Value-at-Risk and Expected Shortfall for Quadratic Portfolio of Securities with Mixture of Elliptic Distributed Risk Factors

Jules SADEFO KAMDEM *†

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Abstract

Generally, in the financial literature, the notion of quadratic VaR is implicitly confused with the Delta-Gamma VaR, because more authors dealt with portfolios that contained derivatives instruments. In this paper, we postpone to estimate both the Expected Shortfall and Value-at-Risk of a quadratic approximate portfolio of securities (i.e equities) without the Δ and Γ Greeks, when the joint log-returns changes with mixture of multivariate elliptic distributions. To illustrate our method, we give special attention to mixture of normal distributions, and mixture of Student t-distributions.

Key Words: VaR, ES, Quadratic portfolios of Equities, Applied Numerical Analysis, Computational Finance, Long/Short Equities strategies.

1 Introduction

Value-at-Risk is a market risk management tool that permits to measure the maximum loss of the portfolio with a certain confidence probability $1-\alpha$, over a certain time horizon such as one day. Formally, if the price of portfolio's $P(t,S(t))$ at time t is a random variable where $S(t)$ represents a vector of risk factors at time t, then the Value-at-Risk (VaR_α) be implicitly given by the formula

$$Prob\{-P(t,S(t))+P(0,S(0))>VaR_\alpha\}=\alpha.$$

Generally, in the financial literature, the notion of quadratic VaR is implicitly confused with the $\Delta-\Gamma$ VaR, because more authors dealt with portfolios that contained derivatives instruments. Also to estimate the VaR_α for portfolios depending non-linearly on the return, or portfolios of non-normally distributed assets, one turns to Monte Carlo methods. Monte Carlo methodology has the obvious advantage of being almost universally applicable, but has the disadvantage of being much slower than comparable parametric methods, when the latter are available.

This paper is concerned with the numerical computation of static Value-at-Risk and Expected Shortfall for portfolios of securities (i.e equities) depending quadratically on securities log-returns. In fact, following the quadratic $\Delta-\Gamma$ Portfolio that contains derivatives instruments, we introduce the notion of *quadratic portfolio of securities (i.e equities)* due to the analytic approximation of Taylor in 2^{nd} order of log-returns for very small variations of time. Quadratic approximations have also been the subject of a number of papers dedicated to numerical computations for VaR (but these have been done for portfolio that contains derivatives instruments). We refer the reader to Cardenas and al.(1997) [2] for a rigorous analytical approach method to compute quadratic VaR using fast Fourier transform. Note that in [1], Brummelhuis, Cordoba,

*Paper based on the M. J. SADEFO KAMDEM Université de Reims Phd Thesis. Author address: Laboratoire de Mathématiques (CNRS UMR 6056) BP 1039 Moulin de la Housse 51687 Reims Cedex FRANCE. sadefo@univ-reims.fr

†The author is currently temporary lecturer at the Mathematics Department (Université d'evry val d'essonne FRANCE).

Quintanilla and Seco have dealt with the similar problem, but their work have been done analytically for $\Delta - \Gamma$ Portfolio VaR when the joint underlying risk factors follow a normal distribution which is a particular case of elliptic distribution.

In this paper, our calculus will be done according to the assumption that the joint securities (i.e equities) log-returns follow a mixture of elliptic distributions. Note that one shortcoming of the simple elliptic distribution is the fact that, all marginal distributions have the same characteristic function. Therefore the notion of mixture of elliptic distributions seem to be an alternative when the assumption of a simple elliptic distribution is not sufficient. To illustrate our method, we will take some examples based on mixture of *multivariate t-student* and mixture of normal distributions. Note also that, Following RiskMetrics, Sadefo-Kamdem [13](2003) have generalized the notion of Δ-normal VaR by introducing the notion of Δ-Elliptic VaR, with special attention to Δ-Student VaR, but this concerned the linear portfolio. In this paper, we will do the same for nonlinear quadratic portfolios without derivatives instruments.

The rest of the paper is organized as follows: In section 2, we introduced the notion of quadratic portfolio of securities(i.e equities) due to the 2^{nd} order Taylor approximation of log returns. Our calculus is made with the more generalized assumptions that the joint underlying log-returns follow a mixture of elliptic distributions. In section 3 following [4], we recall the definition of elliptic distributions and we show that under the hypothesis of elliptic distributions the VaR estimation of such portfolios is reduced to a multiple integral estimation. Next following the paper of Alan Genz [6], we recall the notions of symmetric interpolar rules for multiple integrals over hypersphere and we use this method to reduce our problem to one dimensional integral estimation. In the same section, we illustrate our method by giving an explicit equation with solution Value-at-Risk (VaR) when the joint log-returns follow some particular mixture of elliptic distributions named mixture of multivariate Student t-distributions or the mixture of normal distributions. In these cases the VaR estimation is reduced to finding the zero's of some transcendental equation that contains certain specials functions. In section 5 we treats the Expected Shortfall for general elliptic quadratic portfolios of securities without derivatives instruments and we illustrate with the special case of normal distributions. Finally, in section 6 we give a conclusion.

2 Quadratic Portfolio of Securities(i.e Equities)

A portfolio of n securities is a vector $\theta \in \mathbb{R}^n$; the component θ_i represents the number of holdings of the i^{th} instruments, which in practise does not need to be an integer. So that at some time t the price of the portfolio of n securities is given by:

$$P(t) = \sum_{i=1}^{n} \theta_i \, S_i(t), \tag{1}$$

where $S(t) = (S_1(t), \dots, S_n(t))$ such that

$$P(t) - P(0) = \sum_{i=1}^{n} \theta_i (S_i(t) - S_i(0)) = \sum_{i=1}^{n} S_i(0).\theta_i.(\frac{S_i(t)}{S_i(0)} - 1).$$

Since the log-return $\eta_i(t)$ over the time interval [0, t] of the i-th security (i.e equity) with price S_i is:

$$log(S_i(t)/S_i(0)) = \eta_i(t), \tag{2}$$

we have that

$$S_i(t) - S_i(0) = S_i(0)(\frac{S_i(t)}{S_i(0)} - 1) = S_i(0)(exp(\eta_i(t)) - 1).$$

Also it is straightforward to see that:

$$S(t) = (S_1(0)exp(\eta_1), \dots, S_n(0)exp(\eta_n)).$$

For small fluctuation of time, usually in the literature authors contend themselves with the approximation $\exp(\eta_i) - 1 \approx \eta_i$. *One innovates* of this paper is to take into account the next quadratic correction term.

Therefore by using 2^{nd} order Taylor's expansion of the exponentials due to small fluctuations of returns with time, we have

$$exp(\eta_i(t)) - 1 \approx \eta_i(t) + \frac{\eta_i(t)^2}{2}. \quad (3)$$

Therefore by using (3), the Profit-and-Loss function over the time-interval [0, t] is:

$$\begin{aligned} P(t) - P(0) &= \sum_{i=1}^{n} S_i(0) \cdot \theta_i \cdot (exp(\eta_i(t)) - 1) \\ &\approx \sum_{i=1}^{n} S_i(0)\, \theta_i \left(\eta_i(t) + \frac{\eta_i(t)^2}{2} \right). \end{aligned}$$

If we assume that $\eta = (\eta_1, \ldots, \eta_n)$ follow an elliptic distribution denote by $N_n(\mu, \Sigma, \phi)$, or a mixture of elliptic distributions when the simple elliptic distribution is not sufficient, by following the usual convention of recording portfolio losses by negative numbers, but stating the Value-at-Risk as a positive quantity of money, The VaR_α at confidence level of $1 - \alpha$ is given by solution of the following equation:

$$Prob\{|P(t) - P(0)| \geq VaR_\alpha\} = \alpha.$$

In the probability space of losses $|P(t) - P(0)| = -P(t) + P(0)$, therefore recalling (4), we have that:

$$Prob\{\sum_{i=1}^{n} S_i(0) \cdot \theta_i \left(\eta_i(t) + \frac{\eta_i(t)^2}{2} \right) \leq -VaR_\alpha\} = \alpha. \quad (4)$$

