the next period's excess return has a predictable disturbance component that also comes from the forecast of feedback traders, the covariance depends on the variance of the forecast.

$$E(q_{t+1}) = \mu(1+\mu)\text{Var}(\Pi_{t-1}) / \gamma \Sigma$$

$E(q_{t+1})$ is strictly positive because rational investors always outsmart feedback traders. There is no way that feedback traders can earn a higher expected return than rational investors in this model. Now our problem reduces to finding out $\text{Var}(\Pi_{t-1})$.

Denoting the autocovariance function of $\{\pi_{t+1}\}$ as $\Gamma(h)$, where h is the lag,

$$\text{Var}(\Pi_{t-1}) = k-2 \{ k\text{Var}(\pi_{t+1}) + 2[(k-1)\Gamma(1) + (k-2)\Gamma(2) + \dots \Gamma(k-1)] \}$$

Factoring out $k\text{Var}(\pi_{t+1})$ term from the curly bracket of the right hand side yields

$$\text{Var}(\Pi_{t-1}) = k^{-1}\text{Var}(\pi_{t+1}) \{ 1 + 2k^{-1}[(k-1)\varrho(1) + (k-2)\varrho(2) + \dots \varrho(k-1)] \}$$

where $\varrho(h)$ is the autocorrelation function of $\{\pi_{t+1}\}$. The expression inside the curly bracket of the right hand side is the ratio of the variance of the sum of k periods' excess returns to k times the variance of 1 period excess return. I denote it as $\text{VR}(k)$.

$$\text{VR}(k) = \text{Var}[\pi_{t-1} + \pi_{t-2} + \dots + \pi_{t-k}] / [k\text{Var}(\pi_{t+1})]$$

Using this notation, the variance of the forecast $\text{Var}(\Pi_{t-1})$ can be expressed as

$$\text{Var}(\Pi_{t-1}) = \text{Var}(\pi_{t+1})\text{VR}(k)/k$$

By the way, the variance of the excess return is

¹⁷ This rate is equivalent to the difference of rate of returns for rational investors' wealth and feedback traders' wealth.

$$\text{Var}(\pi_{t+1}) = \mu^2 \text{Var}(\Pi_{t-1}) + \Sigma$$

So the variance of the excess return $\text{Var}(\pi_{t+1})$ can be expressed as

$$\text{Var}(\pi_{t+1}) = \Sigma / [1 - \mu^2 \text{VR}(k)/k]$$

This shows that if $\text{VR}(k)/k$ converges to 0 as k gets large, then $\text{Var}(\pi_{t+1})$ converges to Σ as k gets large.¹⁸ Therefore as long as $\text{VR}(k)/k$ converges to 0 as k gets large, $\text{Var}(\Pi_{t-1})$ converges to 0 as k gets large. This is a generalization of the law of large numbers. The variance ratio $\text{VR}(k)$ is bounded above by k , which occurs when all the autocorrelation coefficients of $\{\pi_{t+1}\}$ up to lag $k-1$ is 1. If the excess return is a white noise like in the efficient market, the variance ratio is 1 and the usual law of large numbers applies. Positive feedback traders make long horizon variance ratios of the "rate of return" less than 1. So a natural conjecture is that they will make long horizon variance ratios of the "excess return" also less than 1. Simulations show that this is indeed the case. I assume 1 year as 1 period in the model, set $r = 0.05$ and generate 24000 observations of excess returns. This size of sample is enough to keep the theoretical standard error of variance ratios under market efficiency less than 0.2 up to 500-year horizon even with non-overlapping sum of excess returns.¹⁹ The result is displayed in the below table.

VR(k)	k							
	10	20	30	40	50	100	200	500
$\mu = 1$	0.62	0.57	0.53	0.52	0.60	0.47	0.46	0.67
$\mu = 2$	0.54	0.44	0.40	0.33	0.36	0.35	0.33	0.25
$\mu = 3$	0.55	0.35	0.32	0.27	0.32	0.28	0.29	0.27

This shows that $\text{VR}(k)$ is less than 1 at least for all the values of k of our interest in the presence of positive feedback traders. It also shows that $\text{VR}(k)$ is almost constant when the memory horizon k increases and the level is decreasing in the wealth ratio of feedback traders to rational investors μ . Specifically, $\text{VR}(k)$ is almost constant at around 0.5 when μ is 1, around 0.4 when μ is 2 and around 0.3 when μ is 3. Therefore, the law of large numbers works better than

¹⁸ 18 As k gets very large, Σ in turn converges to the efficient market level $(1+r)^2 \sigma_e^2 / r^2$. However, there is a difference in the speed of convergence that will become important in the next section. The definition of Σ is $\{1+r + \mu(rk)^{-1}[1-(1+r)^{-k}]\} 2 \sigma_e^2 / r^2$ and this shows that the decaying speed of Σ is deterred by the small quantity r in the expression $(rk)^{-1}$ at the initial stage of k increase. The speed of Σ 's convergence to $(1+r)^2 \sigma_e^2 / r^2$ is much slower than the speed of $\text{Var}(\pi_{t+1})$'s convergence to Σ or the speed of $\text{Var}(\Pi_{t-1})$'s convergence to 0.

¹⁹ 19 The asymptotic standard deviation of variance ratio statistics $\text{VR}(k)$ under market efficiency with non-overlapping sum of k period excess returns is $[2(k-1)/N]^{1/2}$, where N is the sample size. This standard error is increasing with k and decreasing with N . With 24000 yearly excess returns, the standard error of 500 year variance ratio is $[2(500-1)/24000]^{1/2} = 0.20$.

under market efficiency and $\text{Var}(\Pi_{t-1})$ converges to 0 as k gets larger.²⁰ Using the expression of $\text{Var}(\pi_{t+1})$, the variance of the forecast $\text{Var}(\Pi_{t-1})$ can be rewritten as

$$\text{Var}(\Pi_{t-1}) = E/[k/\text{VR}(k)-\mu^2]$$

Therefore the expected required rate of entrance is

$$E(q_{t+1}) = \mu(1+\mu) / \gamma[k/\text{VR}(k)-\mu^2]$$

This shows that $E(q_{t+1})$ also converges to 0 as the memory horizon of positive feedback traders gets longer. More practically, $E(q_{t+1})$ can be quite low under plausible parameter values. In Table 2-1, I calculate $E(q_{t+1})$ for different values of γ , μ and k . I set $\text{VR}(k) = 0.5$ for $\mu=1$, $\text{VR}(k) = 0.4$ for $\mu=2$ and $\text{VR}(k) = 0.3$ for $\mu=3$. It shows that $E(q_{t+1})$ is decreasing in k and γ while it is increasing in μ . For $\mu = 1$ and $k = 40$ which are within the range calibrated in earlier, $E(q_{t+1}) = 0.025$ if $\gamma = 1$, $E(q_{t+1}) = 0.013$ if $\gamma = 2$, $E(q_{t+1}) = 0.008$ if $\gamma = 3$ and $E(q_{t+1}) = 0.006$ if $\gamma = 4$. In annual terms, this means that when positive feedback traders take account of the past 40 years of return history, new entry of only around 1% ~ 2% of positive feedback traders' current wealth each year is enough to keep the wealth ratio of positive feedback traders to rational investors constant over time under plausible values for the index of relative risk aversion.²¹

²⁰ One might be interested in the formal proof of the proposition that $\text{Var}(\Pi_{t-1})$ converges to 0 as k goes to infinity. One way to proceed might be as follows. The proposition that $\text{Var}(\Pi_{t-1})$ converges to 0 as k goes to infinity is equivalent to the proposition that $\text{Var}(\pi_{t+1})$ converges to Σ as k goes to infinity. To prove the latter proposition, one can derive the first k Yule-Walker equations for the autocovariance function. Combined with the variance equation, this forms a system of $k+1$ linear equations in $k+1$ variable.

