Option pricing under transaction costs: an equilibrium approach

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Abstract

This article proposes an equilibrium model for contingent claim pricing under transaction costs. In our model, N investors maximize their expected terminal utility on a finite horizon with respect to trading strategies and holdings in the contingent claim, and an equilibrium is achieved if the contingent claim market clears. Unlike previous studies, our model can produce a commonly agreed contingent claim price among investors and can explain investors demands for contingent claim if an equilibrium exists. The main contribution of this work is the formulation of the model and a detailed proof of the existence of an equilibrium under very mild conditions when the contingent claim is a European style option.

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1 Introduction

Since the publishing of the seminal paper by Black and Scholes (1973), contingent claim (derivative security) pricing theories have been extensively studied. There are, in general, two different approaches to pricing derivatives: equilibrium pricing and arbitrage pricing. The former is based on investors' utility maximization and market clearing. Roughly speaking, each investor in the economy is given an initial endowment, which may allow her to trade with other investors. Under uncertainty, each investor chooses her own trading strategy to maximize her expected utility under her own probability belief at a finite time horizon. Therefore, for given prices of the contingent claims, each investor determines her demand in each contingent claim by solving her expected utility maximization problem. If the congregated demand of each contingent claim is 0, we say that the market reaches an equilibrium and the corresponding prices of the contingent claims are thus called equilibrium prices.

Arbitrage usually means "a free lunch", or more specifically, it is a portfolio which requires no initial cash infusion but whose cash value is non-negative with probability 1 at any time and is positive with a positive probability at certain time. It has been shown that if arbitrage exists in a market, it cannot reach equilibrium. Arbitrage pricing is to rule out arbitrage opportunities and to obtain the ranges of possible contingent claim prices. In a complete market, it is possible to obtain a unique contingent claim price. Since every payoff can be replicated by a portfolio of the primary securities, such as bond and stock, the contingent claim's price has to be equal to the value of the replicating portfolio if there is no arbitrage. In this case, the arbitrage price is also equal to the equilibrium price. However, in an incomplete market, where certain payoffs can not be replicated, the arbitrage argument at best can only give the upper and lower bounds of the prices at which the claim can be traded. Consequently, equilibrium pricing is more appropriate in incomplete markets if a unique price has to be determined.

The existence of a contingent claim is unjustified in a complete market; this is the so called "catch-22 of option pricing theory". Since the payoff of every contingent claim can be replicated by a portfolio of primary securities, investors can simply take a position in the portfolio rather than buy or sell the contingent claim. However, investors are not living in a world of complete markets, and this results the existence of various contingent claims.

Typically, an incomplete market is the result of the insufficiency of traded primary securities. For example, in the market model where the source of randomness comes from a d-dimensional Brownian motion, it can be shown that if there are less than $d+1$ primary securities, the market cannot be complete. Even if there are enough primary securities, the market may be incomplete for many other reasons, such as portfolio constraints, discrete hedging, and market friction, any of which may account for the demand of contingent claims. Common sources of portfolio constraints come from non-bankruptcy or prohibition of short-selling. Furthermore, continuous hedging can only exist in an idealized market. In reality, nobody can "physically" trade continuously. Therefore, contingent claims may replace some trading strategies which are otherwise infeasible. Market friction, i.e. transaction costs in the context of this work, is another important factor which invalidates the idealized Black-Scholes market. Generally, replicating the payoff of a contingent claim requires continuously adjusting a portfolio; because of transaction costs, inevitably, there will be an extra cash outflow. In the rest of this section, we briefly review previous works on pricing under proportional transaction costs.

There are 3 major streams of pricing methodologies under transaction costs: Non-linear PDE, super-replication and utility maximization. Pricing under transaction costs was first studied by Leland (1985). In Leland's work, usual Black-Scholes type hedging was carried out but with finite hedging frequency. Consequently, high order terms in Taylor expansions generally cannot be truncated. However, when hedging frequency is relatively large, we may still get a good approximation by doing so. The result is a PDE very similar to the Black-Scholes PDE, except for the term of absolute value of the second order partial derivative. In the case of vanilla options and small transaction costs, a very simple solution can be found by merely adjusting the volatility by a certain factor. The PDE method was in due course studied by Avellaneda and Parás (1994), Barles and Soner (1996), and Hoggard *et al.* (1994). The advantage of the PDE method is its simplicity, once a PDE is specified, common finite difference methods are available to carry out numerical solutions easily.

Since the truncation of the high order terms in Taylor expansions is not justified in a continuous time model with a discrete hedging, Boyle and Vorst (1992) replaced the continuous time model with discrete binomial tree, and again, applied the usual replication strategy to obtain the contingent claim price. However, this strategy was later found not necessarily optimal by Edirisinghe *et al.* (1993). [1] In fact, in their work, it was found that some strategies may cost less but can still cover the position in any circumstances. These are commonly called super-replication strategies. Therefore, contingent claim price was defined to be the cost of the least expensive super-replication strategy. A similar model was also discovered by Bensaid *et al.* (1992). Although both works used linear programming as the numerical tool, Edirisinghe *et al.* claimed that their algorithm was more efficient. Nevertheless, it is still cumbersome when periods are many. Their models were later extended to continuous time by Soner *et al.* (1995) and Cvitanić *et al.* (1999). Though the mathematical tools in continuous time models are much more involved, a closed form solution to the optimal super-replication strategy can be found for path-independent European style options. In the special case of vanilla call options, the trivial strategy of buy-and-hold is proved to be optimal. It can be expected that these results should agree with that of Edirisinghe *et al.* if the time interval approaches 0 in the binomial model.

The utility maximization method was first proposed by Hodges and Neuberger (1989). Davis *et al.* (1993) continued their study and re-defined the seller's price as the minimum price at which she sells a positive amount of the claim. The buyer's price is similarly defined as the maximum price at which she buys a positive amount of the claim. Davis and Zariphopoulou (1995) also made efforts to find the buyer's price of American contingent claims. However, they found it difficult to define a seller's price for an American contingent claim. The implementation of the utility-based model inevitably requires dynamic stochastic programming. In the case of 2-asset economy, the result is usually a 3-dimensional non-linear Bellman PDE (4-dimensional if temporal dimension is included) subject to 2 controls, which is computationally expensive, if not intractable. By assuming exponential utility, Davis *et al.* (1993) managed to reduce the PDE to 2-dimensional and carried out a finite difference scheme.

There are two common problems with the aforesaid models. First of all, there is no equilibrium. In Black-Scholes world, the price of contingent claim is a linear functional of the payoff, thus, the buyer and the seller can always agree on a same price. When transaction costs exist, the previously mentioned models generally produce "non-

[1]See also Boyle and Tan (1994) for a simple example.

symmetry" in the price, viewed as a functional of payoff. That is to say, for two cash flows which cancel each other in the future, their current values do not necessarily cancel each other. This accounts for the difference between buyer's price and seller's. In the utility based approach, the situation is even worse–the price even depends on the investor's utility function. Davis *et al.* (1993) questioned this problem and stated that the utility-based approach usually generates higher seller's price and lower buyer's price with Black-Scholes price between them. Consequently, no matter what investors utility functions are, they can never agree on a same price in Davis' model. Secondly, none of the models explains the demand of contingent claims, though evidently, there may be demands for certain kind of contingent claims in an incomplete market. In Davis *et al.* (1993), transaction costs were indeed mentioned as a reason for the demand of the option. However, according to their definition of seller's and buyer's price, nobody is better off by taking a position in the option, thus, we still do not see why investors should invest in it. The above two questions can hardly be answered in models dealing with one party only. We should, therefore, apply microeconomic theory as usual to study the equilibrium in the contingent claim market. If an equilibrium exists, the price is certainly commonly agreed among the investors, while the corresponding demands will answer our second question.

