A NOTE ON POINTWISE PROJECTIVE MODULES

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ABSTRACT. let $\mathcal{R}$ be any ring, and $\mathcal{M}$ is a (left) $\mathcal{R}$–module. $\mathcal{M}$ is said to be *pointwise projective*, if for each epimorphism $\alpha : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are any $\mathcal{R}$–modules, and for each homomorphism $\beta : \mathcal{M} \longrightarrow \mathcal{A}$, then for every $m \in \mathcal{M}$, there exists a homomorphism $\varphi : \mathcal{M} \longrightarrow \mathcal{A}$, which may depend on m, such that $\alpha \circ \varphi(m) = \beta(m)$. In this paper we study some properties of pointwise projective modules.

INTRODUCTION

projective modules have been studied extensively, and many generalization for the concept of a projective module were given, for example, improjective modules [2], quasiprojective modules [10] and nearly–projective modules [7]. Another generalization to the concept of a projective module is the concept of locally projective module, Zimmermann [13], and Azumaya [1]. In this paper we will call a locally projective module, pointwise projective module, and is abbreviated by *pwp. module*, and we study some of the properties of these modules.
In this paper $\mathcal{R}$ stands for a commutative ring with 1, unless otherwise stated, and a module means a (left) $\mathcal{R}$–module.

1. POINTWISE PROJECTIVE MODULES: BASIC RESULTS

Let $\mathcal{R}$ be any ring, not necessary commutative, and $\mathcal{M}$ is a (left) $\mathcal{R}$–module. We recall that the $\mathcal{R}$–module $\mathcal{M}$ is said to be pointwise projective (abbreviated by pwp.), if for each epimorphism $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are any $\mathcal{R}$–modules, and for each homomorphism $\beta : \mathcal{M} \longrightarrow \mathcal{B}$, then for each $m \in \mathcal{M}$, there exists a homomorphism $\varphi : \mathcal{M} \longrightarrow \mathcal{A}$, which may depend on m, such that $\alpha \circ \varphi(m) = \beta(m)$ diagrammatically:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\alpha"] & \mathcal{B} \arrow[r] & 0 \\
 & \mathcal{M} \arrow[u, "\beta"'] \arrow[ul, dotted, "\varphi"] &
\end{tikzcd}$$

see [1]. Note that every projective module is pwp., however, the converse is false as we will see later. Azumaya and Zimmermann gave the following characterization for pwp. modules [1] and [13].

Proposition 1.1. *Let $\mathcal{M}$ be any $\mathcal{R}$–module, the following statements are equivalent:*

1. *$\mathcal{M}$ is a pwp. module.*
2. *Each epimorphism $\alpha : \mathcal{A} \longrightarrow \mathcal{M}$, where $\mathcal{A}$ is any $\mathcal{R}$–module, is pointwise split (i.e. for each $b \in \mathcal{M}$, there exists a homomorphism $\beta : \mathcal{M} \longrightarrow \mathcal{A}$, which may depend on b, such that $\alpha \circ \beta(b) = b$).*
3. *For each $m \in \mathcal{M}$, there are $x_1, x_2, \ldots, x_n \in \mathcal{M}$ and $\varphi_1\varphi_2 \ldots, \varphi_n \in \mathcal{M}^*$ such that $m = \sum_{i=1}^{n} \varphi_i(m)x_i$.*
4. *for each $m_1, m_2, \ldots, m_t \in \mathcal{M}$, there are $x_1, x_2, \ldots, x_n \in \mathcal{M}$ and $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathcal{M}^*$ such that $m_i = \sum_{k=1}^{n} \varphi_k(m_i)x_k$ for all $1 \le i \le t$.*

Remark. Part (4) in the last proposition is called *the Dual–Basis–Lemma for pwp. modules.*

Next we give another characterization of pwp. modules.