$$\begin{aligned} \Gamma(0) &= \mu 2k-2[k\Gamma(0) + 2(k-1)\Gamma(1) + 2(k-2)\Gamma(2) + \dots + 2\Gamma(k-1)] + \Sigma \\ \Gamma(1) &= -\mu k-1[\Gamma(1) + \Gamma(2) + \Gamma(3) + \Gamma(4) + \dots + \Gamma(k)] \\ \Gamma(2) &= -\mu k-1[\Gamma(0) + \Gamma(1) + \Gamma(2) + \Gamma(3) + \dots + \Gamma(k-1)] \\ \Gamma(3) &= -\mu k-1[\Gamma(1) + \Gamma(0) + \Gamma(1) + \Gamma(2) + \dots + \Gamma(k-2)] \\ \Gamma(4) &= -\mu k-1[\Gamma(2) + \Gamma(1) + \Gamma(0) + \Gamma(1) + \dots + \Gamma(k-3)] \\ &\vdots \\ &\vdots \\ \Gamma(k-2) &= -\mu k-1[\Gamma(k-4) + \dots + \Gamma(0) + \Gamma(1) + \Gamma(2) + \Gamma(3)] \\ \Gamma(k-1) &= -\mu k-1[\Gamma(k-3) + \dots + \Gamma(1) + \Gamma(0) + \Gamma(1) + \Gamma(2)] \\ \Gamma(k) &= -\mu k-1[\Gamma(k-2) + \dots + \Gamma(2) + \Gamma(1) + \Gamma(0) + \Gamma(1)] \end{aligned}$$

In matrix form, this looks like the following.

$$\Gamma_k = A_k \Gamma_k + B_k$$

where Γ_k is the autocovariance vector and A_k is a $k+1$ by $k+1$ matrix and B_k is a constant vector that has Σ as its first element and 0 as its all the other elements. The solution is $\Gamma_k = (I_k - A_k)^{-1} B_k$. If one can show that the sequence $\|I_k - (I_k - A_k)^{-1}\|$ converges to 0 as k goes to infinity, then it follows that one can always find out k such that any integer bigger than k makes $|\Gamma_k(0) - \Sigma|$ less than an arbitrary positive real number. One may want to use the fact that inversion is a continuous function in the space of invertible matrices of a fixed size. However, there are two problems in proceeding this way. The first problem is that the size of the matrix $I_k - A_k$ grows as k grows. What the continuity of inversion in the space of invertible matrices of a fixed size says is that one can always make the norm of the difference of any two inverse matrices of a fixed size smaller than an arbitrarily positive real number by making the norm of the difference of the two matrices smaller than a positive real number. Therefore, even if one can show the sequence $\|A_k\|$ converges to 0 as k goes to infinity, the continuity of inversion does not necessarily imply that the sequence $\|I_k - (I_k - A_k)^{-1}\|$ converges to 0 as k goes to infinity. The second problem is that the sequence $\|A_k\|$ does not converge to 0 as k goes to infinity. I do not pursue this issue further in this paper.

The result of this section crucially depends on the fact that the variance of irrational traders' forecast on the next period's return gets close to 0 when they use a consistent estimator for the true expected return with a large sample size. Negative feedback traders use the average of past periods' return multiplied by -1 as their forecast of the next period's return. They make the variance ratio of returns bigger than 1 because they introduce positive serial correlation of returns at all lags. On the other hand, a long memory horizon makes them forecast with a large sample size. So there are two competing forces at work. Simulations show that the variance ratios are bigger than 1 for all values of k , but they do not increase with k .

k		10	20	30	40	50	100	200	500
VR(k)	$\mu = -0.5$	1.75	1.78	1.70	2.02	1.97	2.06	2.23	1.50

As the effect of negative feedback traders with extended knowledge on the variance ratio is negligible compared to the effect on the length of memory horizon, the result of this section also applies to the negative feedback traders.²² Noise traders of the type analyzed in De Long et al. (1990a) do not generate this result because the variance of their forecast on the next period's return is assumed to be constant in the model.

7. The Deviation of Prices from Fundamentals

When the variance of the forecast $\text{Var}(\Pi_{t+1})$ is close to 0, the variance of the conditional expected excess return $\text{Var}[E_t(\pi_{t+1})]$ is also close to 0 because the variance of the conditional expected excess return comes from the variance of the predictable disturbance component in the next period's return, which is the forecast of feedback traders. This means the conditional expected excess return hardly varies around its expectation, which is the unconditional expected excess return 0. This in turn means the price equation is close to the one under market efficiency. Therefore, the serial correlation in excess returns will be close to 0 at every lag. More specifically,

²¹ 21 The choice of the length of 1 period affects the expected required rate of return per fixed real time mainly because it affects the sample size k of feedback traders with memory horizon fixed m real time. To see the problem, I assume $\gamma = 2$, $\mu = 1$ and $\text{VR}(k) = 1$. Then $E(q_{t+1}) = 1/(k-1)$. I denote the length of 1 period as L . I assume the memory horizon is fixed in real time at M . Then the memory horizon in terms of the period in the model is M/L , which is k . The expected required rate of entrance is per 1 period in the model. So the expected required rate of entrance per 1 unit in real time is $E(q_{t+1})/L$. Rewriting these together yields $E(q_{t+1})/L = 1/(M-L)$, where $M > L > 0$. As long as M is fixed, this expected rate per unit real time increases with L because the sample size of feedback traders decreases. However, this also shows that simply decreasing L towards 0 does not make the expected rate per unit real time converge to 0. Although the increase of sample size by shortening the length of 1 period makes the expected rate per 1 period in the model converge to 0, the expected rate per unit real time converges to $1/M$. As long as the length of 1 period is held considerably shorter than the memory horizon in real time, what really matters for the expected rate per real time is only the memory horizon in real time. In all examples in this paper I use 1 year both as the unit of real time and as the length of 1 period for simplicity.

²² 22 Simply multiplying μ by -1 does not yield the correct expression for $E(q_{t+1})$ in the presence of negative feedback traders. We need to modify $E(q_{t+1})$ as $E(q_{t+1}) = \mu(\mu-1) / \gamma[k/\text{VR}(k)-\mu^2]$ before multiplying μ by -1. Setting $\mu = -0.5$ and $\text{VR}(k) = 1.8$, I report $E(q_{t+1})$ for different values of k and γ in Table 2-1.

the autocorrelation coefficients of excess return are the ratio of the autocovariances of excess return to the variance of excess return. The only source of the autocovariances of excess return is the forecast component in excess returns, so the absolute value of the autocovariances are bounded above by $\mu^2 \text{Var}(\Pi_{t-1})$ while the variance of excess return is bounded below by Σ . The absolute value of the autocorrelation coefficients are bounded above by the quotient of these two. I denote this upper bound as z .

$$z = \mu^2 \text{Var}(\Pi_{t-1}) / \Sigma = \mu^2 / [k / \text{VR}(k) - \mu^2]$$

This shows that as the memory horizon of positive feedback traders gets longer, all the autocorrelation coefficients of excess return converge to 0. If $\mu = 1$ and $\text{VR}(k) = 0.5$, then $z = 0.053$ when $k = 10$, $z = 0.026$ when $k = 20$, $z = 0.017$ when $k = 30$, $z = 0.013$ when $k = 40$ and $z = 0.01$ when $k = 50$. By the same argument as in the last section, this result applies also to the negative feedback traders with extended knowledge. By the way, the absence of serial correlation is not a sufficient condition for the market efficiency. So there can be cases where there is no serial con-elation in excess returns while irrational investors lose a lot to rational investors.²³

In spite of the near efficient price equation, simulations show that prices, which is the solution to this equation, deviate widely from fundamentals in the presence of positive feedback traders with very long memories.²⁴ To explain this puzzling fact, I rewrite the price equation.

$$E_t[p_{t+1} + d_{t+1} - (1+r)p_t] = -\mu \Pi_{t-1}$$

The solution of this equation without rational bubbles is

$$P_t = d_t / r + \mu \Pi_{t-1} (1+r)^{-1} + \mu E_t [\Pi_t (1+r)^{-2} + \Pi_{t+1} (1+r)^{-3} + \Pi_{t+2} (1+r)^{-4} + \dots]$$

This shows that the price is determined by the expectation of the infinitely extended future dividend stream, the current forecast of positive feedback traders and the expectation of

²³ 23 One can construct an example of no serial correlation in excess returns in the presence of predictable disturbance component in the next period's excess return either by adjusting the covariance between innovations of irrational forecast and dividend or by adjusting the persistence of irrational forecast. In this case, rational investors still can take advantage of the irrational forecast and earn significantly higher expected return than irrational investors although there is no serial correlation in returns. By the way, simply increasing the persistence of irrational forecast does not necessarily contribute to making the serial correlation in excess returns 0 even in the absence of any covariance between innovations of irrational forecast and dividend. The effect of the persistence of irrational forecast on the serial correlation in excess returns depends on other properties of the process that the irrational forecast follows. However, increasing the persistence of irrational forecast unambiguously contributes to increasing the persistence of price deviation from fundamentals that the irrational forecast generates.

