Static Hedging of Multivariate Derivatives by Simulation

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Abstract

We propose an approximate static hedging procedure for multivariate derivatives. The hedging portfolio is composed of statically held simple univariate options, optimally weighted minimizing the variance of the difference between the target claim and the approximate replicating portfolio. The method uses simulated paths to estimate the weights of the hedging portfolio and is related to control variates techniques. We report numerical results showing the performance of this static hedging procedure on bivariate options on the maximum of two assets and on 2- and 7-dimensional portfolio options. It is shown that, in the presence of transaction costs, Value at Risk and Expected Shortfall of the dynamically hedged positions can be higher than the ones obtained by a static hedge.

KEYWORDS: Option hedging, Risk management, Monte Carlo simulation, Control variates.

1 Introduction

Arbitrage pricing theory can be used to obtain the market value of complex derivatives. If arbitrages are ruled out and a replicating portfolio exists then its cost must be equal to that of the contingent claim. The famous Black–Scholes (BS) pricing formula is based on arbitrage arguments and explicitly provide a delta-based hedging strategy to replicate a plain put/call option.

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However the success is not guaranteed as continuous time rebalancing of the hedging portfolio is in order. On one side this is practically infeasible forcing the hedger to discretize the strategy and on the other the situation is worsened if transaction costs are taken into account. A substantial research effort has tried to solve some of the problematic aspects of practical derivatives hedging. [Boyle and Emanuel, 1980] is an early paper that deals with discretization error in implementing the Black–Scholes strategy. [Leland, 1985] propose a discrete modified BS-type strategy that allows in the limit (but see also [Kabanov and Safarian, 1997]) to replicate the claim allowing for transaction costs. This is obtained by inflating the volatility of the stock that in turn forces to charge a bigger price to the buyer and to reduce the weight of the required rebalacings. Leland's paper shows that quite often the hedger need 10 to 20% of the BS claim price to be able to fulfill his obligations. Another impressive illustration of the difficulties in concrete option hedging is in [Green and Figlewsky, 1999], that shifts the attention on model risk. It is argued that BS model might be inappropriate for a variety of reasons, the most investigated being probably the estimation and modelization of the volatility of the underlying stock. The paper shows that huge risk is involved in practical option hedging. This is true also when extreme care is used to estimate volatilities or when realized *ex post* volatility is (unfairly) plugged into the strategy. Moreover, inflating the volatility up to 50% of the calculated estimate, reduces the mean loss but is less useful to reduce extreme shortfalls.

Other papers, [Derman et al., 1995, Carr et al., 1998], describe a different paradigm, named *static hedging*. If the claim is statically redundant, then it exists a portfolio of traded assets that replicates the target payoff without the need to rebalance in continuous time. A very simple example is given by the put/call parity equation, that can be trivially employed to hedge a call (put) option by a portfolio in the stock and in a put (call) option. It is of course not always possible to find such a simple solution, but [Derman et al., 1995] shows how to approximately hedge a barrier claim using plain options with different strikes and maturities. This approximation is exact solely on the nodes of a binomial tree that models the underlying dynamics but nevertheless the static hedging that is obtained has no extra cost besides the initial purchase.

Recent work by Ben-Ameur et al. describe a *partial hedging* technique, that actually hedges a 'portion' of the claim. This can be done when the amount needed for a perfect hedge is not available or the agent prefers to use only a fraction of the price for hedging purposes, [Ben-Ameur et al., 2001]. Of course some default risk is introduced and it is understood that, should the hedging portfolio fail to provide the required sum, the agent is resorting to other resources to fill the gap. An example, in the case of a European call option, is given by a knock out option with the same strike that will hedge the claim only if the terminal stock price is below the knockout barrier. Needless to say, this is a cheap but risky way to hedge against risk and it turns out that this behavior is coherent with a risk-seeking agent [Follmer and Leukert, 1999]. In a more recent paper [Follmer and Leukert, 2000] show how to optimally hedge a contingent claim minimizing, under a budget constraint, the expected shortfall weighted by a loss function l that depends on the risk attitude of the agent. It is shown that the optimal solution is to hedge a modified claim, that in the case of a European call smoothly varies from the knockout option in the risk-seeking case to a call with a bigger strike for the most risk averse agent. Note that the idea of hedging a modified claim is also exploited for the apparently different purpose of reducing the effects of misspecified models [Ahn et al., 1997].