Proposition 1.2. *Let $\mathcal{M}$ be any $\mathcal{R}$–module, then $\mathcal{M}$ is pwp. if and only if for each free $\mathcal{R}$–module $\mathcal{F}$, every epimorphism $\alpha : \mathcal{F} \longrightarrow \mathcal{M}$ is pointwise split.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Let $\mathcal{A}$ be any $\mathcal{R}$–module with epimorphism $\beta : \mathcal{A} \longrightarrow \mathcal{M}$, let $\{a_i \big| i \in I\}$ be a set of generators for $\mathcal{A}$, and let $\mathcal{F}$ be a free $\mathcal{R}$–module with basis $\{z_i \big| i \in I\}$. Define $\gamma : \mathcal{F} \longrightarrow \mathcal{M}$ is an epimorphism, hence it is a pointwise split, i.e. for each $m \in \mathcal{M}$, there exists a homomorphism $\varphi : \mathcal{M} \longrightarrow \mathcal{F}$, φ may depend on m, such that $\beta \circ \gamma \circ \varphi(m) = m$. Hence by (1.1), $\mathcal{M}$ is pwp. $\square$

It is known that $\oplus_{i\in I}\mathcal{M}_i$ is a projective $\mathcal{R}$–module if and only if $\mathcal{M}_i$ is a projective $\mathcal{R}$–module for all $i \in I$ [4, p.118]. A similar result holds for pwp. modules.

Proposition 1.3. *$\oplus_{i\in I}\mathcal{M}_i$ is a pwp. $\mathcal{R}$–module if and only if $\mathcal{M}_i$ is a pwp. $\mathcal{R}$–module for all $i \in I$.*

Proof. Let $\alpha \in \oplus_{i\in I}\mathcal{M}_i$ be written as $\alpha = [m_i]$; $m_i \in \mathcal{M}_i$ for all $i \in I$.

($\Rightarrow$) In the diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\alpha"] & \mathcal{B} \arrow[r] & 0 \\
\oplus\mathcal{M}_i \arrow[u, "h"] \arrow[r, shift left, "\gamma_j"] & \mathcal{M}_i \arrow[u, "\beta"'] \arrow[ul, dashed, "\varphi"'] \arrow[l, shift left, "\eta_i"] &
\end{tikzcd}$$

Define $\gamma_j : \oplus_{i\in I}\mathcal{M}_i \to \mathcal{M}_i$ as $\gamma_j([m_i]) = m_j$. By assumption, for all $f \in \oplus_{i\in I}\mathcal{M}_i$ there exists a homomorphism $h : \oplus_{i\in I}\mathcal{M}_i \to \mathcal{A}$ such that $\alpha \circ h(f) = \beta \circ \gamma_j(f)$. Define $\eta_i : \mathcal{M}_i \to \oplus_{i\in I}\mathcal{M}_i$ as

$$[\eta_i(m)](j) = \begin{cases} m & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\varphi : \mathcal{M}_i \to \mathcal{A}$ as $\varphi(m) = h \circ \eta_i(m)$. Let $m \in \mathcal{M}_i$,

$$\alpha \circ \varphi(m) = \beta \circ \gamma_j \circ \eta_i(m) = \beta(m),$$

because $\gamma_j \circ \eta_i = I$.

($\Leftarrow$) In the diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\alpha"] & & \mathcal{B} \arrow[r] & 0 \\
\mathcal{M}_i \arrow[u, "h_i"] \arrow[rr, "\eta_i"'] & & \oplus\mathcal{M}_i \arrow[u, "\beta"'] \arrow[ull, dashed, "k"]
\end{tikzcd}$$

By assumption, for each $m \in \mathcal{M}_i$, there exists a homomorphism $h_i : \oplus_{i\in I}\mathcal{M}_i \to \mathcal{A}$ as $k(f) = \sum_{i\in I} h_i f(i)$, clearly k is well–defined. Now,

$$\alpha \circ k(f) = \sum_{i\in I} \alpha \circ h_i(f) = \beta\left(\sum_{i\in I} f(i)\right) = \beta(f),$$

by the definition of η_i. $\square$

Recall that an $\mathcal{R}$–module $\mathcal{M}$ is called *torsionless* if $\cap \ker f = 0$, where the intersection is taken over all $f \in \mathcal{M}^* = Hom(\mathcal{M}, \mathcal{R})$. it is known that every projective module is torsionless. The Dual–Basis–Lemma for pwp. modules gives

Proposition 1.4. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$–module, then $\mathcal{M}$ is torsionless.*

Note, from now on we will assume that the ring $\mathcal{R}$ is commutative with 1. We shall investigate the behaviour of the pwp. property under localization.