²⁴ 24 This is different from the coexistence of price deviation from fundamentals and no serial correlation in excess returns that does not necessarily imply the absence of any predictable disturbance component in the next period's excess return. As the source of price deviations from fundamentals is the presence of predictable disturbance component in the next period's excess return, any example of no serial correlation in excess returns in the presence of predictable disturbance component in the next period's excess return can be an example of the coexistence of price deviation from fundamentals and no serial correlation in excess returns.

the infinitely extended future forecast stream of positive feedback traders. So if the forecast follows an extremely persistent process, an infinitesimal fluctuation of the forecast can result in a significant fluctuation of prices around fundamental values. As the autocorrelation coefficients of excess return get close to 0, the excess return series can be approximated by a white noise and because positive feedback traders' forecast is a moving average of this white noise excess returns, it follows approximately an MA process of the order equal to the memory horizon subtracted by 1. The autocorrelation function of this MA(k-1) process is

$$\begin{aligned} \rho(h) &= (k-h)/k \quad , \quad h = 0, 1, 2, \dots, k-1 \\ &= 0 \quad , \quad h > k-1 \end{aligned}$$

When the memory horizon is very long, the autocorrelation function decays extremely slowly and the forecast is extremely persistent, although its variance is very small. Therefore, prices can deviate widely from fundamentals with an infinitesimal variance of the forecast. As the deviation of prices from fundamentals is determined by the forecast, the deviation of prices from fundamentals is also extremely persistent. And by the argument present earlier, the extreme persistence of volatility follows. More specifically, I derive the deviation component in the price when the memory horizon is long enough such that we can approximate the forecast series with the MA(k-1) process. I denote the deviation component in the price as U. A little bit of algebra shows that the deviation component is

$$\begin{aligned} D_t &= \mu k^{-1} r^{-1} \{ (1+r)^{-1} [1-(1+r)^{-k}] \pi_t + [1-(1+r)^{-k}] \pi_{t-1} + [1-(1+r)^{-k}] \pi_{t-2} + \\ &\dots + [1-(1+r)^{-2}] \pi_{t-k+1} + [1-(1+r)^{-1}] \pi_{t-k} \} \end{aligned}$$

This shows that the deviation series follows approximately an MA(k) process when the memory horizon is long enough. Therefore the deviation of prices from fundamentals is also extremely persistent when the memory horizon is very long.²⁵ I now calculate the variance of this deviation component to see the magnitude of the price deviation from fundamentals.

$$\begin{aligned} \text{Var}(D_t) &= \mu^2 k^{-2} r^{-2} \{ (1+r)^{-2} [1-(1+r)^{-k}]^2 + k-2r^{-1} [1-(1+r)^{-k}] + r^{-1} (2+r)^{-1} [1-(1+r)^{-2k}] \} \\ &\quad * [1-\mu^2 \text{VR}(k)/k]^{-1} \{1+r+\mu k^{-1} r^{-1} [1-(1+r)^{-k}]\}^2 \sigma_\varepsilon^2 / r^2 \end{aligned}$$

This can be rewritten as

$$\begin{aligned} \text{Var}(D_t) &= \mu^2 (rk)^{-1} \{ 1 - (rk)^{-1} [2(1-e^{-rk}) - (2+r)^{-1} (1-e^{-2rk})] + k^{-1} (1+r)^{-2} (1-e^{-rk})^2 \} \\ &\quad * [1-\mu^2 \text{VR}(k)/k]^{-1} [1+r+\mu (rk)^{-1} (1-e^{-rk})]^2 \sigma_\varepsilon^2 / r^3 \end{aligned}$$

The conditional variance of the 1 period ahead fundamental value is $\sigma_\varepsilon^2 / r^2$. When $r = 0.05$ annually, the term $\sigma_\varepsilon^2 / r^3$ corresponds to the conditional variance of the 20 year ahead

²⁵ 25 I do not derive the correlation coefficients of this MA(k) process because they look messy. But the calculation is straightforward. As the autocorrelation functions of an MA process do not change if one changes the signs of all of the coefficients in white noise terms, all the persistence arguments in this section applies also to the negative feedback traders.

fundamental value. Therefore the term in front of σ_ϵ^2/r^3 measures the relative magnitude of the variance of the deviation to the conditional variance of the 20-year ahead fundamental value. I denote this term as $\Theta(k)$. As k goes to infinity, the $\Theta(k)$ converges to 0. Therefore, the variance of the deviation eventually vanishes. However, the decaying speed of $\Theta(k)$ when k increases is deterred by the small quantity r in the expression and therefore the decay of the variance of the deviation with increasing k is much slower than other quantities such as the variance of the feedback traders' forecast or the expected required rate of entrance or the autocorrelation coefficients of excess return.

The decaying pattern is also different from the other quantities in that it first increases and then decreases as k increases. These are the effect of the summation of the infinitely extended future expected forecast stream when the forecast series gets more and more persistent. The solid curves in Figure 2-1 show the $\Theta(k)$ as a function of k between 5 and 300 for $\mu = 1, 2, 3$ with $r = 0.05$. I set $VR(k) = 0.r$ for $\mu = 1$, $VR(k) = 0.4$ for $\mu = 2$ and $VR(k) = 0.3$ for $\mu = 3$. It shows that as the wealth ratio of feedback traders to rational investors μ increases, the variance of the deviation increases. Specifically, the value of $\Theta(k)$ at its maximum is 0.5 when $\mu = 1$, 4 when $\mu = 2$ and 17 when $\mu = 3$. This means the variance of the deviation at its maximum is 50%, 400%, 1700% of the conditional variance of the fundamental value 20 years ahead when μ is 1, 2, 3, respectively.

The variance of the deviation reaches its maximum when the length of the memory horizon is on the order of the inverse of the interest rate, which is around 20 years. The figure also shows that the variance of the deviation decays very slowly after passing its maximum as the memory horizon gets longer. At the memory horizon of 200 years, the variance of the deviation is still 10%, 50%, 200% of the conditional variance of the fundamental value 20 years ahead when μ is 1, 2, 3, respectively. To see the essential part of $\Theta(k)$ more clearly, I use farther approximations. In deriving the deviation component in the price, I ignore the second term in the price because it is negligible compared to the third term when k is large enough. Then the deviation component is

$$D_t = \mu k^{-1} r^{-1} (1+r)^{-1} \{ [1-(1+r)^{-k}] \pi_t + [1-(1+r)^{-(k-1)}] \pi_{t-1} + \dots + [1-(1+r)^{-1}] \pi_{t-k+1} \}$$

In calculating the variance of this deviation component, I approximate $\text{Var}(\pi_{t+i})$ with Σ because the proportion of the $\mu^2 \text{Var}(\Pi_{t-1})$ term in $\text{Var}(\pi_{t+i})$ is negligible when k is large enough.²⁶ Then the variance of the deviation component reduces to

$$\text{Var}(D_t) = \mu^2 k^{-2} r^{-2} (1+r)^{-2} \{ k - [(3+2r) + (1+r)^{-2k} - 2(2+r)(1+r)^{-k}] / r(2+r) \}$$

²⁶ Specifically, the proportion of the $\mu^2 \text{Var}(\Pi_{t-1})$ term in $\text{Var}(\pi_{t+i})$ is $\mu^2 VR(k)/k$. If $\mu=1$ and $VR(k) = 0.5$ this proportion is 0.05 when $k = 10$, 0.025 when $k = 20$, 0.017 when $k = 30$ and 0.01 when $k = 50$.

$$* \{(1+r) + \mu k^{-1} r^{-1} [1 - (1+r)^{-k}]\}^2 \sigma_\varepsilon^2 / r^2$$

After using approximations $(1+r)^k \approx e^{rk}$, $(1+r) \approx 1$, $(2+r) \approx 2$ and $(3+2r) \approx 3$, this is simplified to

$$\text{Var}(D_t) = \{\mu^2 (rk)^{-1} [1 - (3 + e^{-2rk} - 4e^{-rk}) / 2rk] [1 + \mu(1 - e^{-rk}) / rk]^2\} (\sigma_\varepsilon^2 / r^3)$$

The term inside the curly bracket is the simplified version of $\Theta(k)$. The dotted curves in Figure 2-1 show the simplified $\Theta(k)$ as a function of k between 5 and 300 for $\mu = 1, 2, 3$ with $r = 0.05$.²⁷

8. Conclusion

Positive feedback traders with extended knowledge can explain at least three puzzling aspects of international securities markets. First, positive feedback traders with extended knowledge can generate the long run mean reversion of returns. Through simulations of a model where positive feedback traders and rational investors coexist, I calibrated the wealth ratio of positive feedback traders to rational investors to be around 1 ~ 1.5 and the memory horizon of positive feedback traders to be around 10 - 40 years. I also estimated the parameter.