Using the following elementary algebra,

$$\eta_i(t) + \frac{\eta_i(t)^2}{2} = \frac{1}{2}\left((\eta_i(t) + 1)^2 - 1 \right), \quad (5)$$

the equation 4) becomes

$$Prob\{\sum_{i=1}^{n} S_i(0) \cdot \frac{\theta_i}{2} (\eta_i(t) + 1)^2) \leq -VaR_\alpha + \sum_{i=1}^{n} \frac{\theta_i}{2} \cdot S_i(0)\} = \alpha. \quad (6)$$

By posing $\mathbb{X} = (\eta_1 + 1, \ldots, \eta_n + 1) = (X_1, \ldots, X_n)$, it is straightforward that $\mathbb{X}$ is an elliptic distribution due to the fact that it is an affine map of an elliptic distribution η. We note $\mathbb{X} \sim N(\mu + 1, \Sigma, \phi^{'})$ with a continuous density function $h_1(x)$. Let Λ be the diagonal matrix with eigenvalues $\alpha_i = \frac{\theta_i\, S_i(0)}{2} = \Lambda_{ii}$,

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i (\eta_i(t) + 1)^2 &= \mathbb{X}\, \Lambda\, \mathbb{X}^t \\ &= \sum_{j=1}^{n} \Lambda_{ii}\, X_i^2 \end{aligned}$$

then (6) becomes

$$Prob\{\mathbb{X}\, \Lambda\, \mathbb{X}^t \leq K\} = \alpha, \quad (7)$$

where $K = -VaR_\alpha + \sum_{i=1}^{n} \alpha_{ii} = \frac{P(0)}{2} - VaR_\alpha$.

Remark 2.1 It is straightforward that, the sign of K will depend to the diagonal matrix Λ. Because definitively the mathrix Λ is diagonal, K will depend to the sign and the multiplicity of Λ_{ii}, for i=1,...,n.

Remark 2.2 For example, the diagonal matrix Λ will have positive eigenvalues, and some negatives eigenvalues, if our portfolio is constitute by long equities and some short equities. In that case, we will be concerned by an integral over hyperboloid. These kinds of integrals have been the subject, when some find the Value-at-Risk of a $\Delta - \Gamma$ portfolios that contains derivatives products. For more details (see Sadefo-Kamdem and Genz [12], Brummelhuis and al. [1] and some references therein).

Remark 2.3 Mathematically, observe that we are in the similar situation as for the quadratic $\Delta - \Gamma$ approximation of a portfolio of derivatives, although here we are dealing with a portfolio securities (i.e simple equities). The computation of our model need as inputs the quantity θ_i and the initial equities prices $S_i(0)$ for each $i = 1, \ldots, n$ as given in (1). Note that in our model, there is no Δ and Γ which are the necessary inputs to compute the Delta-Gamma VaR, because our portfolio is without derivatives instruments.

Remark 2.4 Since we have the similarity with the $\Delta - \Gamma$ portfolio, some methods such as Monte Carlo Methods, analytical methods that have been developed to serve the estimation of $\Delta - \Gamma$ VaR, are useful if we need to estimate the VaR of a quadratic portfolio of equities as we have introduced. For more details see for example Rouvinez [11], Duffie and Pan [5], Glasserman, Heidelberger and Shahabuddin [7], Sadefo-Kamdem and Genz [12].

Remark 2.5 Also observe that following Duffie and Pan [5], it will be straightforward possible to compute the Value-at-Risk for quadratic portfolios of securities (i.e equities) as we have introduced by taken into account the jump diffusions.

3 Reduction to a transcendental equation

In this section, we will reduce the problem of computation of the Value-at-Risk for quadratic portfolio of equities to the study of the asymptotic behavior of the density function distribution over the hyper-sphere.

3.1 Notions of Elliptic Distributions

The following definitions will be given as in [4](2002) .

3.1.1 Spherical Distribution

Definition 3.1 A random vector $X = (X_1, X_2, \ldots, X_n)^t$ has a *spherical distribution* if for every orthogonal map $U \in R^{n\times n}$ (i.e. maps satisfying $UU^t = U^tU = I_{n\times n}$)

$$UX =_d X.^1$$

If X has a density f(x) then this is equivalent to $f(x) = g(x^t x) = g(\|x\|^2)$ for some function $g : R_+ \longrightarrow R_+$, so that the spherical distributions are best interpreted as those distributions whose density is constant on spheres.

Elliptical distributions extend the multivariate normal $N_n(\mu, \Sigma)$, for which μ is mean and Σ is the covariance matrix. Mathematically, they are the affine maps of spherical distributions in $\mathbb{R}^n$.

3.1.2 Elliptic Distribution

Definition 3.2 Let $T : R^n \longrightarrow R^n$, $y \longmapsto Ay + \mu$, $A \in R^{n\times n}$, $\mu \in R^n$. X has an *elliptical distribution* if X = T (Y) and $Y \sim S_n(\phi)$. If Y has a density $f(y) = g(y^t y)$ and if A is regular (det(A) $\neq 0$ so that $\Sigma = A^t\ A$ is strictly positive), then $X = AY + \mu$ has a density

$$h(x) = g((x-\mu)^t \Sigma^{-1} (x-\mu))/\sqrt{det(\Sigma)} \tag{8}$$

[1] $=_d$ denote equality in distribution, and we note $X \sim S_n(\phi)$.

and the contours of equal density are now ellipsoids. An elliptical distribution is fully described by its mean, its covariance matrix and its characteristic generator.

- Any linear combination of an elliptically distributed random vector is also elliptical with the same characteristic generator ϕ. If $Y \sim N_n(\mu, \Sigma, \phi)$, $b \in \mathbf{R}^m$ and $B \in \mathbf{R}^{m\times n}$ then $B \cdot Y + b \sim N_m(B\mu + b, B\Sigma B^t, \phi)$.

3.2 Equation with solution VaR

Since $\mathbb{X}$ is an elliptic distribution, its density take the following form:

$$h_1(x) = h((x - \mathbb{1}));$$

where $\mathbb{1}$ is the vector of unities and h is the density function of η which take the form :

$$h(x) = g((x-\mu)\Sigma^{-1}(x-\mu)^t)/\sqrt{det(\Sigma)}.$$

Following (6), VaR_α will be the solution of the following equation:

$$I(K) := Prob\{\ \mathbb{X}\, \Lambda\, \mathbb{X}^t\ \geq\ -VaR_\alpha + \sum_1^n \alpha_{ii}\} = 1 - \alpha.$$

In terms of our elliptic distribution parameters we have to solve the following equation:

$$I(K) = \int_{\{\ \mathbb{X}\,\Lambda\,\mathbb{X}^t\ \geq\ K\ \}} h_1(x)dx = 1 - \alpha \tag{9}$$

with $\mathbb{X} \sim E_n(\mu + 1, \Sigma, \phi)$, $A\ A^t = \Sigma$ and $I(K)$ becomes

$$I(K) = \int_{\{\ \mathbb{X}\,\Lambda\,\mathbb{X}^t \geq K\ \}} g((x-\mu-\mathbb{1})^{\rm t}\Sigma^{-1}(\mathrm{x}-\mu-\mathbb{1}))\frac{\mathrm{dx}}{\sqrt{\det(\Sigma)}}.$$

Remark 3.3 For more simplification, we will restrict ourselves with the case where all $\alpha_{ii} > 0$ for i=1, ..., n. This means that in the next steps, we work with the portfolios that contains only longs equities, although the preceded results are available, if we treat a portfolios of long/short equities. We postpone to study completely the case of portfolio with long/short equities in a sequel and later paper. Note that long/short equities strategies are used by some Hedge Funds managers.