Our model is based on Cvitanić and Karatzas' (1996) framework, and many conventions are carried on in this work. This work is organized as follows, in Section 2, we will present our model and formulate the seller's and buyer's problems for both European and American contingent claims, and define the equilibrium; in Section 3, we demonstrate how our model agrees with Black-Scholes model when there is no transaction cost; in Section 4, we discuss the existence of the equilibrium, and propose a way to make the model possibly computationally tractable; Section 5 gives a brief summary of this work and proposes future research directions.

2 The model

2.1 The market

We consider a market consisting of two primary securities only. One is a bank account with rate of return $r(t)$, its price evolves as

$$dB(t) = B(t)r(t)dt \tag{1}$$

independent of investor's probability belief. The other is a risky stock, whose price, $S(t)$, is a stochastic process. For an investor with probability measure $\mathbb{P}$, its price evolves as

$$dS(t) = S(t)[b(t)dt + \sigma(t)dW(t)] \tag{2}$$

where $W(t)$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to a continuous and complete filtration $\{\mathcal{F}_t, t \in [0, \infty[\}$. The coefficients $r(t), b(t), \sigma(t)$ are generally stochastic, we impose the conditions that they are uniformly bounded on $[0, \infty[\otimes\Omega$, and that $\sigma(t)$ is bounded away from 0 on $[0, \infty[\otimes\Omega$. When there are transaction costs, the stock can only be purchased at a price $(1+\lambda)S(t)$ or sold at a price $(1-\mu)S(t)$ at time t, where $\lambda, \mu \geq 0; \mu < 1$.

We introduce another probability measure $\mathbb{P}_0$, under which the stock price evolves as

$$dS(t) = S(t)[r(t)dt + \sigma(t)dW_0(t)] \tag{3}$$

where $W_0(t)$ is a standard Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P}_0)$. Clearly, $\mathbb{P}_0$ and $\mathbb{P}$ are related by

$$\mathbb{P}_0(A) = E[Z_0(T)\mathbf{1}_A] \quad \forall A \in \mathcal{F}_T \tag{4}$$

where $E[\cdot]$ is the expectation operator under $\mathbb{P}$, and

$$Z_0(t) = \exp\left[-\int_0^t \frac{b(u) - r(u)}{\sigma(u)} dW(u) - \frac{1}{2}\int_0^t \left(\frac{b(u) - r(u)}{\sigma(u)}\right)^2 du\right] \tag{5}$$

When there is no transaction cost, $\mathbb{P}_0$ is the so-called risk neutral measure and plays an essential role in arbitrage pricing theory. In the presence of transaction costs, $\mathbb{P}_0$ offers nothing special. It is introduced here just to adopt Cvitanić and Karatzas' convention. We will denote $E_0[\cdot]$ the expectation operator under $\mathbb{P}_0$.

2.2 European contingent claims

We then introduce an European contingent claim into this market.

Definition A *European contingent claim* is a contract which pays $\xi(T)$ in bank account and $\eta(T)$ in stock at maturity T, where $\xi(T)$ and $\eta(T)$ are $\mathcal{F}_T$ measurable and $E_0[\xi^2(T)+\eta^2(T)]<\infty$. It is called an option if $\xi(T)+f(\eta(T))\geq 0$ $\mathbb{P}_0$-a.s. and $\xi(T)+f(\eta(T))>0$ with a positive probability under $\mathbb{P}_0$, where

$$f(y)=\left(1+\frac{\lambda-\mu}{2}\right)y-\frac{\lambda+\mu}{2}|y|.$$

In this section we formulate an investor's problem if she takes a position in the contingent claim.

Definition A *utility function* is an increasing, concave function $U(x)$ with $-\infty<U(x)<+\infty$ in $]0,\infty[$, $U(x)=-\infty$ in $]-\infty,0[$, and $U(0)=\lim_{x\downarrow 0}U(x)$.

Remark In order to study the dual problem, Cvitanić and Karatzas have more restrictions on the continuity and differentiability of U, but the above definition is sufficient for this work. U being finite and concave in $]0,\infty[$ ensures its continuity in $]0,\infty[$. Together with the definition of $U(0)$, it further implies that U is closed. U is not necessarily differentiable or strictly concave, though being finite and concave in $]0,\infty[$ also implies that it is differentiable almost everywhere in $]0,\infty[$.

With this definition of the utility function, the investor is to form some trading strategy to maximize her expected terminal utility under her own probability measure.

Definition A *portfolio* or *trading strategy* is an $\{\mathcal{F}_t\}$ adapted vector process $\pi(t)=(L(t),M(t))$ where $L(t)$ and $M(t)$ both have left continuous, non-decreasing sample paths, $L(t)+M(t)<\infty$ a.s. and $L(0)=M(0)=0$.

Remark $L(t)$ and $M(t)$ are in dollar amount instead of share, one may think of that $dL(t)/S(t)$ is the number of shares of the stock purchased between t and $t+dt$, while $dM(t)/S(t)$ is the number of shares sold between t and $t+dt$. When $\lambda=\mu=0$, it is convenient to define the trading strategy $\pi_t=L_t-M_t$.

With this trading strategy, the wealth of the investor generally depends on the dollar amounts in both the bank account and the stock.

Definition A *wealth process* generated by a portfolio $\pi(t)$ is an $\{\mathcal{F}_t\}$ adapted vector process $(X^{\epsilon,\gamma,\pi}(t), Y^{\epsilon,\gamma,\pi}(t))$, where

$$\begin{aligned} X^{\epsilon,\gamma,\pi}(t) &= x - \gamma P - (1+\lambda)L(t) + (1-\mu)M(t) \\ &\quad + \int_0^t X^{\epsilon,\gamma,\pi}(u)r(u)du + \gamma\xi(T)\mathbf{1}_{t\geq T} \qquad (6) \\ Y^{\epsilon,\gamma,\pi}(t) &= y + L(t) - M(t) \\ &\quad + \int_0^t Y^{\epsilon,\gamma,\pi}(u)[b(u)du + \sigma(u)dW(u)] + \gamma\eta(T)\mathbf{1}_{t\geq T} \qquad (7) \end{aligned}$$

$\epsilon = (x, y) \in \mathbb{R}_+^2 - \{(0,0)\}$ is the investor's initial endowment, γ is the investor's holding in the claim and P is the price of the claim.

Remark It is convenient to define a cash value function $C(x, y) = x + f(y)$ for a wealth pair. When $\lambda = \mu = 0$, C may replace the two-dimensional wealth process.