Second, I showed that positive feedback traders with extended knowledge partly explain the equity premium puzzle. I illustrated that the extra equity premium of 2% ~ 4% that positive feedback traders with extended knowledge generate in the simulation is the effect from market inefficiency and can come about without any increase in the covariance of aggregate consumption growth and stock returns, implying that positive feedback traders with extended knowledge can drive a wedge between the equity premium and the covariance of aggregate consumption growth and stock returns.

Third, I showed that positive feedback traders with extended knowledge explain the negative relationship between volatility change and stock price level change, and the extreme persistence of volatility at the same time. I showed that these are due to the transitory component in prices that positive feedback traders with extended knowledge generate when return volatility is constant along the fundamentals path.

²⁷ 27 One may wonder how irrational investors affect derivative prices. If we regard the yield stream from the underlying asset as nonstochastic until the expiration date of derivatives, the equilibrium price of any derivative is tied to the price of the underlying asset by the condition of no riskless arbitrage opportunity. Moreover, it turns out that the relationship between derivative prices and the underlying asset price is not affected by the expected rate of capital gain. This seems to suggest that derivative prices as a function of the underlying asset price are not affected by agents' risk aversion, and therefore not affected by the presence of irrational investors. However, in the case of continuous time derivative pricing there is a subtle problem that derivative prices are affected by changes in the underlying asset price process itself although they are not affected by changes in the expected rate of capital gain. This problem can be neglected in the presence of positive feedback traders with very long memories since the transitory component in the underlying asset price that they generate is very persistent. In this case, the underlying asset price process can be regarded as not being affected by their presence at least in the short run until the expiration date of derivatives.

Also, this paper showed that positive feedback traders with very long memories make international securities markets a seemingly paradoxical place where they hardly lose to rational investors although they are not rational and their infinitesimal forecast error causes prices to deviate widely from fundamentals. I showed that the source of the first phenomenon is the fact that positive feedback traders with very long memories are using a consistent estimator of the true unconditional expected return with a large sample size as then-forecast of the next period's return even if they are not aware of doing so. I showed that the source of the second phenomenon is the fact that the forecast of positive feedback traders with very long memories is extremely persistent as their forecast is a moving average of very many periods' return.

Table 1-1: Annual Return Standard Deviations and Variance Ratios for $\mu = 0.5$

Horizon of Positive Feedback Traders with Extended Knowledge (k)	Annual Return Standard Deviation	Return Horizon (months)							
		1	24	36	48	60	72	84	96
12	0.159	1.339	0.745	0.671	0.664	0.634	0.629	0.607	0.614
60	0.175	1.051	0.943	0.852	0.785	0.711	0.612	0.564	0.562
120	0.173	1.033	0.943	0.904	0.878	0.818	0.769	0.780	0.699
240	0.167	1.006	0.992	0.925	0.938	0.885	0.911	0.852	0.852
360	0.159	1.015	0.967	0.958	0.946	0.935	0.922	0.965	0.937
480	0.157	1.014	0.993	0.928	0.926	0.921	0.861	0.839	0.892
600	0.152	1.044	0.970	0.966	0.971	0.942	0.973	0.989	0.967
720	0.153	0.941	0.955	0.955	1.025	0.915	0.891	0.844	0.919
840	0.149	0.983	0.998	0.947	0.959	0.916	0.933	0.829	0.904
960	0.149	0.983	1.005	0.994	0.997	1.001	1.003	1.022	1.009
1080	0.148	1.012	1.007	1.075	1.033	1.073	1.041	0.975	0.994
1200	0.144	0.998	1.018	1.001	0.988	1.003	0.981	0.981	0.965

Table 1-2: Annual Return Standard Deviations and Variance Ratios for $\mu = 1$

Horizon of Positive Feedback Traders with Extended Knowledge (k)	Annual Return Standard Deviation	Return Horizon (months)							
		1	24	36	48	60	72	84	96
12	0.195	1.646	0.656	0.542	0.504	0.468	0.463	0.429	0.428
60	0.227	1.099	0.904	0.753	0.661	0.560	0.427	0.390	0.382
120	0.232	1.072	0.926	0.899	0.789	0.734	0.693	0.689	0.543
240	0.231	1.022	0.982	0.964	0.932	0.858	0.852	0.888	0.730
360	0.208	1.040	0.948	0.937	0.904	0.918	0.879	0.782	0.804
480	0.198	1.007	0.945	0.894	0.926	0.938	0.800	0.875	0.858
600	0.184	1.012	0.999	0.969	0.935	0.993	1.009	0.933	0.929
720	0.184	0.944	0.942	1.011	0.896	0.958	0.842	0.839	0.978
840	0.176	0.961	1.013	1.005	1.164	1.036	1.054	1.079	1.019
960	0.164	1.013	0.956	0.924	0.992	0.934	0.971	0.962	0.965
1080	0.166	0.993	0.985	1.019	0.980	0.953	1.005	0.945	0.991
1200	0.170	0.973	0.988	1.018	1.018	1.083	1.116	1.095	1.079

Table 1-3: Annual Return Standard Deviations and Variance Ratios for $\mu = 1.5$

Horizon of Positive Feedback Traders with Extended Knowledge (k)	Annual Return Standard Deviation	Return Horizon (months)							
		1	24	36	48	60	72	84	96
12	0.238	1.879	0.599	0.476	0.418	0.398	0.376	0.383	0.322
60	0.293	1.102	0.895	0.722	0.585	0.451	0.362	0.271	0.290
120	0.302	1.046	0.904	0.829	0.771	0.825	0.668	0.619	0.489
240	0.273	1.016	0.929	0.938	0.894	0.807	0.780	0.746	0.772
360	0.250	1.001	1.002	0.926	0.987	0.955	0.855	0.772	0.859
480	0.248	1.021	0.993	1.007	0.950	0.971	0.941	0.996	0.916
600	0.211	1.035	1.013	0.923	0.935	0.938	0.900	0.845	0.867
720	0.208	1.032	0.985	0.986	0.994	1.003	0.975	0.925	0.964
840	0.199	1.023	0.982	0.968	0.963	0.949	0.975	0.927	0.987
960	0.206	1.031	1.027	1.043	1.090	1.003	1.124	1.129	1.024
1080	0.188	0.989	1.027	1.074	1.016	1.017	1.092	1.005	1.013
1200	0.181	0.993	0.964	0.971	0.938	1.010	0.990	1.059	1.022

Table 1-4: Annual Return Standard Deviations and Variance Ratios for $\mu = 2$

Horizon of Positive Feedback Traders with Extended Knowledge (k)	Annual Return Standard Deviation	Return Horizon (months)							
		1	24	36	48	60	72	84	96
12	0.298	1.977	0.615	0.399	0.339	0.296	0.292	0.264	0.247
60	0.367	1.119	0.809	0.690	0.506	0.396	0.266	0.243	0.238
120	0.377	1.072	0.898	0.816	0.716	0.668	0.649	0.471	0.468
240	0.404	1.048	0.932	0.852	0.947	0.918	0.759	0.867	0.823
360	0.332	0.907	0.963	1.053	0.972	0.908	0.997	0.959	0.998
480	0.345	0.951	0.986	0.963	0.933	0.936	1.012	0.932	0.952
600	0.278	0.974	1.126	1.074	1.135	1.227	1.232	0.890	0.971
720	0.262	0.979	1.033	1.001	1.037	0.993	0.961	1.129	1.068
840	0.216	1.045	0.993	1.021	0.983	1.050	1.012	1.032	0.975
960	0.209	1.036	1.032	0.993	0.963	0.999	0.998	0.987	0.945
1080	0.198	0.977	1.012	1.057	1.029	1.125	1.025	1.058	1.017
1200	0.205	1.006	0.994	0.964	0.946	1.016	0.927	1.013	0.970

Table 1-5: Annual Return Standard Deviations and Variance Ratios for $\mu = -0.9$