If we assume that Λ is a diagonal matrix with all positive diagonal values, we decompose $\Lambda = \Lambda^{1/2}\Lambda^{1/2}$ therefore the equation (9) becomes

$$\begin{aligned} I(K) &= \int_{\{<\Lambda^{1/2}(Az+\mu+\mathbb{1}),\Lambda^{1/2}(\mathrm{Az}+\mu+\mathbb{1})>\ \geq\ \mathrm{K}\}} g(\|z\|^2)\ dz \\ &= \int_{\{\ \|\Lambda^{1/2}(Az+\mu+\mathbb{1})\|_2^2\ \leq\ K\ \}} g(\|z\|^2)\ dz. \end{aligned}$$

Cholesky decomposition states that $\Sigma = AA^t$ when Σ is supposed to be positive, therefore if we change the variable $z = A^{-1}(y - \mu - \mathbb{1})$, the preceded integral becomes :

$$I(K) = \int_{(Az+\mu+\mathbb{1})^{\rm t}\Lambda(\mathrm{Az}+\mu+\mathbb{1})\ \geq\ \mathrm{K}\}} g(\|z\|^2)dz.$$

If we carry out the following decomposition $(Az + \mu + \mathbb{1})^{\rm t}\Lambda(\mathrm{Az} + \mu + \mathbb{1}) = (\mathrm{z} + \mathrm{v})^{\rm t}\mathrm{D}(\mathrm{z} + \mathrm{v}) + \delta$, with $D = A^t\ \Lambda\ A$, $v = A^{-1}(\mu + \mathbb{1})$ and $\delta = 0$, after some elementary calculus,

$$I(K) = \int_{\{(z+v)^tD(z+v)\ \geq\ K-\delta\}} g(\|z\|^2)dz.$$

Since we have restricted ourselves to the portfolio such that $\alpha_{ii} > 0$, for a given reasonable $1 > \alpha > 0$ the equation (7) have a sense, if we suppose that $K > 0$. This means that in the following steps, we suppose that the solution VaR_α of the equation (7) will be under $P(0)/2$.

Therefore by suggesting $R > 0$ such that $K = R^2$, $z + v = u$, $dz = du$, we find that

$$I(R^2) = \int_{\{u^t.D.u \geq R^2\}} g(\|u - v\|^2)dz = 1 - \alpha.$$

By introducing the variable $z = D^{1/2}u/R$, we have :

$$I(R^2) = R^n \int_{\{\|z\| \geq 1\}} g(\|RD^{\frac{-1}{2}} z - v\|^2) \frac{dz}{\sqrt{det(D)}}.$$

Next, by using spherical variables $z = r.\xi$, where $\xi \in S_{n-1}$, $dz = r^{n-1} dr d\sigma(\xi)$ for which $d\sigma(z)$ is a elementary surface of z on $S_{n-1} = \{\xi | \xi \in \mathbb{R}^n, \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 = 1\}$, also by introducing the function $J(r, R)$ such that

$$J(r, R) = \int_{S_{n-1}} g(\|rRD^{\frac{-1}{2}} \xi - v\|^2) d\sigma(\xi), \tag{10}$$

we obtain the following expression:

$$\begin{aligned} R^{-n}\, I(R^2) &= \int_1^\infty r^{n-1} \Big[\int_{S_{n-1}} g(\|rRD^{\frac{-1}{2}} \xi - v\|^2) d\sigma(\xi) \Big] \frac{dr}{\sqrt{det(D)}} \\ &= \int_1^\infty r^{n-1} J(r, R) \frac{dr}{\sqrt{det(D)}}. \end{aligned} \tag{11}$$

Next by introducing the function

$$H(s) = s^n \int_1^\infty r^{n-1} J(r, s) dr, \tag{12}$$

our goal will be to solve the following equation

$$H(s) = (1 - \alpha)\sqrt{det(D)}. \tag{13}$$

In the following section, we propose to approximate J(r,R) by applied the numerical methods giving in the paper of Alan Genz (2003), (see [6] for more details).

3.3 Numerical approximation of J(r,R)

In this section, we estimate the integral J(r,R) by a numerical methods given by Alan Genz in [6].

3.4 Some interpolation rules on S_{n-1}

The paper [6] of Alan Genz, give the following method. Suppose that we need to estimate the following integral

$$J_1(f) = \int_{S_{n-1}} f(z) d\sigma(z)$$

where $d\sigma(z)$ is an element of surface on $S_{n-1} = \{z | z \in \mathbb{R}^n, z_1^2 + z_2^2 + \cdots + z_n^2 = 1\}$.

In effect, let be the n-1 simplex by $T_{n-1} = \{x | x \in \mathbb{R}^{n-1}, 0 \leq x_1 + x_2 + \cdots + x_{n-1} \leq 1\}$ and for any $x \in T_{n-1}$, define $x_n = 1 - \sum_{i=1}^{n-1} x_i$. Also $t_p = (t_{p_1}, \ldots, t_{p_{n-1}})$ if points t_0, t_1,..., t_m are given, satisfying the condition : $|t_p| = \sum_{i=1}^n t_{p_i} = 1$ whenever $\sum_{i=1}^n p_i = m$, for non-negative integers p_1,..., p_n, then the Lagrange interpolation formula (sylvester [15]) for a function g(x) on T_{n-1} is given by

$$L^{(m,n-1)}(g,x) = \sum_{|p|=m} \prod_{i=1}^{n} \prod_{j=0}^{p_i-1} \frac{x_i - t_j}{t_{p_i} - t_j} g(t_p)$$

$L^{(m,n-1)}(g,x)$ is the unique polynomial of degree m which interpolates g(x) at all of the C^m_{m+n-1} points in the set $\{x|x = (t_{p_1}, \ldots, t_{p_{n-1}}), |p| = m\}$. Silvester provided families of points, satisfying the condition $|t_p| = 1$ when $|p| = m$, in the form $t_i = \frac{i+\mu}{m+\theta n}$ for i=0,1,. . . ,m, and μ real. If $0 \leq \theta \leq 1$, all interpolation points for $L^{(m,n-1)}(g,x)$ are in T_{n-1}. Sylvester derived families of interpolators rules for integration over T_{n-1} by integrating $L^{(m,n-1)}(g,x)$. Fully symmetric interpolar integration rules can be obtained by substitute $x_i = z_i^2$, and $t_i = u_i^2$ in $L^{(m,n-1)}(g,x)$, and define

$$M^{(m,n)}(f,z) = \sum_{|p|=m} \prod_{i=1}^{n} \prod_{j=0}^{p_i-1} \frac{z_i^2 - u_j^2}{u_{p_i}^2 - u_j^2} f\{u_p\}$$

where $f\{u\}$ is a symmetric sum defined by

$$f\{u\} = 2^{-c(u)} \sum_s f(s_1u_1, s_2u_2, \ldots, s_nu_n)$$

with c(u) the number of nonzero entries in $(u_1, \ldots, u_n)$, and the $\sum_s$ taken over all of the signs combinations that occur when $s_i = \pm 1$ for those i with u_i different to zero.