The ith investor's problem is then expressed as follows

$$\max_{\gamma\in\mathbb{R}} \max_{\pi(t)\in\mathcal{A}(x,y,\gamma)} E\left[U\left(X^{\epsilon,\gamma,\pi}(T) + f(Y^{\epsilon,\gamma,\pi}(T))\right)|\mathcal{F}_0\right] \qquad (8)$$

where $\mathcal{A}$ is a set of *admissible trading strategies.*

Definition

$$\mathcal{A} = \{\pi(t)|X^{\epsilon,\gamma,\pi}(t) + f(Y^{\epsilon,\gamma,\pi}(t)) \geq 0 \text{ a.s. } \forall t \in [0,T]\}$$

which is the so-called non-bankruptcy condition.

Remark We will use the convention that the maximum on an empty set is $-\infty$. A more proper way to express the investor's problem is

$$\max_{(\gamma,\pi(t))} E\left[U\left(X^{\epsilon,\gamma,\pi}(T) + f(Y^{\epsilon,\gamma,\pi}(T))\right)|\mathcal{F}_0\right] \qquad (9)$$

$$\text{s.t. } (\gamma, \pi(t)) \in \{(\gamma,\pi)|\gamma \in \mathbb{R}, \pi \in \mathcal{A}(x,y,\gamma)\} \qquad (10)$$

Strictly speaking, even when (9) has a solution, (8) might not, because the nested maximization might have no solution for some $\gamma \in \mathbb{R}$. Nevertheless, we will use the stage-wise maximization through this work for its clarity in identifying the problem.

2.3 American contingent claims

We now introduce an American contingent claim instead of a European one into this market.

Definition An *American contingent claim* is a contract that expires at time T, and pays $\xi(t)$ in the bank account and $\eta(t)$ in stock if the buyer chooses to exercise at some time $t \in]0,T]$, where $\xi(t)$ and $\eta(t)$ are $\mathcal{F}_t$ measurable random variables, and $E_0[\xi^2(t)+\eta^2(t)] < \infty$, $\forall t \in]0,T]$. It is called an *option* if $\xi(t)+f(\eta(t)) \geq 0$ $\mathbb{P}_0$-a.s. $\forall t \in]0,T]$, and $\exists t \in]0,T]$, $\xi(t)+f(\eta(t)) > 0$ with a positive probability under $\mathbb{P}_0$.

There are some distinct features of American contingent claims because of the possible early exercise. For this reason, we have to consider the difference between buyer's and seller's objectives. The buyer's objective is to choose an optimal time to exercise the contract. If the investor acts as a buyer, i.e., her holding in the claim is $\gamma \geq 0$,then her wealth process is a pair

$$
\begin{aligned}
X^{\epsilon,\gamma,\pi}(t) &= x - \gamma P - (1+\lambda)L(t) + (1-\mu)M(t) \\
&\quad + \int_0^t X^{\epsilon,\gamma,\pi}(u) r(u) du + \gamma \xi(\tau) \mathbf{1}_{t \geq \tau} \qquad (11) \\
Y^{\epsilon,\gamma,\pi}(t) &= y + L(t) - M(t) \\
&\quad + \int_0^t Y^{\epsilon,\gamma,\pi}(u)[b(u)du + \sigma(u)dW(u)] + \gamma \eta(\tau) \mathbf{1}_{t \geq \tau} \qquad (12)
\end{aligned}
$$

where τ is an arbitrary $\{\mathcal{F}_t\}$-stopping time determined by the buyer. Therefore, for given γ, the buyer's problem is thus

$$
\max_{\tau} \max_{\pi(t) \in \mathcal{A}(x,y,\gamma)} E[U(X^{\epsilon,\gamma,\pi}(t) + f(Y^{\epsilon,\gamma,\pi}(t)))|\mathcal{F}_0] \qquad (13)
$$

The seller's objective is to fulfill her obligation, no matter when the buyer exercises the contract, rationally or irrationally. It is relatively difficult to formulate the seller's problem, but we can consider the following way. The seller forms two portfolios, one is to fulfill her obligation, which, at time 0, cannot be expected to contribute to her terminal consumption; the other is for her own consumption, which contributes to her terminal utility. The former requires an initial infusion in cash and stock, and suitable management to cover her position.

Definition We say $\delta = (g,h) \in \mathbb{R}^2$ *hedges* an American contingent claim $(\xi(t), \eta(t))$ if there is a trading strategy $\phi(t)$ such that $\forall t \in]0,T]$,

$$X^{\delta,0,\phi}(t) - \xi(t) + f(Y^{\delta,0,\phi}(t) - \eta(t)) \geq 0$$

$\mathbb{P}_0$-a.s.

Therefore, given $\gamma \leq 0$, the seller's problem is

$$\max_{(g,h)} \max_{\pi} E[U(X^{(x-g-\gamma P, y-h),0,\pi}(T) + f(Y^{(x-g-\gamma P, y-h),0,\pi}(T)))|\mathcal{F}_0] \tag{14}$$

$$\text{s.t. } (g,h) \in \mathcal{H}(\gamma\xi(t), \gamma\eta(t)), \pi \in \mathcal{A}(x-g-\gamma P, y-h, 0) \tag{15}$$

where

$$\mathcal{H}(\cdot,\cdot) := \left\{(g,h) \in \mathbb{R}^2 | (g,h) \text{ hedges the American contingent claim } (\cdot,\cdot)\right\}.$$

Remark The set $\mathcal{H}(\cdot,\cdot)$ is independent of the individual's utility.

At the final stage, the investor determines to be a buyer or a seller, her ultimate problem is thus

$$\max_{\gamma \in \mathbb{R}} \Phi(x - \gamma P, y) \tag{16}$$

where $\Phi(x-\gamma P, y)$ is defined by (13) if $\gamma \geq 0$ and by (14) if $\gamma \leq 0$. It is easy to verify that the two definitions for the case $\gamma = 0$ are consistent.

2.4 The equilibrium

We consider that the contingent claim is traded only among N investors. The ith investor has an initial endowment $\epsilon_i = (x_i, y_i)$, a utility function $U_i(\cdot)$ and a probability measure $\mathbb{P}_i$, under which the stock price evolves as

$$dS(t) = S(t)[b_i(t)dt + \sigma(t)dW_i(t)] \tag{17}$$

where $W_i(t)$ is a standard Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P}_i)$. The $\mathbb{P}_i$'s are then related by

$$\mathbb{P}_i(A) = E_j[Z_{ij}(T)\mathbf{1}_A] \quad \forall A \in \mathcal{F}_T; i,j = 0, \cdots, N \tag{18}$$

where $E_j[\cdot]$ is the expectation operator under $\mathbb{P}_j$,

$$Z_{ij}(t) = \exp\left[-\int_0^t \theta_{ij}(u)dW(u) - \frac{1}{2}\int_0^t \theta_{ij}^2(u)du\right] \tag{19}$$

and

$$\theta_{ij} = \frac{b_j(u) - b_i(u)}{\sigma(u)}. \tag{20}$$

Remark Obviously, $\mathbb{P}_0, \mathbb{P}_1, \cdots, \mathbb{P}_N$ are mutually equivalent, we therefore will not specify the probability measure when we say something "a.s." or "with a positive probability" afterwards.