Horizon of Positive Feedback Traders with Extended Knowledge (k)	Annual Return Standard Deviation	Return Horizon (months)							
		1	24	36	48	60	72	84	96
12	0.045	0.197	1.847	2.614	3.302	3.904	4.471	4.962	5.359
60	0.039	0.532	1.519	2.029	2.535	3.056	3.595	4.026	4.499
120	0.047	0.655	1.383	1.776	2.176	2.568	2.915	3.355	3.693
240	0.058	0.801	1.182	1.398	1.572	1.783	2.017	2.196	2.427
360	0.063	0.901	1.090	1.249	1.288	1.369	1.494	1.595	1.751
480	0.074	0.921	1.050	1.120	1.183	1.294	1.351	1.442	1.497
600	0.076	0.971	1.049	1.084	1.141	1.171	1.218	1.309	1.332
720	0.084	1.003	1.059	1.081	1.128	1.162	1.180	1.159	1.178
840	0.081	0.909	1.091	1.123	1.249	1.213	1.323	1.373	1.459
960	0.085	1.012	1.037	1.031	1.091	1.128	1.077	1.147	1.191
1080	0.082	0.987	1.049	1.072	1.101	1.119	1.177	1.229	1.264
1200	0.088	0.956	1.051	1.079	1.149	1.259	1.261	1.285	1.312

Table 1-6: Unrestricted OLS Regression

Year (k)	Adjusted R²	β_k (t statistic)	D.W.
1	0.018	- 0.0161 (-1.814)	1.957
2	0.024	0.117 (1.324)	1.945
3	0.018	-0.049 (-0.528)	1.942
4	0.009	-0.013 (-0.131)	1.936
5	0.003	-0.024 (-0.238)	1.932
6	0.012	0.149 (1.539)	1.908
7	0.002	-0.059 (-0.617)	1.912
8	0.003	0.086 (0.835)	1.918
9	0.001	-0.078 (-0.734)	1.919
10	-0.004	-0.079 (-0.756)	1.952
11	0.018	-0.187 (-1.838)	1.999
12	0.029	-0.149 (-1.442)	2.052
13	0.057	-0.213 (-2.025)	2.075
14	0.052	-0.081 (-0.744)	2.089
15	0.040	-0.007 (-0.067)	2.082
16	0.029	-0.149 (-1.442)	2.052
17	0.051	-0.204 (-1.797)	2.129
18	0.057	-0.155 (-1.312)	2.142
19	0.041	-0.064 (-0.495)	2.108
20	0.049	-0.092 (-0.676)	2.128
21	0.041	-0.102 (-0.707)	2.148
22	0.022	-0.058 (-0.376)	2.163
23	0.032	-0.222 (-1.391)	2.182
24	0.018	-0.046 (-0.281)	2.186
25	0.008	-0.120 (-0.688)	2.169
26	0.011	0.117 (0.657)	2.221
27	0.012	0.122 (0.672)	2.178
28	0.055	-0.425 (-2.285)	2.041
29	0.091	0.306 (1.618)	2.059

30	0.079	-0.124 (-0.632)	2.071
31	0.057	-0.045 (-0.225)	2.076
32	0.045	-0.225 (-1.046)	2.052
33	0.063	-0.189 (-0.852)	2.075
34	0.034	0.087 (0.379)	2.074
35	0.002	0.003 (0.015)	2.018
36	0.005	-0.019 (-0.078)	1.976
37	0.019	-0.308 (-1.241)	1.996
38	-0.011	-0.003 (-0.012)	1.996
39	-0.036	0.132 (0.534)	1.981
40	-0.066	0.126 (-0.486)	1.991
41	-0.099	-0.149 (-0.559)	1.955
42	-0.110	0.273 (1.002)	1.992
43	-0.139	0.234 (0.842)	1.924
44	-0.118	-0.413 (-1.491)	2.329
45	-0.119	-0.364 (-1.262)	2.034
46	-0.153	0.053 (0.166)	2.057
47	-0.212	0.106 (0.299)	2.024
48	-0.244	-0.010 (-0.028)	1.957
49	-0.239	0.318 (0.833)	1.853
50	-0.164	-0.728 (-1.900)	2.050
51	-0.172	-0.499 (-1.209)	1.948
52	-0.214	0.444 (0.972)	1.811
53	-0.138	-0.718 (-1.464)	1.831
54	-0.277	0.074 (0.134)	1.855
55	-0.198	0.462 (0.769)	2.111
56	-0.197	-0.293 (-0.449)	2.449
57	-0.413	-0.173 (-0.223)	2.718
58	-0.689	-0.518 (-0.561)	2.987
59	-1.405	-0.187 (-0.169)	2.911
60	-2.893	-0.385 (-0.263)	3.297

Table 1-7: Equity Premium

k	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$	$\mu = 2$	$\mu = -0.9$
12	0.88	2.39	4.62	7.84	-0.78
60	0.83	2.09	4.24	6.94	-1.05
120	0.84	2.17	4.06	7.11	-1.48
240	0.84	1.96	3.63	6.09	-1.63
360	0.79	1.78	3.17	5.31	-1.73
480	0.71	1.76	3.15	4.16	-1.34
600	0.63	1.55	2.29	3.53	-1.13
720	0.76	1.57	2.38	3.34	-0.75
840	0.55	1.42	2.07	2.59	-1.43
960	0.59	1.31	2.11	2.40	-1.27
1080	0.72	1.24	1.78	2.17	-1.71
1200	0.74	1.24	1.72	2.18	-1.32