Lemma 3.4 *If*

$$w_p = J_1 \left(\prod_{i=1}^{n} \prod_{j=0}^{p_i-1} \frac{z_i^2 - u_j^2}{u_{p_i}^2 - u_j^2} \right)$$

then

$$J_1(f) = R^{(m,n)}(f) = \sum_{|p|=m} w_p f\{u_p\}$$

$$f\{u\} = 2^{-c(u)} \sum_s f(s_1u_1, s_2u_2, \ldots, s_nu_n),$$

with c(u) the number of nonzero entries in $(u_1, \ldots, u_n)$, and the $\sum_s$ taken over all of the signs combinations that occur when $s_i = \pm 1$ for those i with u_i different to zero.

The proof is given in [6] by (Alan Genz (2003)) as follow:
Let $z^k = z_1^{k_1} z_2^{k_2} \cdot z_n^{k_n}$. J_1 and R are both linear functionals, so it is sufficient to show that $R^{(m,n)}(z^k) = J_1(z^k)$ whenever $|k| \leq 2m+1$. If k has any component k_i that is odd, then $J_1(z^k) = 0$, and $R^{(m,n)}(z^k) = 0$ because ever term u_q^k in each of the symmetry sums $f\{u_p\}$ has a cancelling term - u_q^k. Therefore, the only monomials that need to be considered are of the form z^{2k}, with $|k| \leq m$. The uniqueness of $L^{(m,n-1)}(g,x)$ implies $L^{(m,n-1)}(x^k, x) = x^k$ whenever $|k| \leq m$, so $M^{(m,n)}(z^{2k}, z) = z^{2k}$, whenever $|k| \leq m$. Combining these results:

$$\begin{aligned} J_1(f) &= M^{(m,n)}(f,z)) \\ &= \sum_{|p|=m} w_p \, f\{u_p\} \\ &= R^{(m,n)}(z^k) \end{aligned}$$

whenever $f(z) = z^k$, with $|k| \leq 2m+1$, so $R^{(m,n)}(f)$ has polynomial degree 2m+1. For more details (cf. Genz [6]).

3.5 Application to Numerical approximation of $J(r, R)$

Since one of our goals is to estimate the integral (10), it is straightforward that the lemma (3.4) is applicable to the function f such that

$$f(z) = g(\|rRD^{\frac{-1}{2}}z - v\|^2).$$

Next, since we have that

$$f\{u_p\} = g(\|rRD^{\frac{-1}{2}}(s.u_p)^t - v\|^2),$$

by introducing the approximate function J_{u_p} that depend to the choice of u_p, (9) becomes

$$\begin{aligned} J(r,R) &\approx \sum_{|p|=m}\sum_{s} w_p \quad g(\|rRD^{\frac{-1}{2}}(s\cdot u_p) - v\|^2) \\ &=: J_{u_p}(r,R), \end{aligned} \tag{14}$$

where we denote $s \cdot u_p = (s_1 u_1, \dots, s_n u_n)^t$.

Remark 3.5 $J_{u_p}(r, R)$ is the numerical approximation of $J(r, R)$ as given in (14), is depend to the choice of interpolation points u_p on hypersphere. Recall that $J(r, R)$ was a fixed function that depend to R and the density function of our elliptic distribution.

By introducing H_{u_p}, the approximate function of H as define in (12) that depend of J_{u_p}, such that

$$\begin{aligned} H_{u_p}(s) &:= s^n \int_1^{\infty} r^{n-1} J_{u_p}(r,s)dr \\ &\approx H(s). \end{aligned} \tag{15}$$

When replace $H(s)$ in (13) by $H_{u_p}(s)$, we then prove the following result:

Theorem 3.6 *If we have a quadratic portfolio of securities (i.e longs equities) such that the Profit & Loss function over the time window of interest is, to good approximation, given by* $\Delta\Pi \approx \sum_{i=1}^{n} S_i(0)\cdot\theta_i(\eta_i(t) + \frac{\eta_i(t)^2}{2})$*, with portfolio weights* $\theta_i > 0$*. Suppose moreover that the joint log-returns is a random vector* $(\eta_1, \dots, \eta_n)$ *that follows a continuous elliptic distribution, with probability density as in (8), where* μ *is the vector mean and* Σ *is the variance-covariance matrix, and where we suppose that* $g(s^2)$ *is integrable over* $\mathbb{R}$*, continuous and nowhere 0. Then the approximate portfolio's quadratic elliptic* VaR^g_{α,u_p} *at confidence* $(1-\alpha)$ *is given by*

$$VaR^g_{\alpha,u_p} = \frac{P(0)}{2} - R^2_{g,u_p} \tag{16}$$

where R_{g,u_p} *is the unique solution of the equation*

$$H_{u_p}(s) = (1-\alpha)\cdot\sqrt{det(D)} = \frac{(1-\alpha)}{2^{n/2}}\sqrt{det(\Sigma)\prod_{i=1}^{n}\theta_i\cdot S_i(0)}. \tag{17}$$

In this case, we assume that our losses will not be greater than half-price of the portfolio at time 0.

Remark 3.7 The precedent theorem give to us an approximate Quadratic Portfolio Value-at-Risk (VaR^g_{α,u_p}) that depend to our choice of interpolation points on hypersphere, α and the function g. Therefore it is clear that the best choice of interpolation point will depend to the g function in (8).

With some simple calculus we have the following remark:

Remark 3.8

$$J_{u_p}(r,R) = \sum_{|p|=m} w_p \sum_s g\Big(a(s,u_p,R)\cdot r^2 - 2\cdot b(s,u_p,R,D,v)\cdot r + c(v)\Big) \tag{18}$$

for which $a(s,u_p,R,D) = \|RD^{\frac{-1}{2}}(s\cdot u_p)\|^2$, $b(s,u_p,R,D,v) = R < D^{\frac{-1}{2}}(s\cdot u_p), v >$, $c(v) = \|v\|^2$. Sometimes, for more simplification we will note a, b, c.

Since inequality of Schwartz give that $b^2 - ac < 0$, we use the change of variable by posing $b1 = \frac{b^2-ac}{a} < 0$, $u = r - \frac{b}{a}$, by using the binom of Newton, and by introducing the function $G^{j,g}$ for $j = 0,\dots,n-1$, such that we have the following remark:

Remark 3.9

$$J(r,R) = \sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n-1}{j} (b/a)^{n-1-j} G^{j,g}_{u_p,s}(R) \tag{19}$$

with

$$G^{j,g}_{u_p,s}(R) = \int_{1-\frac{b}{a}}^{\infty} z^j \cdot g(az^2 - b_1)dz \tag{20}$$

for which a, b and c are defined in (3.8).

By replace $d = b/a = \frac{<D^{\frac{-1}{2}}(s\cdot u_p),v>}{R\|D^{\frac{-1}{2}}(s.u_p)\|^2}$ by its value in (3.9), we obtain

$$G^{j,g}_{u_p,s}(R) = \int_{1-\frac{<D^{\frac{-1}{2}}(s\cdot u_p),v>}{R\|D^{\frac{-1}{2}}(s\cdot u_p)\|^2}}^{\infty} z^j\ g\Big(R^2\ \|D^{\frac{-1}{2}}(s\cdot u_p)\|^2 z^2 - \frac{< D^{\frac{-1}{2}}(s\cdot u_p), v >^2}{\|D^{\frac{-1}{2}}(s\cdot u_p)\|^2} + \|v\|^2\Big)dz, \tag{21}$$

and we therefore have the following theorem:

Theorem 3.10 *If we have a quadratic portfolio of securities (i.e longs equities) for which the joint securities log-returns changes with continuous elliptic distribution with pdf distribution as in* (8)*, then the approximate quadratic elliptic* $VaR^{\Lambda}_{\alpha,u_p}$ *at confidence* $(1-\alpha)$ *is given by*

$$VaR^{\Lambda}_{\alpha,u_p} \ = \ \frac{P(0)}{2} \ - \ R^2_{g,u_p} \tag{22}$$

where R_{g,u_p} *is the unique solution of the equation*

$$\sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n-1}{j} R^{j+1} \Big(\frac{< D^{\frac{-1}{2}}(s\cdot u_p), v >}{\|D^{\frac{-1}{2}}(s\cdot u_p)\|^2}\Big)^{n-1-j} G^{j,g}_{u_p,s}(R) = \frac{(1-\alpha)}{2^{n/2}} \sqrt{det(\Sigma)\prod_{i=1}^{n}\theta_i\cdot S_i(0)}. \tag{23}$$

In this case, we assume that our losses will not be greater than half-price of the portfolio at time 0.