Proposition 1.5. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$–module, let $\mathcal{S}$ be a multiplicatively closed subset of $\mathcal{R}$, then $\mathcal{S}^{-1}\mathcal{M}$ is a pwp. $\mathcal{S}^{-1}\mathcal{R}$–module. In particular, $\mathcal{M}_P$ is a pwp. $\mathcal{R}_P$–module for each prime ideal P of $\mathcal{R}$.*

Proof. let $n/s \in \mathcal{S}^{-1}\mathcal{M}$, by (1.1) $n = \sum_{i=1}^{k} \varphi_i(n)x_i$, $x_i \in \mathcal{M}$, $\varphi_i \in \mathcal{M}^*$. Then

$$\frac{n}{s} = \sum_{i=1}^{k} \frac{\varphi_i(n)}{s} x_i.$$

Define $g_i : \mathcal{S}^{-1}\mathcal{M} \to \mathcal{S}^{-1}\mathcal{R}$ by $g_i(n/s) = \varphi_i(n)/s$, for all i; $1 \le i \le k$. Note that,

$$\frac{n}{s} = \sum_{i} g_i\left(\frac{n}{s}\right)\frac{x_i}{1}.$$

Hence by (1.1), $\mathcal{S}^{-1}\mathcal{M}$ is pwp. $\square$

Remark. The converse of the last proposition is not true, see example (5.5).

2. The Jacaobson Radicals of Pointwise Projective Modules

Azumaya [1] proved that if $\mathcal{M}$ is a pwp. $\mathcal{R}$–module then $J(\mathcal{M}) = J(\mathcal{R})\mathcal{M}$, thus every pwp. module has a maximal submodule. Also he proved that $J(\mathcal{M}) = \mathcal{M}$ only if $\mathcal{M} = 0$, thus we have

Proposition 2.1. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$–module, then $\mathcal{M}$ has a unique maximal submodule if and only if $J(\mathcal{M})$ is a maximal submodule of $\mathcal{M}$.*

Proof. Since $\mathcal{M}$ is pwp., then $\mathcal{M}$ has a maximal submodule, say U. If U is unique, then $U = J(\mathcal{M})$. Now, if $J(\mathcal{M})$ is a maximal in $\mathcal{M}$, then $J(\mathcal{M})$ is a unique maximal submodule. □

Recall that a non–zero submodule $\mathcal{N}$ of an $\mathcal{R}$–module $\mathcal{M}$ is call *large* if $\mathcal{N} \cap \mathcal{K} \neq 0$ for every non–zero submodule $\mathcal{K}$ of $\mathcal{M}$ [4, p.106]. A large ideal of $\mathcal{R}$ is just a large submodule of the $\mathcal{R}$–module $\mathcal{R}$. Put $Z(\mathcal{M}) = \{m \in \mathcal{M} \big| ann(m) \text{ is large in } \mathcal{R}\}$, $Z(\mathcal{M})$ is called *the singular submodule of* $\mathcal{M}$ [4, p.138]. It is known that if $\mathcal{A}_i$ is large in $\mathcal{M}$; $i = 1, 2, \cdots, n$, then $\cap_{i=1}^n \mathcal{A}_i$ is large in $\mathcal{M}$ [4, p.109]. We have

Proposition 2.2. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$–module, then $Z(\mathcal{M}) = Z(\mathcal{R})\mathcal{M}$.*