Figure 1 - 1 $\mu = 1$ and $k = 20$ years

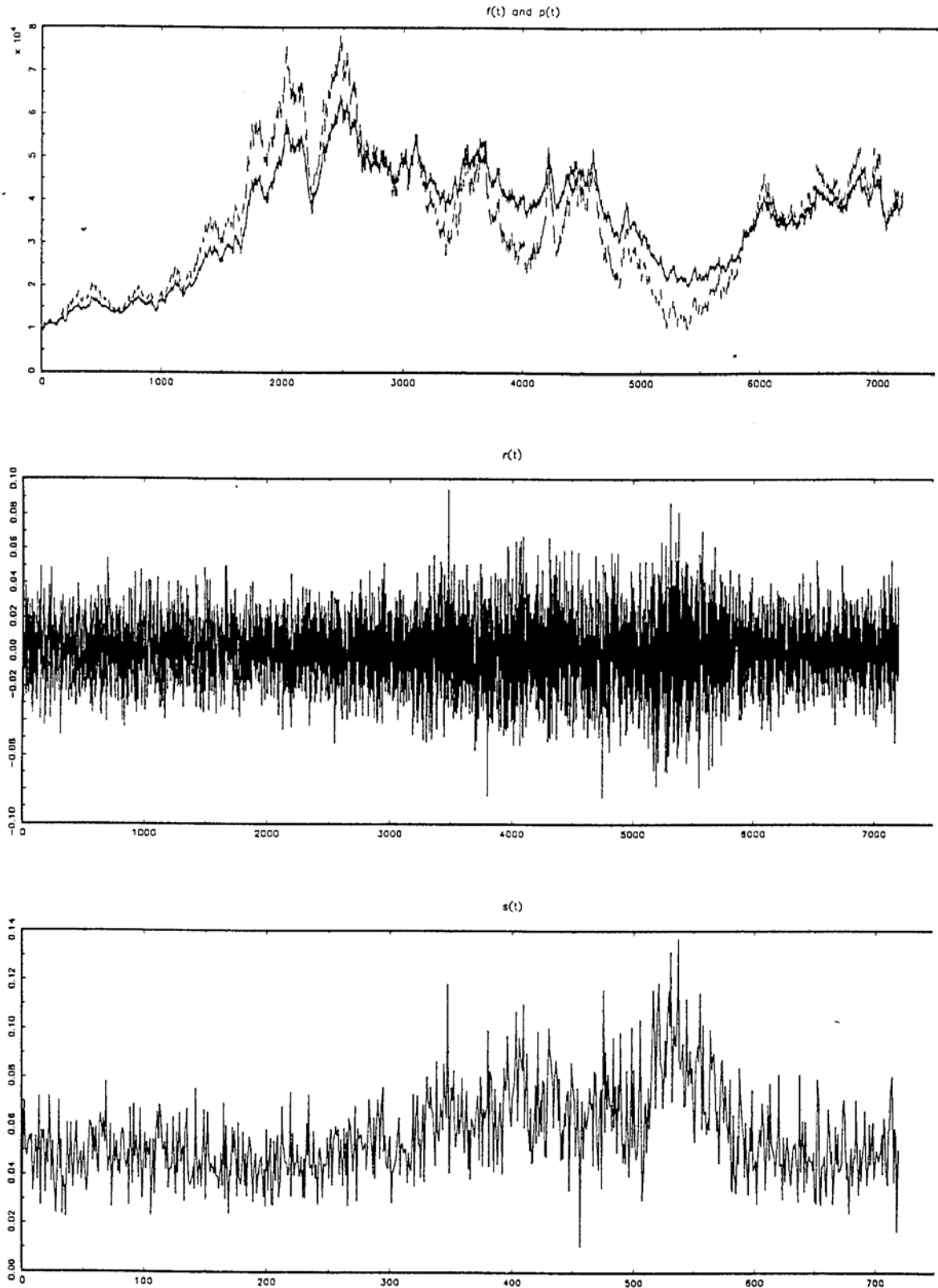


Figure 1 - 2 $\mu = 1$ and $k = 30$ years

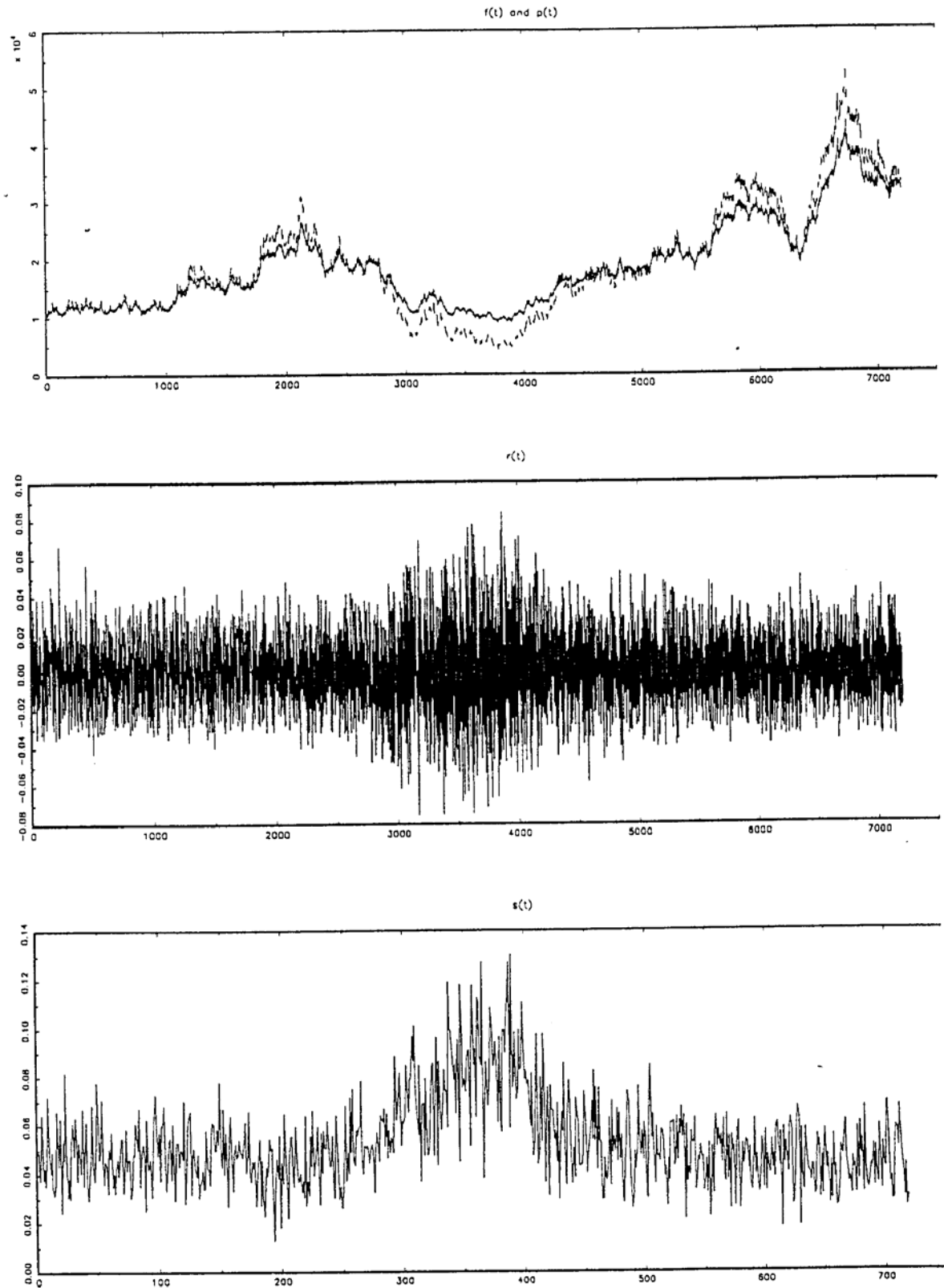


Figure 1 - 3 $\mu = 1.5$ and $k = 20$ years