Remark 3.11 We have reduced our problem to one dimensional integral equation. Therefore, to get an explicit equation to solve, we need to estimate $G^{j,g}_{u_p,s}(R)$ that depend to R with parameters g, u_p, v and D.

In the case of normal distribution or t-distribution, it will suffices to replace g in the expression of (20), an to estimate the one dimensional integral (20).

3.5.1 The case of normal distribution

In the case of normal distribution, the pdf is given by:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n|\Sigma|}} exp(-\frac{1}{2}(x-\mu)\Sigma^{-1}(x-\mu)^t) \tag{24}$$

and specific is given as follow

$$g(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{x}{2}} = C(n)\, e^{-\frac{x}{2}},$$

therefore it suffices to replace g in (20) then

$$G^j_{u_p,s}(R) = (2\pi)^{-\frac{n}{2}} e^{\frac{b_1}{2}} \int_{1-\frac{b}{a}}^{\infty} u^j e^{-\frac{au^2}{2}} du. \tag{25}$$

If $1-\frac{b}{a} > 0$ (it is the case when R is sufficiently big such that $|v| < R\|D^{\frac{-1}{2}}(s.u_p)\|$), and by replacing g in the formula (21), it becomes

$$\frac{G^j_{u_p,s}(R)}{\exp(-\frac{\|v\|^2}{2})(2\pi)^{-\frac{n}{2}}} = \exp\,(\frac{< D^{\frac{-1}{2}}(s.u_p), v >^2}{R\|D^{\frac{-1}{2}}(s^t u_p)\|^2})(2/a)^{\frac{1+j}{2}}\Gamma(\frac{j+1}{2}, \frac{(R\|D^{\frac{-1}{2}}(s^t u_p)\|)^2}{2}(1-\frac{< D^{\frac{-1}{2}}(s.u_p), v >}{R\|D^{\frac{-1}{2}}(s^t u_p)\|^2})^2\,). \tag{26}$$

Also, since $a = (R\|D^{\frac{-1}{2}}(s.u_p)\|)^2$, we therefore deduce the following theorem:

Theorem 3.12 *If we have a portfolio of securities (i.e longs equities), such that the Profit & Loss function over the time window of interest is, to good approximation, given by* $\Delta\Pi \approx \sum_{i=1}^n S_i(0)\cdot\theta_i(\eta_i(t)+\frac{\eta_i(t)^2}{2})$, *with portfolio weights* θ_i. *Suppose moreover that the joint log-returns is a random vector* $(\eta_1,\ldots,\eta_n)$ *that follows a continuous multivariate normal distribution with density function in (24), with the vector mean* μ *, and variance-covariance matrix* Σ*, the Quadratic Value-at-Risk* ($VaR^{\Lambda}_{\alpha,u_p}$) *at confidence* $1-\alpha$ *is given by the following formula*

$$VaR^{\Lambda}_{\alpha,u_p} \;=\; -R^2_{u_p,\alpha} \;+\; \frac{P(0)}{2}$$

for which $R_{u_p,\alpha}$ *is the unique solution of the following transcendental equation.*

$$2(1-\alpha)\frac{\sqrt{det(D)}}{(2\pi)^{\frac{n}{2}}} = \sum_{|p|=m} w_p \sum_s \frac{(< D^{\frac{-1}{2}}(s.u_p), v >)^{(n-j-1)}}{\|D^{\frac{-1}{2}}(s.u_p)\|^{(2n-1-j)}}\, e^{\frac{b_1}{2}} \sum_{j=0}^{n-1}\binom{n-1}{j}\Gamma\left(\frac{j+1}{2}, \frac{a}{2}(1-\frac{b}{a})^2\right), \tag{27}$$

for which $b_1 = \frac{(< D^{\frac{-1}{2}}(s.u_p), v>)^2}{\|D^{\frac{-1}{2}}(s.u_p)\|^2} - \|v\|^2$, $\frac{b}{a} = \frac{< D^{\frac{-1}{2}}(s.u_p), v>}{R\|D^{\frac{-1}{2}}(s.u_p)\|^2}$, $a = R^2\|D^{\frac{-1}{2}}(s.u_p)\|^2$. *In this case, we implicitly assume* $VaR_{\alpha;u_p} \leq P(0)/2$.

Recall the definition of the incomplete Γ-function:

$$\Gamma(z,w) = \int_w^{\infty} e^{-s} s^{z-1} ds. \tag{28}$$

3.5.2 Case of t-student distribution

If our elliptic distribution is in particular chosen as the multivariate t-student distribution, we will have density function given by

$$g(x) = \frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\nu/2)\sqrt{(\nu\pi)^n}}\Big(1+\frac{x}{\nu}\Big)^{(\frac{-\nu-n}{2})} \tag{29}$$

therefore by replacing g in (23), we obtain the equation

$$\sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n-1}{j} (b/a)^{n-1-j} \int_{1-\frac{b}{a}}^{\infty} u^j \Big(1 + \frac{au^2 - b_1^2}{\nu}\Big)^{(\frac{-\nu-n}{2})} du = \frac{(1-\alpha)\sqrt{det(D)}}{C(n,\nu)R^n}. \qquad (30)$$

suggesting $c_1 = \nu - b_1^2$, our equation is reduce to

$$\sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n-1}{j} (b/a)^{n-1-j} \int_{1-\frac{b}{a}}^{\infty} u^j \Big(au^2 + c_1\Big)^{(\frac{-\nu-n}{2})} du = \frac{(1-\alpha)\sqrt{det(D)}}{\nu^{\frac{\nu+n}{2}} C(n,\nu)R^n}. \qquad (31)$$

Changing variable in integral of 32 according to $v = u^2$ and $\beta = \frac{a}{c_1}$, we find that

$$R^n \sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n-1}{j} (b/a)^{n-1-j} c_1^{\frac{-n-\nu}{2}} \int_{(1-\frac{b}{a})^2}^{\infty} v^{\frac{j+1}{2}-1} \Big(\beta v + 1\Big)^{(\frac{-\nu-n}{2})} du = \frac{(1-\alpha)\pi^{n/2}\Gamma(\nu/2)\sqrt{det(D)}}{\nu^{\frac{\nu+n}{2}}\Gamma(\frac{\nu+n}{2})}. \qquad (32)$$

For the latter integral equation, we will use the following formula from [8]:

Lemma 3.13 *(cf. [8], formula 3.194(2)). If $|arg(\frac{u}{\beta})| < \pi$, and $Re(\nu_1) > Re(\mu) > 0$, then*

$$\int_u^{+\infty} x^{\mu-1}(1+\beta x)^{-\nu_1} dx = \frac{u^{\mu-\nu_1}\beta^{-\nu_1}}{\nu_1 - \mu} {}_2F_1(\nu_1, \nu_1-\mu; \nu_1-\mu+1; -\frac{1}{\beta \cdot u}). \qquad (33)$$

Here ${}_2F_1(\alpha;\beta,\gamma;w)$ is the hypergeometric function.