Proof. Let $m \in Z(\mathcal{M})$, by (1.1) $m = \sum_{i=1}^n \varphi_i(m) y_i$; $y_i \in \mathcal{M}$ and $\varphi_i \in \mathcal{M}^*$. Clearly, $ann(m) \subseteq ann(\varphi_i(m))$, hence $ann(\varphi_i(M))$ is large in $\mathcal{R}$. Thus $\varphi_i(m) \in Z(\mathcal{R})$, therefore $m \in Z(\mathcal{R})\mathcal{M}$. Mow, let $y \in Z(\mathcal{R})\mathcal{M}$, then $y = \sum_{i=1}^n r_i m_i$; $r_i \in Z(\mathcal{R})$, $m_i \in \mathcal{M}$. Thus $\cap_{i=1}^n ann(r_i)$ is large in $\mathcal{R}$ and it is a subset of $ann(y)$, i.e. $ann(y)$ is large in $\mathcal{R}$. Hence $y \in Z(\mathcal{M})$. □

Lemma 2.3. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$–module, and let I be an ideal of $\mathcal{R}$ such that $\mathcal{M} = I\mathcal{M}$, then for all $a \in \mathcal{M}$, $ann(a) + I = \mathcal{R}$*

Proof. Assume that $ann(a) + I \neq \mathcal{R}$, then there exists a maximal ideal $\mathcal{A}$ of $\mathcal{R}$ such that $ann(a) + I \subseteq \mathcal{A}$. Hence $\mathcal{M} = I\mathcal{M} \subseteq \mathcal{A}\mathcal{M}$ and thus $\mathcal{M} = \mathcal{A}\mathcal{M}$. By (1.5) and [1] $\mathcal{M}_{\mathcal{A}} = J(\mathcal{M}_{\mathcal{A}})$, and by [1] $\mathcal{M}_{\mathcal{A}} = 0$. Hence there exists $r \in \mathcal{R} - \mathcal{A}$ such that $ra = 0$, and that is contradiction. □

Corollary 2.4. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$–module, and let I be an ideal of $\mathcal{R}$ such that $\mathcal{M} = I\mathcal{M}$, then for all $a \in \mathcal{M}$, $a \in Ia$.*

Proof. By (2.3) for all $a \in \mathcal{M}$, $ann(a) + I = \mathcal{R}$, thus there exist $r \in ann(a)$ and $b \in I$ such that $r + b = 1$, hence $ba = a$. □

3. The Trace of Pointwise Projective Module

Recall that the trace of an $\mathcal{R}$–module $\mathcal{M}$ is defined by $\sum f(\mathcal{M})$, where the sum is taken over all $f \in \mathcal{M}^*$, and is denoted by $T(\mathcal{M})$ (simply T). It is known that if $\mathcal{M}$ is a projective $\mathcal{R}$–module, then $T\mathcal{M} = \mathcal{M}$, and $ann(T) = ann(\mathcal{M})$ [3].

Proposition 3.1. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$–module, then:*

(1) $T\mathcal{M} = \mathcal{M}$.
(2) $T^2 = T$.
(3) $ann(T) = ann(\mathcal{M})$.
(4) T *is a pure ideal.*

Proof. (1) Clearly $T\mathcal{M} \subseteq \mathcal{M}$. It follows by (1.1) $\mathcal{M} \subseteq \mathcal{M}$.

(2) Clearly $T^2 \subseteq T$. Let $t = \sum_{i=1}^k \ell_i(m_i)$; $\ell_i \in \mathcal{M}^*$ and $m_i \in \mathcal{M}$. By assumption $m_i = \sum_{i=1}^n \varphi_j(m_i) y_i$; for all i, $1 \leq i \leq k$, where $y_i \in \mathcal{M}$, $\varphi_i \in \mathcal{M}^*$. Thus $t = \sum_i \sum_j \varphi_j(m_i)\ell_i(y_j)$, hence $t \in T^2$.
(3) The proof is simple.
(4) Let $x \in T$, then $x = \sum_i \varphi_i(a_i)$; $\varphi_i \in \mathcal{M}^*$ and $a_i \in \mathcal{M}$. By (2.4) and (1) above $a_i = y a_i$; $y \in T$. Thus $x = \sum_i \varphi_i(y a_i) = y\left(\sum_i \varphi_i(a_i)\right) = yx$.