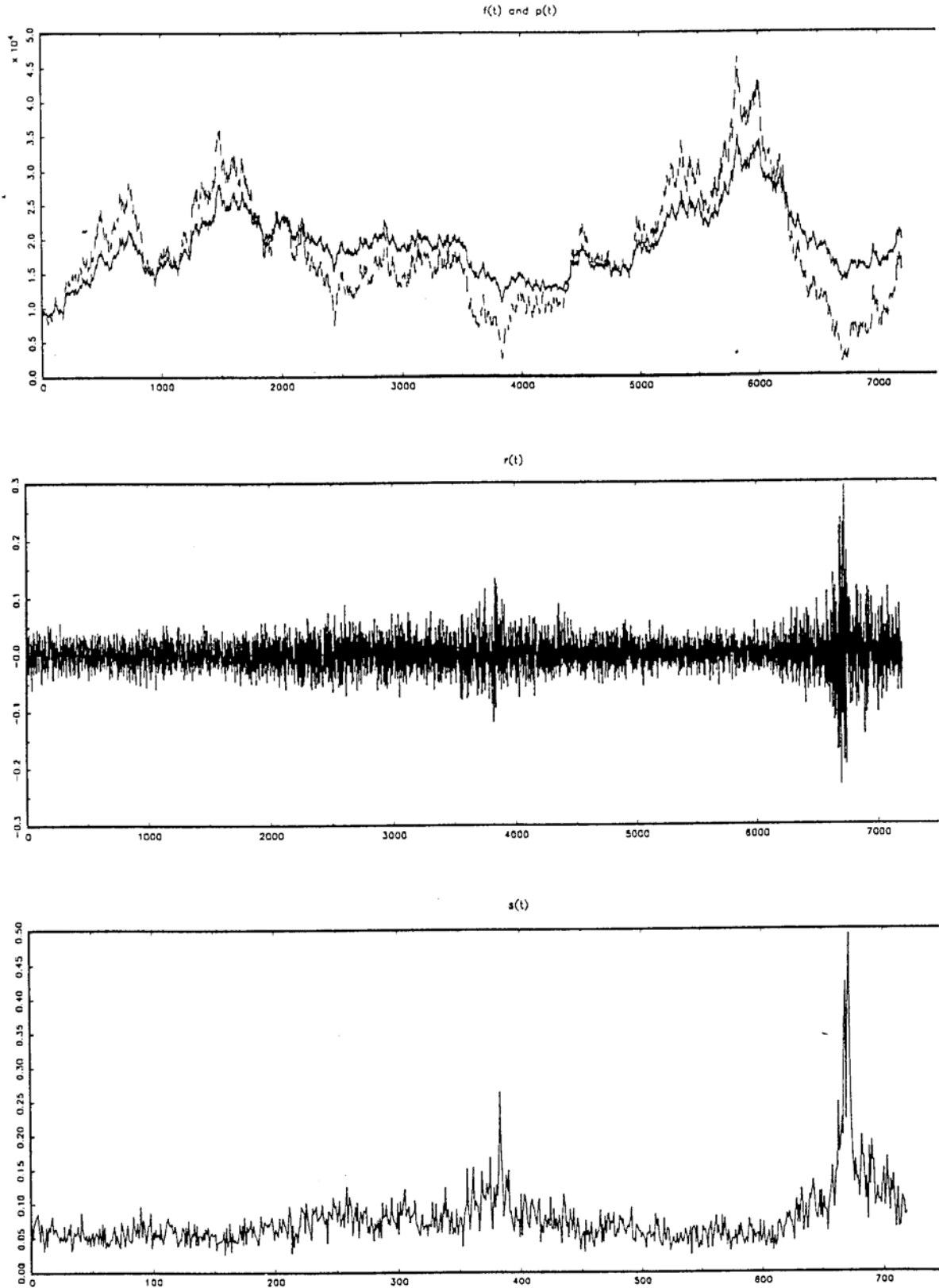


Figure 1 - 4 $\mu = 1.5$ and $k = 30$ years

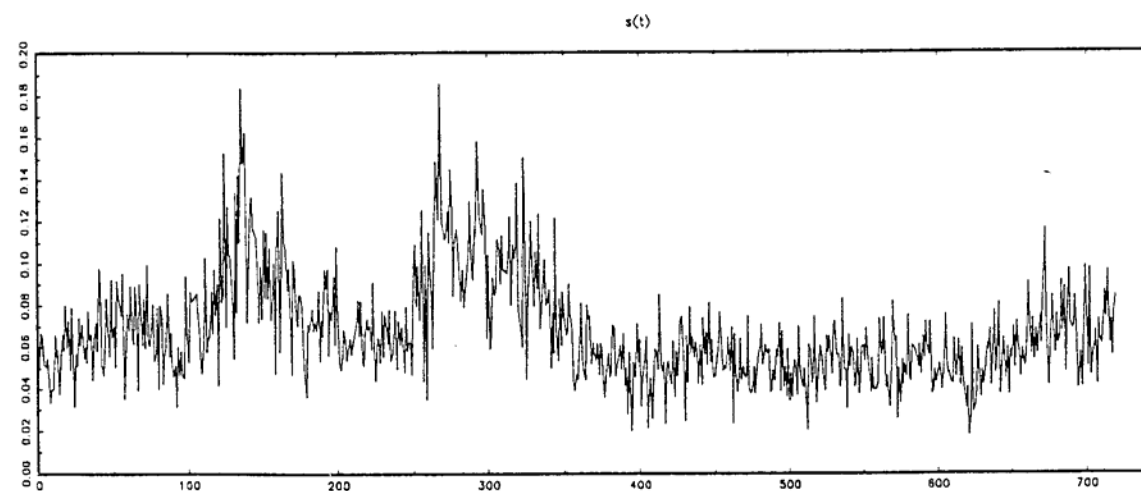
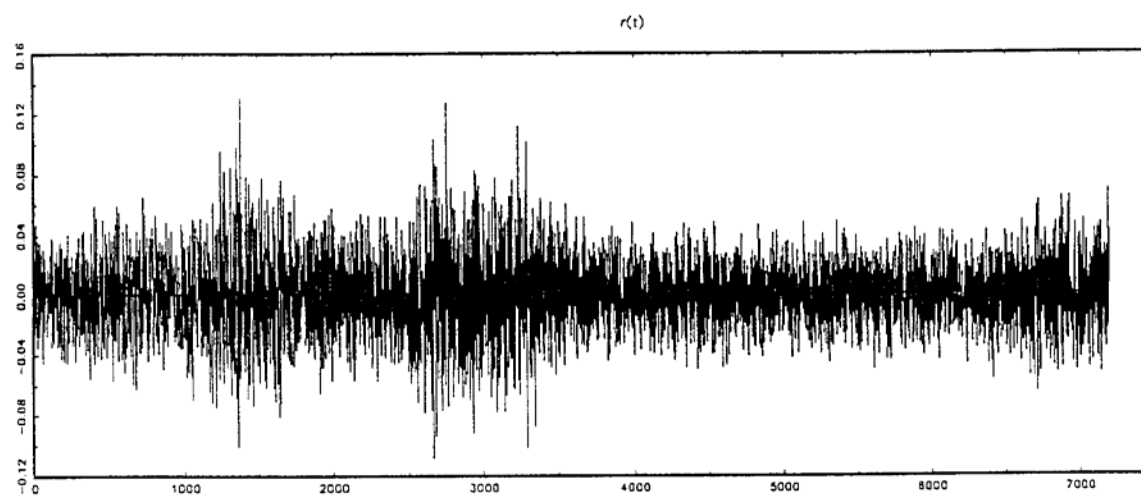
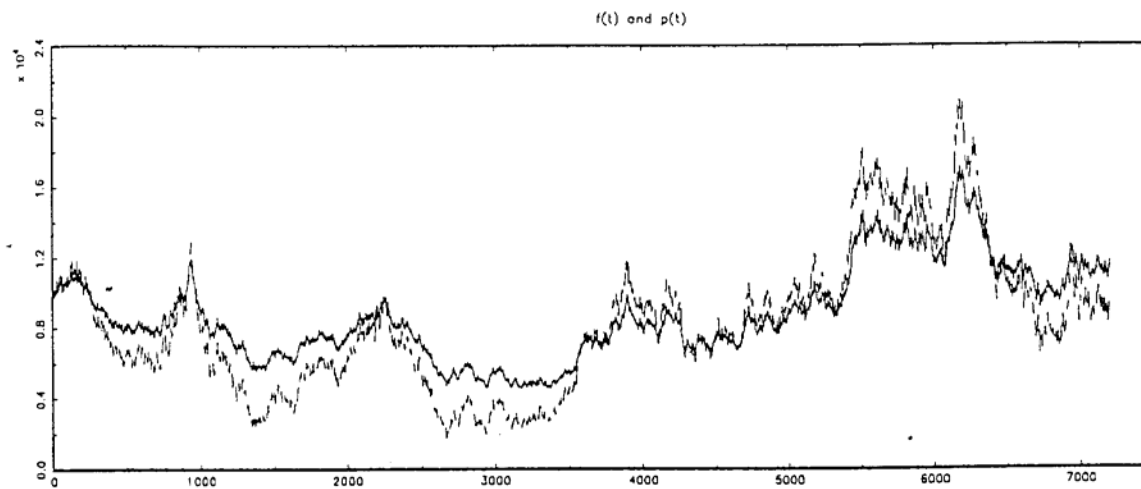