In our case, $\nu_1 = \frac{\nu+n}{2}$, $u = (1-b/a)^2$, $\nu_1 - \mu = \frac{n+\nu-j-1}{2}$, $\nu_1 - \mu + 1 = \frac{n+\nu-j+1}{2}$, so if we replace in (32), we deduce the following theorem:

Theorem 3.14 *If we have a portfolio of securities (i.e equities), such that the Profit & Loss function over the time window of interest is, to good approximation, given by $\Delta\Pi \approx \sum_{i=1}^n S_i(0)\cdot\theta_i(\eta_i(t) + \frac{\eta_i(t)^2}{2})$, with portfolio weights $\theta_i > 0$. Suppose moreover that the joint log-returns is a random vector $(\eta_1, \ldots, \eta_n)$ that follows a continuous multivariate t-distribution with density function given by (29), with vector mean μ , and variance-covariance matrix Σ, then the Quadratic Value-at-Risk ($VaR^{\Lambda}_{\alpha,u_p}$) at confidence $1-\alpha$ is given by the following formula*

$$VaR^{\Lambda}_{\alpha,u_p} = \frac{P(0)}{2} - R^2_{u_p,\alpha}$$

for which $R_{u_p,\alpha}$ is the unique solution of the following transcendental equation.

$$\frac{R^n}{(1-\alpha)} \sum_{|p|=m} w_p \sum_s \sum_{j=0}^{n-1} \binom{n}{j} \frac{(b/a)^{n-1-j}}{(\nu-b_1^2)^{\frac{-n-\nu}{2}}} \frac{{}_2F_1\left[\frac{n+\nu}{2}, \frac{n+\nu-j-1}{2}; \frac{n+\nu-j+1}{2}; \frac{b_1^2-\nu}{a(1-\frac{b}{a})^2}\right]}{(n+\nu-j-1)\sqrt{det(\Sigma)\prod_{i=1}^n \theta_i \cdot S_i(0)}} = \frac{(\pi/2)^{n/2}\Gamma(\nu/2)}{\nu^{\frac{\nu+n}{2}}\Gamma(\frac{\nu+n}{2})}, \qquad (34)$$

for which $b_1 = \frac{(<D^{\frac{-1}{2}}(s.u_p),v>)^2}{\|D^{\frac{-1}{2}}(s.u_p)\|^2} - \|v\|^2$, $\frac{b}{a} = \frac{<D^{\frac{-1}{2}}(s.u_p),v>}{R\|D^{\frac{-1}{2}}(s.u_p)\|^2}$, $a = R^2\|D^{\frac{-1}{2}}(s.u_p)\|^2$. In this case, we implicitly assume that $VaR_{\alpha,u_p} \le P(0)/2$

Remark 3.15 Note that, Hypergeometric ${}_2F_1$'s have been extensively studies, and numerical software for their evaluation is available in Maple and in Mathematica.

4 Quadratic VaR with mixture of elliptic Distributions

Mixture distributions can be used to model situations where the data can be viewed as arising from two or more distinct classes of populations; see also [9]. For example, in the context of Risk Management, if we divide trading days into two sets, quiet days and hectic days, a mixture model will be based on the fact that returns are moderate on quiet days, but can be unusually large or small on hectic days. Practical applications of mixture models to compute VaR can be found in Zangari (1996), who uses a mixture normal to incorporate fat tails in VaR estimation. In this section, we sketch how to generalize the preceding section to the situation where the joint log-returns follow a mixture of elliptic distributions, that is, a convex linear combination of elliptic distributions.

Definition 4.1 We say that $(X_1, \ldots, X_n)$ has a joint distribution that is the mixture of q elliptic distributions $N(\mu_j, \Sigma_j, \phi_j)^2$, with weights $\{\beta_j\}$ (j=1,..,q ; $\beta_j > 0$; $\sum_{j=1}^{q} \beta_j = 1$), if its cumulative distribution function can be written as

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \sum_{j=1}^{q} \beta_j F_j(x_1, \ldots, x_n)$$

with $F_j(x_1, \ldots, x_n)$ the cdf of $N(\mu_j, \Sigma_j, \phi_j)$.

Remark 4.2 In practice, one would usually limit oneself to $q = 2$, due to estimation and identification problems; see [9].

We will suppose that all our elliptic distributions $N(\mu_j, \Sigma_j, \phi_j)$ admit a pdf :

$$f_j(x) = |\Sigma_j|^{-1/2} g_j((x - \mu_j){\Sigma_j}^{-1}(x - \mu_j)^t) \tag{35}$$

for which each g_j is continuous integrable function over $\mathbb{R}$, and that the g_j never vanish jointly in a point of $\mathbb{R}^q$. The pdf of the mixture will then simply be $\sum_{j=1}^{q} \beta_j f_j(x)$.

Let $\Sigma_j = A_j^t\ A_j$, following (11), we introduce $J_k(r, R)$ such that

$$\begin{aligned} \alpha R^{-n} &= \sum_{k=1}^{q} \int_1^\infty r^{n-1} \Big[\int_{S_{n-1}} g_k(\|rRD_k^{\frac{-1}{2}}\xi - v_k\|^2) d\sigma(\xi) \Big] \frac{dr}{\sqrt{det(D_k)}} \\ &= \sum_{k=1}^{q} \int_1^\infty r^{n-1} J_k(r, R)\ dr. \end{aligned} \tag{36}$$

Next following (20), we introduce the function

$$G^{j,g}_{u_p,s,k}(R) = R^n \int_{1-\frac{b_k}{a_k}}^{\infty} z^j\ g_k(a_k z^2 - b_{1k})\ dz, \tag{37}$$

where $a_k = \|RD^{\frac{-1}{2}}(s.u_{pk})\|^2$, $b_k = R < D_k^{\frac{-1}{2}}(s.u_{pk}), v >$,$c_k = \|v_k\|^2$,$b_{1k} = \frac{b_k^2 - a_k c_k}{a_k}$. We have then deduce the following corollary:

Theorem 4.3 *If we have a portfolio of securities (i.e equities) such that the Profit & Loss function over the time window of interest is, to good approximation, given by* $\Delta\Pi \approx \sum_{i=1}^{n} S_i(0) \cdot \theta_i(\eta_i(t) + \frac{\eta_i(t)^2}{2})$, *with portfolio weights* θ_i. *Suppose moreover that the joint log-returns is a random vector* $(\eta_1, \ldots, \eta_n)$ *is a mixture of q elliptic distributions, with density*

$$h(x) = \sum_{j=1}^{q} \beta_j |\Sigma_j|^{-1/2} g_j((x - \mu_j)\Sigma_j^{-1}(x - \mu_j)^t),$$

[2] or $N(\mu_j, \Sigma_j, g_j)$ if we parameterize elliptical distributions using g_j instead of ϕ_j

where μ_j is the vector mean, and Σ_j the variance-covariance matrix of the j-th component of the mixture. We suppose that each g_j is integrable function over $\mathbb{R}$, and that the g_j never vanish jointly in a point of $\mathbb{R}^m$. Then the value-at-Risk, or quadratic mixture-elliptic VaR*, at confidence $1-\alpha$ is given as the solution of the transcendental equation*

$$\sum_{k=1}^{q} \sum_{|p|=m} w_p \sum_{s} \sum_{j=0}^{n-1} \binom{n-1}{j} (b_k/a_k)^{n-1-j} \frac{G^{j,g}_{u_p,s,k}\Big((\frac{P(0)}{2} - VaR^g_{\alpha,u_p})^{1/2}\Big)}{\sqrt{det(\Sigma_k) \prod_{i=1}^n \theta_i \cdot S_i(0)}} = \frac{(1-\alpha)}{2^{n/2}}, \tag{38}$$

for which $G^{j,g}_{u_p,s,k}$ is defined in (37). In this case, we assume that our losses will not be greater than half-price of the portfolio at time 0.