□

Before we give the last proposition in this section we recall some notation. For each $a \in \mathcal{M}$, $\mathcal{D}_a = ann(ann(a))$, $\bar{\mathcal{D}} = \sum_{a\in\mathcal{M}} \mathcal{D}_a$. It is easy to see if $\mathcal{M}$ is any module, then $T \subseteq \mathcal{D}_a$.

Proposition 3.2. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$–module, then $T = \bar{\mathcal{D}}$.*

Proof. Let $m \in \mathcal{M}$, $x \in \mathcal{D}_m$. By (2.3) and (3.1) there exist $r \in ann(m)$ and $t \in T$ such that $r + t = 1$ thus $xt = x$, hence $\mathcal{D}_m \subseteq T$, and $T = \bar{\mathcal{D}}$. □

Example 3.3. For the $\mathbb{Z}$–module $\mathbb{Q}$ of rational, $T(\mathbb{Q}) = 0$ and $\mathcal{D}_a(\mathbb{Q}) = \mathbb{Z}$ for each $a \in \mathbb{Z}$. This shows that the last proposition is false without pointwise projectivity.

4. Projective Modules And Pointwise Projective Modules

As we saw in (§1) every projective module is a pwp. module, but the converse is not true as we see in (5.4). In this section we study conditions under which pwp. modules are projective module.

Proposition 4.1. *Every finity generated pwp. $\mathcal{R}$–module is a projective module.*

Proof. Let $\{m_1, m_2, \cdots, m_t\}$ be a finite set of generators fo the $\mathcal{R}$–module $\mathcal{M}$. By (1.1) $m_j = \sum_{i=1}^{k} \varphi_i(m_j)x_i$; $x_i \in \mathcal{M}$, $\varphi_i \in \mathcal{M}^*$ for all j; $1 \leq j \leq t$. Let $y \in \mathcal{M}$, then $y = \sum_{j=1}^{t} r_j m_j = \sum_{j=1}^{t}\sum_{i=1}^{k} r_j\varphi_i(m_j)x_i = \sum_i \varphi_i(y)x_i$. Thus by the Dual–Basis–Lemma $\mathcal{M}$ is a projective module. □

Before we give other conditions under which pwp. modules are projective modules, we start by the following notation and proposition; let S be the endomorphism ring of an $\mathcal{R}$–module $\mathcal{M}$. Define $[,] : \mathcal{M} \times \mathcal{M}^* \longrightarrow S$ as $[m, f] = f_m$ where $f_m(a) = f(a)m$ for all $a \in \mathcal{M}$. Let Δ be the ideal of S generated by $Im[,]$ [12]. Let $ann_S(m) = \{f \in S \,|\, f(m) = 0\}$ where $m \in \mathcal{M}$.

Proposition 4.2. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$–module, then for all $m \in \mathcal{M}$, $ann_S(m) + \Delta = S$.*

Proof. Let $m \in \mathcal{M}$, by (1.1) $m = \sum_{k=1}^{n} \varphi_k(m)y_k$; $\varphi_k \in \mathcal{M}^*$ and $y_k \in \mathcal{M}$. Thus $m = \sum_{k=1}^{n} [y_k, \varphi_k](m)$. Hence $I - \sum_k [y_k, \varphi_k] \in ann_S(m)$ where I is the identity endomorphism of $\mathcal{M}$. Therefore $ann_S(m) + \Delta = S$. □

It is known that an $\mathcal{R}$–module $\mathcal{M}$ is finitely generated projective if and only if $S = \Delta$ [9]. By this statement and (4.2) we have;

Proposition 4.3. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$–module, that contains an element which is S–torsion free then $\mathcal{M}$ is a finitely generated projective $\mathcal{R}$–module.*