For the ith investor, her utility maximization problem can be formulated by just replacing (x, y) with (x_i, y_i), $b(t)$ with $b_i(t)$, $W(t)$ with $W_i(t)$, $E[\cdot]$ with $E_i[\cdot]$, $\mathcal{A}$ with $\mathcal{A}_i$, and $\mathcal{H}$ with $\mathcal{H}_i$ in the previous maximization problems.

If for each investor, the maximization problem has solution(s), we may denote the optimal holding(s) in the claim as a correspondence of argument P, i.e., for the ith investor, the set of optimal holding(s) in the claim is $\hat{\gamma}_i(P)$, which generally has more than one element. If for some P, $\exists \gamma_i \in \hat{\gamma}_i(P)$ for every i, and

$$\sum_{i=1}^{N} \gamma_i = 0 \tag{21}$$

the price P is called an *equilibrium price*. Since all investors behave as price takers in the money market and stock market, we generally cannot expect these markets to clear among the N investors. In reality, one may think that these markets are much larger than the contingent claim market, so that the N investors' congregated demands in these markets are balanced by others.

For a given contingent claim, if an equilibrium exists but not at $(\gamma_1, \cdots, \gamma_N) = (0, \cdots, 0)$, we say that the contingent claim is desirable. Therefore, to determine whether a contingent claim is desirable, we generally need to find the equilibrium first.

3 Complete market

In this section, we are to show that this model agrees with the Black-Scholes model when the market is complete, that is to say, $\lambda = \mu = 0$, and there is no portfolio constraint, i.e., the non-bankruptcy condition is not imposed on maximization problems. Since the market is complete, each payoff can be replicated with a portfolio, coupled with an initial endowment.

In the case of a European contingent claim, we denote the replicating portfolio π_i with initial endowment $C_i(0)$ for the ith investor. If an equilibrium price P exists, then it has to be $C_i(0)$. Otherwise, we let $\hat{\pi}_i$ be her optimal trading strategy, $\hat{\gamma}_i$ be her optimal holding in the claim, and $\hat{C}_i(T)$ be her final wealth. If $P > C_i(0)$, obviously, she could hold $\hat{\gamma}_i - 1$ shares in the claim, and follow a strategy $\hat{\pi}_i + \pi_i$ to reach final wealth level $\hat{C}_i(T) + (P - C_i(0)) \exp\left(\int_0^T r(t)dt\right) > \hat{C}_i(T)$ a.s. Since her utility function is increasing, her expected utility would be greater in this way. This contradicts our previous assumption of the optimality of $\hat{\gamma}_i$, therefore we cannot have $P > C_i(0)$. Similarly, $P < C_i(0)$ does not hold either. The only possibility is $P = C_i(0)$, where $C_i(0)$ is actually independent of investor's probability measure according to Black and Scholes' theory, and is equal to the risk-neutral price. Hence, if the market is complete and an equilibrium exists, the equilibrium price of an European contingent claim can only be the Black-Scholes price.

Now we will show that it also works for American contingent claims. We find that it is more convenient to consider the seller only. Assuming that the ith investor is a seller, and denoting $c_i = \min\{c = g + h | (g, h) \in \mathcal{H}_i(\xi(t), \eta(t))\}$, which is actually attainable according to Karatzas and Shreve (1998), we argue that a solution to the seller's problem can be found only if the American contingent claim's price $P = c_i$. Obviously, by the definition of c_i, she would not spend more than c_i to cover a share of the American contingent claim. If $P \neq c_i$, we assume that her optimal holding in the claim is $\hat{\gamma}_i < 0$, her super-replicating portfolio is $-\hat{\gamma}_i\hat{\pi}_i$ with initial endowment $-\hat{\gamma}_i c_i$, and the consumption portfolio reaches a terminal wealth $C_i(T)$. Then if $P < c_i$, she could hold 0 in the claim, follow the same strategy for the consumption portfolio, and reach a final consumable wealth $C_i(T) + \hat{\gamma}_i(P - c_i) \exp\left(\int_0^T r(t)dt\right) > C_i(T)$ a.s.; if $P > c_i$, she could sell one more share, i.e., hold $\hat{\gamma}_i - 1 < 0$ shares, use the portfolio $(-\hat{\gamma}_i + 1)\hat{\pi}_i$ with initial endowment $(-\hat{\gamma}_i + 1)c_i$ to hedge the contingent claim, follow the same strategy for the consumption portfolio, and reach a final consumable wealth $C_i(T) + (P - c_i) \exp\left(\int_0^T r(t)dt\right) > C_i(T)$ a.s. In either case, we arrive at a contradiction to our previous assumption, therefore we conclude that $P = c_i$ is the only possibility.

Although the above is enough, we can also show this from the buyer's problem. Again, we assume that the ith investor is a buyer. In Black-Scholes model, the buyer chooses an optimal stopping time τ_i such that she maximizes her expected discounted payoff under $\mathbb{P}_0$, and exercises the contingent claim according to the policy of τ_i. We argue that τ_i also solves her utility maximization problem. [2] Otherwise, we assume that her expected utility is maximized by a stopping time $\hat{\tau}_i$ and a trading strategy $\hat{\pi}_i$, and denote her terminal wealth $C_i(T)$. The two payoffs $\xi(\tau_i)+\eta(\tau_i)$ and $\xi(\hat{\tau}_i)+\eta(\hat{\tau}_i)$ can be replicated by portfolio ψ_i with initial endowment ζ_i, and portfolio $\hat{\psi}_i$ with initial endowment $\hat{\zeta}_i$, respectively. Because τ_i is the one which maximizes the expected discounted payoff under $\mathbb{P}_0$, $\hat{\zeta}_i < \zeta_i$ according to risk neutral pricing theory. Hence, she could exercise the option according to policy τ_i, follow the trading strategy $\hat{\pi}_i - \psi_i + \hat{\psi}_i$, and reach terminal wealth $C_i(T) + (\zeta_i - \hat{\zeta}_i)\exp\left(\int_0^T r(t)dt\right) > C_i(T)$ a.s. This results in a contradiction to the assumption that $\hat{\tau}_i$ solves her utility maximization problem. Therefore we conclude that τ_i also solves her utility maximization problem. Once the optimal stopping time is determined, the American contingent claim is just like a European one, and there is a solution to the buyer's problem only if the contingent claim's price $P = \zeta_i$.

Karatzas and Shreve (1998) has shown that ζ_i and c_i are actually equal and are constants independent of the investor's probability measure, the American contingent claim price in the Black-Scholes model can be thus defined by either of them. Again, if the market is complete and an equilibrium exists, the equilibrium price of an American contingent claim can only be the Black-Scholes price too.

The question is then reduced to whether an equilibrium exists in the complete market described above. This has been well understood and the answer is "yes". Since every cash flow can be replicated by a trading strategy coupled with an initial endowment in a complete market, the implication is that the maximizers γ_i are arbitrary provided that they satisfy the market clearing condition (21). This indicates that a contingent claim is unnecessary in a complete market; because investors are indifferent between investing or not–even if they do in-

[2] Davis and Zariphopoulou (1995) had a proof for the buyer, however, their proof is invalid because the optimal stopping time in the utility maximization problem was not shown to be the same as the one in the expected discounted payoff maximization problem.

vest, they are not concerned with how much they invest.