Remark 4.4 One might, in certain situations, try to model with a mixture of elliptic distributions which all have the same variance-covariance and the same mean, and obtain for example a mixture of different tail behaviors by playing with the g_j's.

The preceding can immediately be specialized to a mixture of normal distributions: the details will be left to the reader.

4.1 Application with mixture of Student t-Distributions

We will consider a mixture of q Student t-distributions such that, the k^{th} density function $i = 1, \ldots, q$ will be given by

$$g_k(x) = \frac{\Gamma(\frac{\nu_k+n}{2})}{\Gamma(\nu_k/2).\pi^{n/2}} \Big(1 + \frac{x}{\nu_k}\Big)^{(\frac{-\nu_k-n}{2})} = C(n,\nu_k)\Big(1 + \frac{x}{\nu_k}\Big)^{(\frac{-\nu_k-n}{2})}, \tag{39}$$

and $\Sigma_k = A_k^t\, A_k$ therefore by replacing g_k by g in (40) and since integration is a linear operation, we obtain the following theorem

Theorem 4.5 *If we have a portfolio of securities (i.e equities) such that the Profit & Loss function over the time window of interest is, to good approximation, given by $\Delta\Pi \approx \sum_{i=1}^n S_i(0)\cdot\theta_i(\eta_i(t) + \frac{\eta_i(t)^2}{2})$, with portfolio weights θ_i. Suppose moreover that the joint log-returns is a random vector $(\eta_1, \ldots, \eta_n)$ is a mixture of q t-distributions, with density*

$$h(x) = \sum_{j=1}^{q} \beta_j |\Sigma_j|^{-1/2} \frac{\Gamma(\frac{\nu_j+n}{2})}{\Gamma(\nu_j/2).\pi^{n/2}} \Big(1 + \frac{(x-\mu_j)\Sigma_j^{-1}(x-\mu_j)^t}{\nu_j}\Big)^{-\frac{n+\nu_j}{2}},$$

where μ_j is the vector mean, and Σ_j the variance-covariance matrix of the j-th component of the mixture. We suppose that each g_j is integrable function over $\mathbb{R}$, and that the g_j never vanish jointly in a point of $\mathbb{R}^m$. Then the value-at-Risk, or quadratic mixture-student VaR*, at confidence $1-\alpha$ is given by :*

$$R^2_{u_p} = -VaR_\alpha + \frac{P(0)}{2}$$

for which R_{u_p} is the unique positive solution of the following equation:

$$\sum_{k=1}^{q} \sum_{|p|=m} \frac{w_p\Gamma(\frac{\nu_k+n}{2})}{\Gamma(\nu_k/2)} \sum_{s} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{R^n (b_k/a_k)^{n-1-j}}{((\nu_k - b_{1k}^2)/\nu_k)^{\frac{-n-\nu}{2}}} \frac{{}_2F_1\left[\frac{n+\nu_k}{2}, \frac{n+\nu_k-j-1}{2}; \frac{n+\nu_k-j+1}{2}; \frac{b_{1k}^2-\nu}{a_k(1-\frac{b_k}{a_k})^2}\right]}{(n+\nu_k-j-1)\sqrt{|\Sigma_k| \prod_{i=1}^n \theta_i \cdot S_i(0)}} = \frac{(1-\alpha)}{(\pi/2)^{\frac{-n}{2}}}, \tag{40}$$

for which $b_{1k} = \frac{(<D_k^{\frac{-1}{2}}(s.u_{pk}), v_k>)^2}{\|D_k^{\frac{-1}{2}}(s.u_{pk})\|^2} - \|v_k\|^2$, $\frac{b_k}{a_k} = \frac{<D_k^{\frac{-1}{2}}(s.u_{pk}), v_k>}{R\|D_k^{\frac{-1}{2}}(s.u_{pk})\|^2}$, $a_k = R^2\|D_k^{\frac{-1}{2}}(s.u_{pk})\|^2$ and $det(\Sigma_k) = |\Sigma_k|$. In this case, we implicitly assume that our losses will not be greater than $P(0)/2$.

5 Elliptic Quadratic Expected Shortfall for portfolio of securities

Expected Shortfall is a sub-additive risk statistic that describes how large losses are on average when they exceed the VaR level. Expected Shortfall will therefore give an indication of the size of extreme losses when the VaR threshold is breached. We will evaluate the expected Shortfall for a quadratic portfolio of securities under the hypothesis of elliptically distributed risk factors. Mathematically, the expected Shortfall associated with a given VaR is defined as:

$$\text{Expected Shortfall } = \mathbb{E}(-\Delta\Pi| - \Delta\Pi > VaR),$$

see for example [9]. Assuming again a multivariate elliptic pdf $f(x) = |\Sigma|^{-1}g((x-\mu)\Sigma^{-1}(x-\mu)^t)$, the Expected Shortfall at confidence level $1-\alpha$ is given by:

$$\begin{aligned}
-ES_\alpha &= \mathbb{E}(\Delta\Pi \mid \Delta\Pi \le -VaR_\alpha) \\
&= \frac{1}{\alpha}\mathbb{E}\left(\Delta\Pi \cdot 1_{\{\Delta\Pi \le -VaR_\alpha\}}\right) \\
&= \frac{1}{\alpha}\int_{\{(x,\Lambda.x)-P(0)/2 \le -VaR_\alpha\}} ((x,\Lambda.x) - P(0)/2)\ h_1(x)\ dx \\
&= \frac{|\Sigma|^{-1/2}}{\alpha}\int_{\{(x,\Lambda.x) \le -VaR_\alpha + P(0)/2\}} ((x,\Lambda.x) - P(0)/2)\ g((x-\mu-1)\Sigma^{-1}(x-\mu-1)^t)dx.
\end{aligned}$$

Using the definition of VaR_α and by replace $\Delta\Pi = (X,\Lambda.X) - \frac{P(0)}{2}$, with random vector X define in section 2,

$$ES_\alpha = \frac{P(0)}{2} - \frac{|\Sigma|^{-1/2}}{\alpha}\int_{\{(x,\Lambda.x) \le -VaR_\alpha + P(0)/2\}} (x,\Lambda.x)\ g((x-\mu-1)\Sigma^{-1}(x-\mu-1)^t)dx. \tag{41}$$

Let $\Sigma = A^t\ A$, as before. Doing the same linear changes of variables as in section 2 and section 3, we arrive at:

$$\begin{aligned}
ES_\alpha &= \frac{P(0)}{2} - \frac{R^{n+2}|D|^{-1/2}}{\alpha}\int_0^1 r^{n+1}\Big[\int_{S_{n-1}} g(\|rRD^{\frac{-1}{2}}\xi - v\|^2)d\sigma(\xi)\Big]dr \\
&= \frac{P(0)}{2} - \frac{R^{n+2}|D|^{-1/2}}{\alpha}\int_0^1 r^{n+1}J(r,R)dr \\
&\approx \frac{P(0)}{2} - \frac{R^{n+2}|D|^{-1/2}}{\alpha}\int_0^1 r^{n+1}J_{u_p}(r,R)dr \\
&= \frac{P(0)}{2} - \frac{R^{n+2}|D|^{-1/2}}{\alpha}\sum_{|p|=m}\sum_s w_p \quad \int_0^1 r^{n+1}g(\|rRD^{\frac{-1}{2}}(s.u_p)^t - v\|^2) \quad dr,
\end{aligned}$$