5. Regular Modules and Pointwise Projective Modules

Let $\mathcal{M}$ be any $\mathcal{R}$–module, $\mathcal{M}$ is called Z–regular if for each $x \in \mathcal{M}$, there exists $h \in \mathcal{M}^*$ such that $x = h(x)x$ [11]. By using (1.1), we have;

Proposition 5.1. *Every Z–regular module is a pwp. $\mathcal{R}$–module.*

Zimmermann [13] proved that every pwp. module over a regular ring is a Z–regular module. We will obtain this result by a different way. But first recall that an $\mathcal{R}$–module $\mathcal{M}$ is called F–regular if every submodule of $\mathcal{M}$ is pure [5]. It is known that if $\mathcal{M}$ is a projective F–regular $\mathcal{R}$–module, than $\mathcal{M}$ is Z–regular [5]. Let us note that a Z– regular module may not be projective [11]. However we have

Proposition 5.2. *Let $\mathcal{M}$ be any $\mathcal{R}$–module, then $\mathcal{M}$ is a F–regular pwp. module if and only if $\mathcal{M}$ is Z–regular.*

Proof. ($\Leftarrow$) It follows by (5.1) and [5].

($\Rightarrow$) Let $\mathcal{M}$ be an F–regular module, let $x \in \mathcal{M}$. By (1.1) $x = \sum_{i=1}^{n} h_i(x)y_i$, where $y_i \in \mathcal{M}$, $h_i \in \mathcal{M}^*$. But $\mathcal{R}x$ is a pure submodule of $\mathcal{M}$, then $x = \sum_i h_i(x)z_i$; $z_i \in \mathcal{R}x$. Thus there exists $r_i \in \mathcal{R}$ such that $x = \sum_{i=1}^{n} h_i(x)r_i x$. Put $h = \sum_{i=1}^{n} r_i h_i \in \mathcal{M}^*$, then $x = h(x)x$. □

It is known that every module over a regular ring is F–regular [8]. Thus we have.

Proposition 5.3. *Let $\mathcal{R}$ be a regular ring, then every pwp. $\mathcal{R}$–module is Z–regular.*

Now we are ready to give examples of pwp. modules that are not projective.

Example 5.4. Let $\mathcal{K}$ be a field, let $i \in I$ where I is an infinite countable set. And let $\mathcal{K}_i = \mathcal{K}$. Put $\mathcal{R} = \prod_{i=1} \mathcal{K}_i$ with the usual operations, $\mathcal{R}$ is a ring. Since $\mathcal{K}$ is a field, then $\mathcal{K}$ is a regular ring, and hence $\mathcal{R}$ is a regular ring. Let $\mathcal{P} = \oplus_{i \in I}\mathcal{K}_i$. Clearly, $\mathcal{P}$ is an ideal of $\mathcal{R}$, also $\mathcal{P}$ is a Z–regular $\mathcal{R}$–module [11]. By (5.1) $\mathcal{P}$ is a pwp. $\mathcal{R}$–module. We claim that $\mathcal{P}$ is not projective $\mathcal{R}$–module. In fact, it can be easily shown that $\mathcal{P}$ is not a direct summand of $\mathcal{R}$, thus it is not a direct summand of $\oplus_{i \in I}\mathcal{R}_i$; $\mathcal{R}_i = \mathcal{R}$. Therefore $\mathcal{P}$ is not a direct summand for any free $\mathcal{R}$–module. Hence by [4, p.265] $\mathcal{P}$ is not projective.

The following example shows that there are locally pwp. modules which are not pwp. modules. The example appeared in [6] for another purpose.