4 Incomplete market

4.1 Existence of equilibrium

When at least one of μ and λ is not zero, we encounter an incomplete market caused by friction. Before applying the equilibrium pricing model, one naturally would ask whether an equilibrium exists. In Appendix, we will show that an equilibrium does exist for European options under very mild conditions. Some details of the proof are borrowed from Appendix A and Appendix B of Cvitanić and Karatzas (1996), readers are strongly recommended to refer to their work for details.

It can be further shown that if an option cannot be exactly replicated, the equilibrium price must be lower than the smallest upper hedging price of the N investors. That is to say, whoever sells the option, she cannot expect to hedge the option with the money that she gets from selling it. Why she still sells? The reason is that the upper hedging price is defined for a seller without initial endowment, while our seller has a positive initial endowment. She can spend part of her initial endowment on hedging the option. Entering the contingent claim market may make her worse off in some states, but from the viewpoint of expected utility maximization, the option serves just as a tool to re-distribute her terminal wealth among states, she does not have to be better off in every state to achieve larger expected utility. In mathematical words, for two random variables C_1 and C_2, $E[U(C_1)|\mathcal{F}_0] > E[U(C_2)|\mathcal{F}_0] \not\Rightarrow C_1 \geq C_2$ a.s.

4.2 Power utility and logarithm utility

Although numerical implementation is beyond the scope of this work, we propose a possible simplification of the model. A naive approach to obtain the equilibrium is by solving the individual's utility maximization problem for a given price P and holding in the claim γ, and then further maximizing with respect to γ. This way, the optimal holding in the claim for investor i can be expressed as a function of P, namely $\gamma_i(P)$, if there is a unique maximizer. Further solving the equation $\sum_{i=1}^{N} \gamma_i(P) = 0$ we can find the equilibrium price P.

However, even for European contingent claims, the first step is a parametric dynamic programming problem with 1 parameter [3] and 4 state variables, though the number of state variables can be reduced to 3 by assuming deterministic interest rate $r(t)$. It is not clear whether numerical computation is realistic.

The simplification is to consider utility functions with constant relative risk aversion $1-\rho$, which are a family of utilities with the form $U(x) = x^\rho$ when $0 < \rho < 1$, $U(x) = \log x$ when $\rho = 0$, and $U(x) = -x^\rho$ when $\rho < 0$. In the context of a market without contingent claims, Shreve and Soner (1994) have applied this family of utility functions to the investor's expected utility maximization problem in an intertemporal model with infinite horizon, and found a closed form solution. However, this can unlikely be the case in our context.

We describe the case $\rho > 0$ only, since other cases are very similar. If $1-\rho_i$ is the ith investor's relative risk aversion, her problem becomes

$$\max_{\gamma\in\mathbb{R}} \max_{\pi(t)\in\mathcal{A}(\epsilon_i,\gamma)} E_i\left[(X^{\epsilon_i,\gamma,\pi}(T) + f(Y^{\epsilon_i,\gamma,\pi}(T)))^{\rho_i}\,|\mathcal{F}_0\right] \tag{22}$$

where $\epsilon_i = (x_i, y_i)$ is the initial endowment. It is easy to find that if $\gamma > 0$,

$$\begin{aligned} &\max_{\pi(t)\in\mathcal{A}(\epsilon_i,\gamma)} E_i\left[(X^{\epsilon_i,\gamma,\pi}(T) + f(Y^{\epsilon_i,\gamma,\pi}(T)))^{\rho_i}\,|\mathcal{F}_0\right] \\ = \ &\gamma^{\rho_i} \max_{\pi(t)\in\mathcal{A}(\frac{1}{\gamma}\epsilon_i,1)} E_i\left[\left(X^{\frac{1}{\gamma}\epsilon_i,1,\pi}(T) + f(Y^{\frac{1}{\gamma}\epsilon_i,1,\pi}(T))\right)^{\rho_i}\,|\mathcal{F}_0\right]; \end{aligned} \tag{23}$$

and if $\gamma < 0$,

$$\begin{aligned} &\max_{\pi(t)\in\mathcal{A}(\epsilon_i,\gamma)} E_i\left[(X^{\epsilon_i,\gamma,\pi}(T) + f(Y^{\epsilon_i,\gamma,\pi}(T)))^{\rho_i}\,|\mathcal{F}_0\right] \\ = \ &|\gamma|^{\rho_i} \max_{\pi(t)\in\mathcal{A}(\frac{1}{|\gamma|}\epsilon_i,-1)} E_i\left[\left(X^{\frac{1}{|\gamma|}\epsilon_i,1,\pi}(T) + f(Y^{\frac{1}{|\gamma|}\epsilon_i,1,\pi}(T))\right)^{\rho}\,|\mathcal{F}_0\right]. \end{aligned} \tag{24}$$

Therefore, for each investor we only need to solve 3 non-parametric dynamic programming problems: for $\gamma > 0$, it is the one in the right hand side of eq (23); for $\gamma < 0$, the one in the right hand side of eq

[3] The dependence on γP exists only in the initial condition of the dynamic programming, so there is actually only one parameter γ.

(24); and for $\gamma = 0$, it is

$$\max_{\pi(t)\in\mathcal{A}(\epsilon_i,0)} E_i\left[\left(X^{\epsilon_i,0,\pi}(T) + f(Y^{\epsilon_i,0,\pi}(T))\right)^{\rho_i} |\mathcal{F}_0\right] \tag{25}$$

When $\gamma \neq 0$, it should be understood that the dynamic programming problems are solved for various initial conditions.

Therefore at the first stage, there are $3N$ dynamic programming problems. Once they are solved, the remaining steps follow, and we can finally determine the equilibrium price in principle. Extreme care may be necessary when γ is close to 0, because the initial endowment $\frac{1}{|\gamma|}\epsilon_i$ approaches infinity.

5 Conclusion

In this work, we proposed an equilibrium model for the pricing of both European and American contingent claims under transaction costs. Unlike other existing models, the investors can agree on a same price if an equilibrium exists. Moreover, the demand for the contingent claim in a market can be explained by the equilibrium demands of investors. We also proved that there exists at least one equilibrium when the contingent claim is a European style option. Further research should explore

1. whether equilibrium exists for contingent claims other than European options;
2. whether the equilibrium is unique, if it exists; if the equilibrium is not unique, what are the sufficient conditions to ensure a unique equilibrium;
3. whether there is some efficient algorithm to locate the equilibrium numerically, if it exists; whether there is a closed form solution to our problem, in view of the discovery of a closed form solution for the super-replication strategy by Cvitanić *et al.* (1999).

A Appendix

In this appendix, we prove that an equilibrium exists when the contingent claim is a European option. For simplicity, we shall drop the subscript which identifies the particular investor, and bring it back only when it is necessary. The proof also exploits convex analysis theories extensively, since a concave function(al) is just a reflection of a convex function(al).