By introducing the function $Q^g_{u_p,s}$ such that

$$Q^g_{u_p,s}(R) = R^{n+2}\int_0^1 r^{n+1}g(\|rRD^{\frac{-1}{2}}(s.u_p)^t - v\|^2) \quad dr \tag{42}$$

we deduce the following theorem:

Theorem 5.1 *Suppose that the portfolio is quadratic in the risk-factors* $X = (X_1,\cdots,X_n)$*:* $\Delta\Pi = (X,\Lambda\cdot X) - \frac{P(0)}{2}$ *and that* $X \sim N(\mu+1,\Sigma,\phi)$*, with pdf* $f(x) = |\Sigma|^{-1}g((x-\mu-1)\Sigma^{-1}(x-\mu-1)^t)$*. If the* VaR_α *is given, then the expected Shortfall at level* α *is given by :*

$$ES_\alpha = \frac{P(0)}{2} - \frac{|D|^{-1/2}}{\alpha}\sum_{|p|=m}\sum_s w_p \quad Q^g_{u_p,s}\Big(\big(\frac{P(0)}{2} - VaR_\alpha\big)^{1/2}\Big). \tag{43}$$

we introduce I_1^g and I_2^g such that

$$R^{-n-2}Q^g_{u_p,s}(R) = \int_0^1 r^{n+1}g(ar^2-2br+c)dr = \int_0^\infty - \int_1^\infty = I^g_{1,u_p,s}(R) - I^g_{2,u_p,s}(R)$$

Following the Integral (20)

$$I^g_{2,u_p,s}(R) = \sum_{j=0}^{n+1}\binom{n+1}{j}(b/a)^{n+1-j}\; G^{j,g}_{u_p,s}(R),$$

for which a,b and c are defined in remark (3.8) and

$$I^g_{1,u_p,s}(R) = \int_0^\infty r^{n+1}g(ar^2-2br+c)\;dr.$$

5.1 Expected Shortfall with normal distribution

In the case of normal distribution, the pdf is given by (24) and the specific g is given as follow

$$g(x) = (2\pi)^{-\frac{n}{2}}e^{-\frac{x}{2}} \;=\; C(n)e^{-\frac{x}{2}}$$

therefore it suffices to replace g in (20) then

$$\begin{aligned} G^j_{u_p,s}(R) &= (2\pi)^{-\frac{n}{2}}\exp\Big(\frac{b^2-ac}{2a}\Big)\int_{1-\frac{b}{a}}^\infty u^j\exp(-\frac{au^2}{2})du \\ &= (2\pi)^{-\frac{n}{2}}\exp\Big(\frac{b^2-ac}{2a}\Big)(2/a)^{\frac{1+j}{2}}\Gamma\Big(\frac{j+1}{2},\frac{a}{2}(1-\frac{b}{a})^2\Big). \end{aligned}$$

By using the following lemma

Lemma 5.2 *(cf. [8], formula 3.462(1)). If Re(ν) > 0, and Re(β) > 0 , then*

$$\int_0^{+\infty} x^{\nu-1}\exp(-\beta x^2-\lambda x)dx = (2\beta)^{-\nu/2}\Gamma(\nu)\exp\Big(\frac{\lambda^2}{8\beta}\Big)\mathbb{D}_{-\nu}\Big(\frac{\lambda}{\sqrt{2\beta}}\Big). \tag{44}$$

Here $\mathbb{D}_{-\nu}$ is the parabolic cylinder function with

$$\mathbb{D}_{-\nu}(z) = 2^{\frac{-\nu}{2}}e^{\frac{-z^2}{2}}\Big[\frac{\sqrt{\pi}}{\Gamma(\frac{1+\nu}{2})}\Phi\Big(\frac{\nu}{2},\frac{1}{2};\frac{z^2}{2}\Big) - \frac{\sqrt{2\pi}z}{\Gamma(\frac{\nu}{2})}\Phi\Big(\frac{1+\nu}{2},\frac{3}{2};\frac{z^2}{2}\Big)\Big],$$

where Φ is the confluent hypergeometric function (for more details see [8] page 1018).

We next obtain

$$I^g_{2,u_p,s}(R) = (2\pi)^{-\frac{n}{2}}\sum_{j=0}^{n+1}\binom{n+1}{j}(b/a)^{n+1-j}\cdot\exp\Big(\frac{b^2-ac}{4a}\Big)(2/a)^{\frac{1+j}{2}}\Gamma\Big(\frac{j+1}{2},\frac{a}{2}(1-\frac{b}{a})^2\Big)$$

and

$$\begin{aligned} I^g_{1,u_p,s}(R) &= (2\pi)^{-\frac{n}{2}}\exp(-\frac{\|v\|^2}{2})\int_0^1 r^{n+1}\exp\Big(-\frac{ar^2-2br}{2}\Big)dr \\ &= (2\pi)^{-\frac{n}{2}}\exp(-\frac{\|v\|^2}{2})a^{\frac{n+2}{2}}\Gamma(n+2)\exp\Big(\frac{b^2}{4a}\Big)\mathbb{D}_{-n-2}\Big(\frac{-b}{\sqrt{a}}\Big), \end{aligned}$$

for which $\mathbb{D}_{-n-2}$ is the parabolic cylinder function.

We have therefore prove the following result:

Theorem 5.3 *Suppose that the portfolio is quadratic in the risk-factors* $X = (X_1, \cdots, X_n)$*:* $\Delta\Pi = (X, \Lambda \cdot X) - \frac{P(0)}{2}$ *and that* X *is a multivariate normal distribution, If the* VaR_α^Λ *is given, then the expected Shortfall at level* α *is given by :*

$$ES_\alpha = \frac{P(0)}{2} - \frac{R^{n+2}|D|^{-1/2}}{\alpha} \sum_{|p|=m} \sum_s w_p \Big[I^g_{1,u_p,s}(R) - I^g_{2,u_p,s}(R) \Big], \tag{45}$$

for which $R = \sqrt{\frac{P(0)}{2} - VaR_\alpha}$

The preceding can immediately be specialized to a mixture of normal distributions. The details will be left to the reader.

5.2 Student t-distribution Quadratic Expected Shortfall

Following the precedent section 3, 4, 5, and particularly the lemma (33), the application can be specialized to a Student t-distribution. The details will be left to the reader.

5.3 How to choose an interpolation points u_p on hypersphere

In order to obtain a good approximation of our integral, one will choose the points of interpolation u_p of our g function such that our approximation is the best as possible.

6 Conclusion

By following the notion of $\Delta - \Gamma$ Portfolio that contains derivatives instruments, we have introduced a Quadratic Portfolios of securities (i.e equities) without the use of Delta and Gamma Greeks. By using the assumption that the joint securities log-returns follow a mixture of elliptic distributions, we have reduced the estimation of VaR of such quadratic portfolio, to the resolution of a multiple integral equation, that contain a multiple integral over hypersphere. To approximate a multiple integral over hypersphere, we propose to use a numerical approximation method given by Alan Genz in[6]. Therefore, the estimation of VaR is reduced to the resolution of one dimensional integral equation. To illustrate our method, we give special attention to mixture of normal distribution and mixture of multivariate t-student distribution. In the case of t-distribution, we need the hypergeometric special function. For given VaR, we also show how to estimate the Expected Shortfall of the Quadratic portfolio without derivatives instruments, when the risk factors follow an elliptic distributions and we illustrate our proposition with normal distribution by using the parabolic cylinder function. Note that this method will be applicable to capital allocation, if we could consider an institution as a portfolio of multi-lines businesses, and if use as risk factors the the joint log returns of such multi-lines businesses values. In the later paper we postpone to give some explicit details when we are concerned with long/short portfolios of equities. A concrete application need the estimation of w_p such that $|p| = m$, therefore we send the reader to Alan Genz [6].

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