Example 5.5. Let $\mathcal{R}$ be a regular ring which has no finitely generated maximal ideal, and thus no maximal ideal which is a direct summand (For example, $\mathcal{R}$ could be the infinite direct product of copies of a field $\mathbb{F}$). Let $\{\mathcal{P}_\alpha | \alpha \in \Lambda\}$ be a family of maximal ideals of $\mathcal{R}$, and let $\mathcal{M} = \prod_\alpha (\mathcal{R}/\mathcal{P}_\alpha)$. Since $\mathcal{R}$ is regular, then $\mathcal{R}$ is locally a field. Hence $\mathcal{M}_{\mathcal{P}_\alpha}$ is a vector space over the field $\mathcal{R}_{\mathcal{P}\alpha}$. Thus $\mathcal{M}_{\mathcal{P}_\alpha}$ is a free $\mathcal{R}_{\mathcal{P}\alpha}$–module. Therefore $\mathcal{M}$ is a locally pwp. module. Now, we show that $\mathcal{M}^* = 0$. In fact, let $f : \mathcal{R}/\mathcal{P}_\alpha \longrightarrow \mathcal{R}$ be an $\mathcal{R}$–homomorphism. Since $\mathcal{R}/\mathcal{P}$ is simple then either $f = 0$ or f is 1-1. If f is 1-1, then $\mathcal{R}/\mathcal{P}_\alpha$ can be identified with an ideal $\mathcal{A}_\alpha$ in $\mathcal{R}$. But $ann(\mathcal{A}_\alpha) = \mathcal{P}_\alpha$, and $\mathcal{P}_\alpha$ is a pure ideal in $\mathcal{R}$ (Because $\mathcal{R}$ is regular), hence $\mathcal{P}_\alpha \cap \mathcal{A}_\alpha = \mathcal{P}_\alpha \dot{\mathcal{A}}_\alpha = 0$. Besides this, by the maximality of $\mathcal{P}_\alpha$, $\mathcal{P}_\alpha + \mathcal{A}_\alpha = \mathcal{R}$. Hence $\mathcal{P}_\alpha$ is a direct summand or $\mathcal{R}$ which is a contradiction. Thus $(\mathcal{R}/\mathcal{P}_\alpha)^* = 0$, and this implies that $\mathcal{M}^* = 0$. Now assume that $\mathcal{M}$ is a pwp. $\mathcal{R}$–module, thus by (1.1) $\mathcal{M} = 0$ which is absurd.

References

[1] Azumaya, G.: *A characterization of semi–perfect rings and modules*, Jain, S. K. (ed.) et al., Ring theory. Proceedings of the biennial Ohio State-Denison mathematics conference, May 14-16, 1992 dedicated to the memory of H. J. Zassenhaus. Singapore: World Scientific. 28-40 (1993).

[2] Bierkenmeier, G. F.: *Modules which are epi–equivalent to projective modules*, Acta. Univ, Carolin. Math. Phys., 24(1983), 9–16.

[3] Jondrup, S. & Trosbory, P. J.: *A remark on pure ideals and projective modules* , Math. Scand., 35(1974), 16–20.

[4] Kasch, F.: *Modules and rings*, cademic press. London, 1982.

[5] Naoum, A. G.: *Regular multiplication modules*, Periodica Mathematica Hungarica, 31(1995), 155–162.

[6] Naoum, A. G.: *1/2 Cancellation modules*. Kyungpook Mathematical Journal, 36(1996), 97–106.
[7] Naoum, A. G. & Al–Mothafar, S. A.: *Nearly projective modules*, Iraqi J.Sci., 42D(2001), 173–190.
[8] Naoum, A. G. & Yasenn, S. M.: *The regular submodule of a mosdule*, Ann. Soc. Math. Polonae, XXXV(1995), 195–201.
[9] Willard, E. R.: *Properties of projective generators*, Math. Ann., 158(1965), 352–364.
[10] Wu, L. E. T. & Jans, J. P.: *On quasi–projectives*, Illinios J. of Math., 11(1967), 439–448.
[11] Zelmanowitz, J.: *Regular modules*, Trans. Amer. Math. Soc., 162(1972), 334–355.
[12] Zimmermann–Huisgen, B.: *Endomorphism ring so self–generators*, Pacific J. Math., 61(1975), 587–601.
[13] Zimmermann–Huisgen, B.: *Pure submodules of direct products of free modules*, Math. Ann., 224(1976), 233–245.