Proposition A.1 *Given γ and P, there is a solution the to investor's problem*

$$\max_{\pi \in \mathcal{A}(x,y,\gamma)} E[U(X^{\epsilon,\gamma,\pi}(T) + f(Y^{\epsilon,\gamma,\pi}(T))|\mathcal{F}_0] < +\infty \tag{26}$$

Proof Denote $v = x - \gamma P$. We say that a triple $(x, y, \gamma) \in \mathbb{R}^3$ hedges a contingent claim $(H_1(T), H_2(T))$ if $\exists \pi \in \mathcal{A}(x, y, \gamma)$ such that $X^{\epsilon,\gamma,\pi}(T) - H_1(T) + f(Y^{\epsilon,\gamma,\pi}(T) - H_2(T)) \geq 0$ a.s. Similar to Cvitanić and Karatzas (1996), we argue that the following set

$$A = \{H(T) \in \mathbf{L}^2_{0+} | (x, y, \gamma) \text{ hedges } (H(T), 0)\} \tag{27}$$

where $\mathbf{L}^2_{0+} := \mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P}_0) \cap \{H | H \geq 0 \text{ a.s.}\}$, is a closed, convex and bounded subset of $\mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P}_0)$. Clearly,

$$\begin{aligned} A &= \{H(T) \in \mathbf{L}^2_{0+} | (x, y, \gamma) \text{ hedges } (H(T), 0)\} \\ &= \{H(T) \in \mathbf{L}^2_{0+} | (v, y, 0) \text{ hedges } (H(T) - \gamma\xi(T), -\gamma\eta(T))\} \end{aligned} \tag{28}$$

which can be shown to be closed, convex and bounded in $\mathbf{L}^2(\boldsymbol{\Omega}, \mathcal{F}, \mathbb{P}_\mathbf{0})$, according to Cvitanić and Karatzas (1996). If $A \neq \emptyset$, with very similar argument to that in Appendix B of Cvitani'c and Karatzas (1996), there exists $\hat{H}(T) \in A$, such that $\forall H(T) \in A$, $E[U(H(T))|\mathcal{F}_0] \leq E[U(\hat{H}(T))|\mathcal{F}_0] < \infty$, and

$$\max_{\pi \in \mathcal{A}(x,y,\gamma)} E[U(X^{\epsilon,\gamma,\pi}(T) + f(Y^{\epsilon,\gamma,\pi}(T))|\mathcal{F}_0] = E[U(\hat{H}(T))|\mathcal{F}_0] < +\infty, \tag{29}$$

the strategy $\hat{\pi}(t)$ which hedges $(\hat{H}(T), 0)$ is optimal and $X^{\epsilon,\gamma,\hat{\pi}}(T) + f(Y^{\epsilon,\gamma,\hat{\pi}}(T)) = \hat{H}(T)$ a.s.; if $A = \emptyset$, then $\mathcal{A}(x, y, \gamma) = \emptyset$, the solution is

$$\max_{\pi \in \mathcal{A}(x,y,\gamma)} E[U(X^{\epsilon,\gamma,\pi}(T) + f(Y^{\epsilon,\gamma,\pi}(T))|\mathcal{F}_0] = -\infty < +\infty. \tag{30}$$

∎

Remark The closeness of the set (28) can be proved very similarly to the proof in Appendix A of Cvitanić and Karatzas (1996). The boundedness can be shown by the definition of the upper hedging price of a contingent claim. The proofs exploit many notations not directly related to this work, and are thus omitted. [4]

[4] Detailed proof available upon request.

The above proof is valid for all European contingent claims including options. Because the maximum is attainable, we can safely define a function

$$\Phi(x-\gamma P,\gamma;y)=\max_{\pi\in\mathcal{A}(x,y,\gamma)}E[U(X^{\epsilon,\gamma,\pi}(T)+f(Y^{\epsilon,\gamma,\pi}(T))|\mathcal{F}_0] \quad (31)$$

In the following studies, y serves only as a parameter rather than an argument of the function Φ, and will be thus dropped. We are more interested in studying Φ as a function of two arguments v and γ rather than x and γ. One may think $\Phi(v,\gamma)$ as the maximal expected utility attainable if the investor is given initial endowment (v,y) and γ shares of the claim for free. It is easy to show that $\Phi(v,\gamma)$ is a proper concave function, with *effective domain*

$$\text{dom }\Phi=\{(v,\gamma)|\Phi(v,\gamma)>-\infty\}\neq\emptyset. \quad (32)$$

A concave function is continuous in the relative interior of its effective domain, next we shall show that for a European option, $\Phi(v,\gamma)$ is a closed concave function.

Proposition A.2 *The function $\Phi(v,\gamma)$ is closed if the contingent claim is an option, i.e. $\xi(T)+f(\eta(T))\geq 0$ a.s. and $\xi(T)+f(\eta(T))>0$ with a positive probability.*

Proof First we shall identify the boundary and the relative interior of dom Φ, i.e. $\partial(\text{dom }\Phi)$ and ri dom Φ. Let $h(|\gamma|,y)<+\infty$ be the upper hedging price of the a contingent claim $(|\gamma|\xi(T),|\gamma|\eta(T))$, which is well defined for an option, according to Cvitanić and Karatzas (1996). It is easy to verify that

$$\partial(\text{dom }\Phi)=\{(v,\gamma)|v=g(\gamma):=-f(y)\mathbf{1}_{\gamma\geq 0}+h(|\gamma|,y)\mathbf{1}_{\gamma<0}\}, \quad (33)$$

and that

$$\text{ri dom }\Phi=\text{int dom }\Phi=\{(v,\gamma)|v>g(\gamma)\} \quad (34)$$

Next we show that $\forall(v,\gamma)\in\overline{\text{dom }\Phi}$, if $v_n\downarrow v$ and $\gamma_n\downarrow\gamma$, $\lim_{n\to\infty}\Phi(v_n,\gamma_n)=\Phi(v,\gamma)$. To show this, $\forall n$, we define the sets

$$A_n=\{H(T)\in\mathbf{L}^2_{0+}|(v_n,y,0)\text{ hedges }(H(T)-\gamma_n\xi(T),-\gamma_n\eta(T))\} \quad (35)$$

which are, obviously, non-empty and monotonically non-increasing in n, therefore we can safely write $\lim_{n\to\infty}A_n=\liminf_{n\to\infty}A_n=\limsup_{n\to\infty}A_n$. We also define

$$A=\{H(T)\in\mathbf{L}^2_{0+}|(v,y,0)\text{ hedges }(H(T)-\gamma\xi(T),-\gamma\eta(T))\} \quad (36)$$