Figure 1 - 5 $\mu = -0.9$ and $k = 10$ years

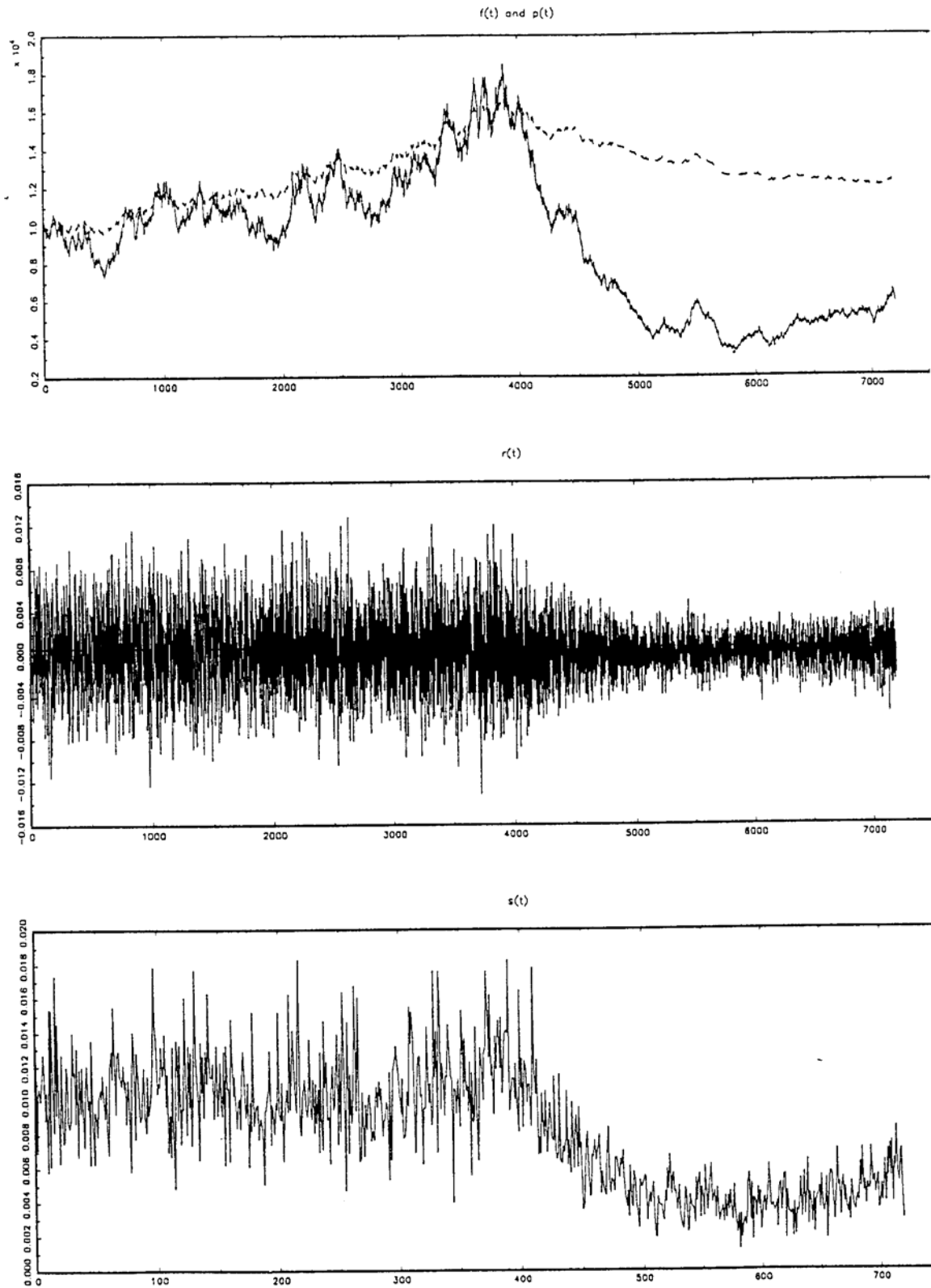


Table 2-1: Expected Required Rate of Entrance $E(q_{t+1})$

$\gamma = 1$

k	10	20	30	40	50	100
$\mu = 1$	0.105	0.051	0.034	0.025	0.02	0.01
$\mu = 2$	0.286	0.13	0.084	0.062	0.049	0.024
$\mu = 3$	0.493	0.208	0.132	0.096	0.076	0.037
$\mu = -0.5$	0.141	0.069	0.046	0.034	0.027	0.013

$\gamma = 2$

k	10	20	30	40	50	100
$\mu = 1$	0.053	0.026	0.017	0.013	0.01	0.005
$\mu = 2$	0.143	0.065	0.042	0.031	0.025	0.012
$\mu = 3$	0.247	0.104	0.066	0.048	0.038	0.019
$\mu = -0.5$	0.071	0.035	0.023	0.017	0.014	0.007

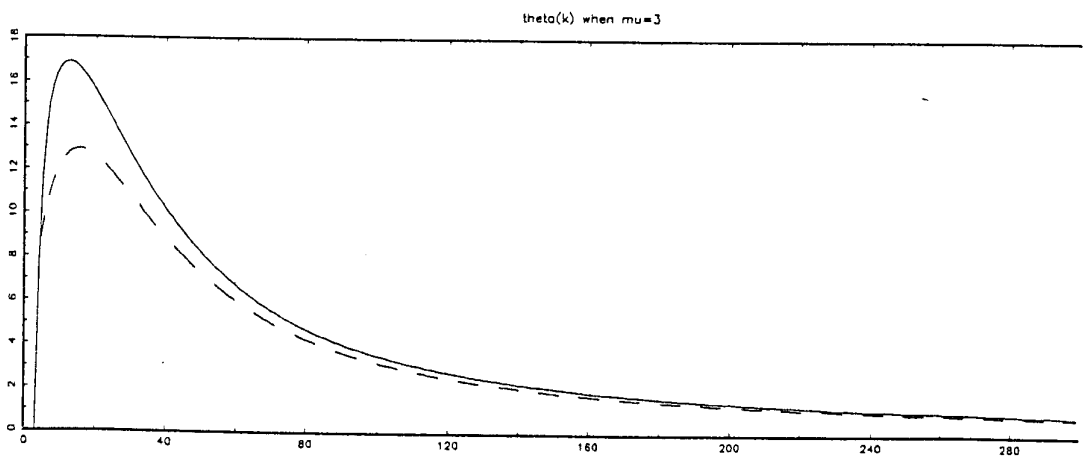
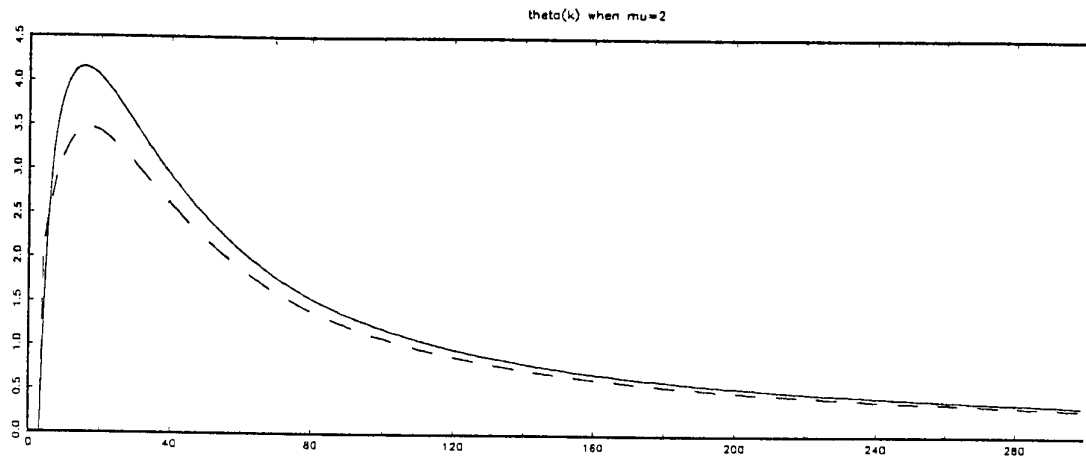
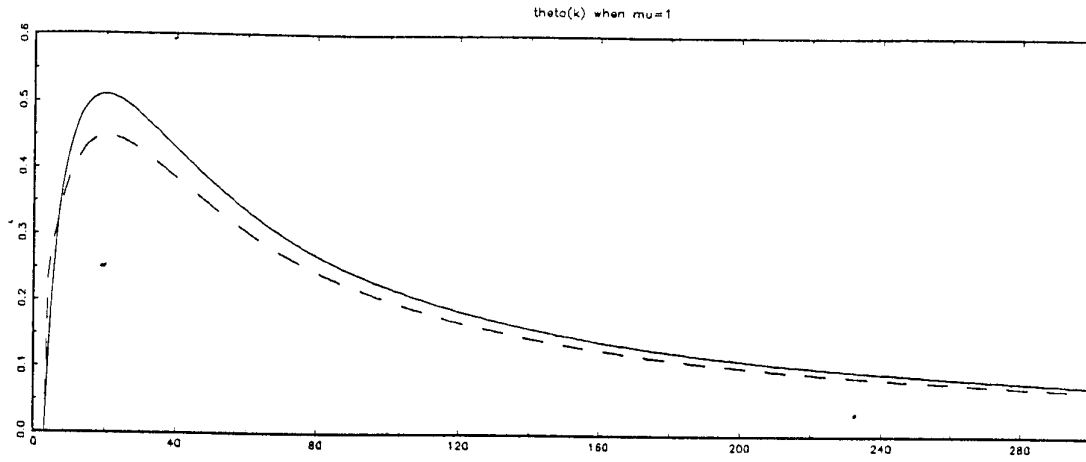
$\gamma = 3$

k	10	20	30	40	50	100
$\mu = 1$	0.035	0.017	0.011	0.008	0.007	0.003
$\mu = 2$	0.095	0.043	0.028	0.021	0.016	0.008
$\mu = 3$	0.164	0.069	0.044	0.032	0.025	0.012
$\mu = -0.5$	0.047	0.023	0.015	0.011	0.009	0.004

$\gamma = 4$

K	10	20	30	40	50	100
$\mu = 1$	0.026	0.013	0.009	0.006	0.005	0.003
$\mu = 2$	0.072	0.033	0.021	0.016	0.012	0.006
$\mu = 3$	0.123	0.052	0.033	0.024	0.019	0.009
$\mu = -0.5$	0.035	0.017	0.012	0.009	0.007	0.003

Figure 2-1 $\Theta(k)$ when $\mu = 1, 2, 3$



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