which is also non-empty. It can be shown that $\lim_{n\to\infty} A_n = A$. In fact, $\forall H(T) \in \liminf_{n\to\infty} A_n$, there exists N_0 such that $\forall n > N_0$, $(v_n, y, 0)$ hedges $(H(T) - \gamma_n \xi(T), -\gamma_n \eta(T))$, and hence $(H(T) - \gamma \xi(T), -\gamma \eta(T))$. By the definition and attainability of the upper hedging price of $(H(T) - \gamma \xi(T), -\gamma \eta(T))$, $(\inf v_n, y) = (v, y)$ can also hedge $(H(T) - \gamma \xi(T), -\gamma \eta(T))$, therefore $H(T) \in A$, and consequently $A \supset \liminf_{n\to\infty} A_n$. But $A \subset A_n \ \forall n$, $A \subset \liminf_{n\to\infty} A_n$. The two conclusions yield $A = \liminf_{n\to\infty} A_n$. From the proof of the existence of $\Phi(v, \gamma)$, we know that $\Phi(v_n, \gamma_n) = E[U(\hat{H}_n(T)|\mathcal{F}_0]$, for some $\hat{H}_n(T) \in A_n$, and that $\Phi(v, \gamma) = E[U(\hat{H}(T)|\mathcal{F}_0]$ for some $\hat{H}(T) \in A$. Since $\hat{H}_n(T)$ are bounded in $\mathbf{L}^2(\Omega, \mathcal{F}, \mathbb{P}_0)$, there is a weakly convergent subsequence, which weakly converges to $\tilde{H}(T)$. Since the A_n's are convex and closed, they are also weakly closed, as a result, $\tilde{H}(T) \in \limsup_{n\to\infty} A_n = A$. For simplicity, we relabel this weakly convergent subsequence and still denote it $\hat{H}_n(T)$. According to Cvitanić and Karatzas (1996), Appendix B, the functional $E[U(\cdot)|\mathcal{F}_0]$ is upper semi-continuous, thus also upper semi-continuous in the weak topology. Consequently,

$$
\begin{aligned}
E[U(\tilde{H}(T))|\mathcal{F}_0] &\geq \limsup_{n\to+\infty} E[U(\hat{H}_n(T))|\mathcal{F}_0] \\
&= \limsup_{n\to+\infty} \Phi(v_n, \gamma_n)
\end{aligned}
$$

thus

$$
\begin{aligned}
\Phi(v, \gamma) &= E[U(\hat{H}(T)|\mathcal{F}_0] \\
&\geq E[U(\tilde{H}(T))|\mathcal{F}_0] \\
&\geq \limsup_{n\to+\infty} \Phi(v_n, \gamma_n).
\end{aligned}
$$

But for an option, $\Phi(v, \gamma)$ is clearly increasing in both v and γ, thus $\Phi(v_n, \gamma_n) \downarrow$ and $\Phi(v, \gamma) \leq \Phi(v_n, \gamma_n)$ for all n, which implies that $\Phi(v, \gamma) \leq \lim_{n\to+\infty} \Phi(v_n, \gamma_n)$. Hence we have $\Phi(v, \gamma) = \lim_{n\to+\infty} \Phi(v_n, \gamma_n)$. As a special result, $\forall (v, \gamma) \in \mathbb{R}^2$, we have

$$
\Phi(v, \gamma) = \lim_{s\downarrow 0} \Phi\left(v + s(v' - v), \gamma + s(\gamma' - \gamma)\right) \tag{37}
$$

for some $(v', \gamma') \in$ ri dom Φ, where the right hand side is exactly the definition of the closure of the concave function Φ, according to Hiriart-Urruty and Lemaréchal (1993), p149, Proposition 1.2.5. ∎

Given P and γ, the investor's maximal expected utility is given by $\Phi(x-\gamma P,\gamma)$, our next step is to maximize $\phi(\gamma) := \Phi(x-\gamma P,\gamma)$ with respect to $\gamma \in \mathbb{R}$. We start with considering the following constrained maximization problem

$$\max_{|\gamma|\leq m} \phi(\gamma) \tag{38}$$

According to Hiriart-Urruty and Lemaréchal (1993), p149, Proposition 1.2.5, $\phi(\gamma)$ is a closed concave function of one argument γ in $\mathbb{R}$, besides, $[-m,m] \cap \text{dom } \phi \neq \emptyset$, $\forall m > 0$, if $P \geq 0$, $(x,y) \in \mathbb{R}^2_+ - \{(0,0)\}$, therefore the above problem always has a solution. We thus denote the set of maximizers $\hat{\gamma}^m(P) \neq \emptyset$ when the price is $P \geq 0$. Generally, $\hat{\gamma}^m(P)$ has more than one member, and $\forall \gamma \in \hat{\gamma}^m(P)$, $(x-\gamma P,\gamma)$ solves

$$\max_{(v,\kappa)\in\mathcal{C}(P)} \Phi(v,\kappa) \tag{39}$$

where $\mathcal{C}(P) = \{(v,\kappa)|v+\kappa P = x, |\kappa| \leq m\}$. It is easy to check that $\mathcal{C}(P) \cap (\text{ri dom } \Phi) \neq \emptyset$.

Proposition A.3 *The correspondence* $\hat{\gamma}^m : [0,+\infty[\to [-m,+m]$ *is upper semi-continuous (u.s.c.).*

Proof Letting $\lim_{n\to\infty} P_n = P \in \mathbb{R}$, and $\gamma_n \in \hat{\gamma}^m(P_n)$, $\lim_{n\to\infty}\gamma_n = \gamma \in \mathbb{R}$ (if γ_n does not converge, we can select a subsequence and relabel accordingly), we argue that $\gamma \in \hat{\gamma}^m(P)$. Otherwise, we can pick some $\delta \in \hat{\gamma}^m(P)$, then $-\infty \leq \Phi(x-\gamma P,\gamma) < \Phi(x-\delta P,\delta)$, which implies that $(x-\delta P,\delta) \in \text{dom } \Phi$. There are two possibilities:

1. $(x-\delta P,\delta) \in \partial(\text{dom } \Phi)$. Again by Hiriart-Urruty and Lemaréchal (1993), p149, Proposition 1.2.5, the function $\Phi(v,\kappa)$ is continuous along the line segment $\mathcal{C}(P)\cap\overline{\text{dom } \Phi}$. Since $\mathcal{C}(P)\cap(\text{ri dom } \Phi) \neq \emptyset$, we can always move $(x-\delta P,\delta)$ a little bit along $\mathcal{C}(P)$ to a point $(x-\gamma' P,\gamma') \in \text{ri dom } \Phi$, such that $\Phi(x-\gamma' P,\gamma') > \Phi(x-\gamma P,\gamma)$, provided that the movement is small enough;
2. $(x-\delta P,\delta) \in \text{ri dom } \Phi$, then let $\gamma' = \delta$.

Therefore we can always find a point $(x-\gamma' P,\gamma') \in \mathcal{C}(P)\cap(\text{ri dom } \Phi)$ such that $\Phi(x-\gamma' P,\gamma') > \Phi(x-\gamma,\gamma)$. Since the function Φ is continuous at $(x-\gamma' P,\gamma')$, $\exists \alpha,\beta > 0$ such that

$$\Phi(v,\kappa) > \Phi(x-\gamma P,\gamma)+\beta,\ \forall (v,\kappa) \in \mathcal{N}_\alpha \tag{40}$$

where $\mathcal{N}_\alpha \equiv \{(v,\kappa)|(v-x+\gamma' P)^2+(\kappa-\gamma')^2 < \alpha^2\}$ is the α-neighborhood of $(x-\gamma' P,\gamma')$. Besides, $\lim_{n\to\infty}\gamma_n=\gamma\neq\gamma'$, the α can be chosen even small enough such that

$$\forall N_0,\ \exists l > N_0 \text{ such that } \gamma_l \notin \mathcal{N}_\alpha. \tag{41}$$

Moreover, it can be shown that

$$\exists N_1, \mathcal{C}(P_n)\cap\mathcal{N}_\alpha\neq\emptyset, \forall n > N_1 \tag{42}$$

which is the result of $\lim_{n\to\infty}P_n=P$. But Φ is closed, i.e.

$$\limsup_{(v',\kappa')\to(v,\kappa)} \Phi(v',\kappa') \le \Phi(v,\kappa) \tag{43}$$

which implies that

$$\limsup_{n\to\infty}\Phi(x-\gamma_n P_n,\gamma_n)\le\Phi(x-\gamma P,\gamma), \tag{44}$$

thus for the previously mentioned $\beta>0$, $\exists N_2$, $\forall n > N_2$, $\Phi(x-\gamma_n P_n,\gamma_n)<\Phi(x-\gamma P,\gamma)+\beta$. By (41), for $N_0=\max(N_1,N_2)$, $\exists l>N_0$ such that $\gamma_l\notin\mathcal{N}_\alpha$, and $\mathcal{C}(P_l)\cap\mathcal{N}_\alpha\neq\emptyset$, therefore, $\forall(v,\kappa)\in\mathcal{C}(P_l)\cap\mathcal{N}_\alpha$, we have

$$\Phi(v,\kappa)>\Phi(x-\gamma P,\gamma)+\beta>\Phi(x-\gamma_l P_l,\gamma_l) \tag{45}$$

which contradicts the assumption that $\gamma_l\in\hat\gamma^m(P_l)$. Therefore we conclude that $\gamma\in\hat\gamma^m(P)$, and have established the u.s.c. ∎

Remark It is easy to observe that $\hat\gamma^m(P)=\{-m\}$, $\forall P>h(1,0)$, according to the definition of $h(1,0)$, and that $\hat\gamma^m(0)=\{m\}$, because the monotonicity of Φ.

Proposition A.4 *There exists at least one equilibrium price* $P\in]0,\infty[$.

Proof We first show that an equilibrium exists if the constraint $|\gamma|\le m$ is imposed upon all investors. We denote $\hat\gamma_i^m(P),\Phi_i(v,\gamma)$ and $h_i(1,0)$ the ith investor's $\hat\gamma^m(P)$, $\Phi(v,\gamma)$ and $h(1,0)$, respectively. Clearly,

$$\hat\gamma_i^m(P)=[\inf\hat\gamma_i^m(P),\sup\hat\gamma_i^m(P)] \tag{46}$$

because of the concavity of Φ_i and the u.s.c of $\hat\gamma_i^m$. We also have $\hat\gamma_i^m(0)=\{m\}$, $\hat\gamma_i^m(h_{\max})=\{-m\}$, $\forall i$, where $h_{\max}:=2\max_{1\le i\le N}h_i(1,0)>$

$h_i(1,0)$, $\forall i$.
We then define the congregated demand Γ^m as a correspondence of argument P by

$$\Gamma^m(P) = \left\{ \sum_{i=1}^{N} \gamma_i | \gamma_i \in \hat{\gamma}_i^m(P), i = 1, \cdots, N \right\} \tag{47}$$

which is also u.s.c and also a closed interval $\Gamma^m(P) = [a(P), d(P)]$, where

$$a(P) = \sum_{i=1}^{N} \inf \hat{\gamma}_i^m(P), \quad d(P) = \sum_{i=1}^{N} \sup \hat{\gamma}_i^m(P). \tag{48}$$

Besides,

$$\Gamma^m(h_{\max}) = \{-Nm\}, \quad \Gamma^m(0) = \{Nm\}. \tag{49}$$

Let $P^m = \sup\{P \in [0, h_{\max}] | a(P) > 0\}$ then $\exists P_n^m \downarrow P^m$, $a(P_n^m) \leq 0$ and $\lim_{n\to\infty} a(P_n^m) \in \Gamma^m(P^m)$ by u.s.c, which implies that $a(P^m) \leq \lim_{n\to\infty} a(P_n^m) \leq 0$; and $\exists Q_n^m \uparrow P^m$, $a(Q_n^m) > 0$ and $\lim_{n\to\infty} a(Q_n^m) \in \Gamma^m(P^m)$ by u.s.c, which implies that $d(P^m) \geq \lim_{n\to\infty} a(Q_n^m) \geq 0$. Hence $0 \in [a(P^m), d(P^m)] = \Gamma^m(P^m)$, i.e., P^m is an equilibrium price. Obviously, $0 < P^m < h_{\max} < \infty$.
We next show that $\forall m > 0$, the equilibrium demands of the option is uniformly bounded. Let $\gamma_1, \cdots, \gamma_N$ be the demands of the N investors, when an equilibrium is achieved at price $P^m > 0$. Since $\Phi_i(x_i - \gamma P, \gamma)$ can only be maximized when $x_i - \gamma P \geq -f(y_i)$, $\forall P > 0$, we find that

$$\gamma_i \leq \frac{x_i + f(y_i)}{P^m} \tag{50}$$

For the ith investor, $\exists Q_i > 0$ such that $\hat{\gamma}_i^m(P) \subset]0, +\infty[$, $\forall P < Q_i$, this is because $\hat{\gamma}_i^m$ is u.s.c and $\hat{\gamma}_i^m(0)$ contains only a single positive element. Moreover, the Q_i is independent of m because of the concavity of $\Phi_i(x_i - \gamma Q_i, \gamma)$ as a function of γ. Therefore, $P^m \geq \min_{1\leq i\leq N} Q_i := Q$, and consequently

$$\gamma_i \leq \frac{x_i + f(y_i)}{P^m} \leq \frac{x_i + f(y_i)}{Q} \leq \max_{1\leq i\leq N} \frac{x_i + f(y_i)}{Q} \equiv K < \infty \tag{51}$$

which implies that

$$\max_{1\leq i\leq N} \gamma_i \leq K, \tag{52}$$

and the above further implies that

$$\min_{1\leq i\leq N} \gamma_i \geq -(N-1)K. \tag{53}$$

Therefore, let $m = NK$, denote the corresponding equilibrium demands $\gamma_1, \cdots, \gamma_N$ and the equilibrium price P^{NK}, then $|\gamma_i| < m$, $\forall i$. Consequently, each γ_i also solves

$$\max_{|\gamma|<m} \Phi(x_i - \gamma P^{NK}, \gamma) \tag{54}$$

but $\Phi(x_i - \gamma P^{NK}, \gamma)$, viewed as a function of one argument γ only, is concave in $\mathbb{R}$, and has a finite local maximum in an open set $]-m, m[$; the conclusion is that the local maximum is also a global maximum, i.e., γ_i also solves

$$\max_{\gamma\in\mathbb{R}} \Phi(x_i - \gamma P^{NK}, \gamma). \tag{55}$$

Together with the market clearing condition held by the γ_i's, we find that the P^{NK} is the desired equilibrium price and γ_i's the corresponding demands. $\blacksquare$

At this stage, we have established the existence of equilibrium in a reasonable general environment.

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