While the previous mentioned works all deal with univariate options, it is obvious that properly hedging is difficult to get for multivariate options as well. Indeed, there are reason to believe that, as each rebalancing requires adjustment of multiple assets in the hedging portfolio, the effect of transaction costs and discretization might be amplified. As far as we know, this multivariate hedging problem has received much less attention than the univariate one, probably for the difficult analytical treatment that is required.

In this paper we will show how a static portfolio to approximately hedge a multivariate derivative can be constructed minimizing the variance of the discrepancy between the payoffs and the revenues of the portfolio. We show that this methods is related to a recent variance reduction method for Monte Carlo option pricing, known as Mean Monte Carlo (MMC), [Pellizzari, 2001]. The paper [Avellaneda et al., 2000] describes a very interesting way to calibrate a Monte Carlo simulation to obtain exact pricing of some benchmark instruments and contains ideas similar to ours. However, the setting is different in that Avellaneda and coauthors still focus mainly on pricing of univariate claims.

In detail, Section 2 presents an example of a portfolio option on two assets clarifying the differences among various hedging methods. It is apparent that, if perfect hedging is practically unattainable or too costly, then approximate static hedging can be explored. In Section 3, the MMC method is briefly covered in order to show that it can be fruitfully interpreted in terms of static hedging and the extension is discussed. We then present a pricing numerical comparison on bivariate max options analyzed in [Boyle, 1988] that extends the binomial approach of Cox-Ross-Rubinstein to the case of two risk sources. Some 2- and 7-dimensional portfolio options are also analyzed with emphasis on downside risk using Value at Risk and Expected Shortfall. Finally, section 5 contains some concluding remarks.

2 A worked example

Suppose you are asked to write an European option on a basket of two risky lognormal stocks whose dynamics is described by

$$dS_i = rS_i dt + \sigma S_i dZ_i, \;\; i = 1, 2,$$

where r, σ_1, σ_2 are constants and Z_1, Z_2 are standard Brownian motions. We denote the values of the asstes at time t by $S_{1t}, S_{2t}, 0 \le t \le T$. The payoff at expiration is

$$f(S_{1T}, S_{2T}) = \max(S_{1T} + S_{2T} - k, 0). \tag{1}$$

Assume we set the initial prices $S_{10} = S_{20} = 100$, volatilities $\sigma_1 = 0.3, \sigma_2 = 0.2$, correlation between returns is $\rho = 0.5$, maturity $T = 1$ year, strike price $k = 190$ and riskless rate $r = 0.1$ (continuously compounded).

As there is no analytic valuation formula for such asset, a preliminary Monte Carlo simulation is performed using 100 replications obtaining the price $\hat{C}_{MC} = 33.89$ equipped with sample standard deviation $\hat{\sigma}_{MC} = 3.26$. Note that 100 simulations are clearly too few to obtain reasonable precision but our main point is totally independent of the number of replicates.

Next, in order to lower the huge standard deviation just quoted, another more refined method (see [Pellizzari, 2001] or the review in the following section) is used to get more accurate Mean Monte Carlo (MMC) price $\hat{C}_{MMC} = 33.34$ with standard deviation $\hat{\sigma}_{MMC} = 0.40$ (again 100 simulations are used). The standard deviation is reduced by 88%, which is *per se* an important achievement. However, this encouraging picture still does not take into account other useful features. In particular the method also reports that the following portfolio of simple call options can be used to (partially) hedge the basket option payoff: consider at $t = 0$ a long position in 0.96 call options on the first asset and 1.01 call options on the second, being respectively 94.88 and 89.71 the strike prices. In order to get this portfolio an additional 5.59 must be borrowed in $t = 0$ and paid back with interest at maturity.

This compound portfolio allows to have at maturity a random sum that can be used to 'hedge' the random payoff of basket option (1). In fact the net difference between the payoff of the basket option and the value of the 'hedging' portfolio is on average null with standard deviation $\exp(rT)\hat{\sigma}_{MMC}\sqrt{100} = 4.4$. Note that the standard deviation of the option payoff is about 36, hence using the previously mentioned 'hedging' portfolio has reduced the risk by a factor of about 8.

Let us rephrase the whole experiment as follows: once you write an option you get the price and commit to the subsequent liability. The two extreme choices are no hedging at all and delta-hedging. On one side we envision huge final risk as the average final net payment is null, but the standard deviation, computed with the risk neutral density, is 36. On the other side it is well known that delta-hedging is imprecise and, more importantly, expensive in a realistic situation. There is a third possible solution, based on the previously described portfolio, that partially hedges the option payoff leaving a risk (measured as standard deviation of the net cashflow) of 4.4. The following Table 1 sums up the situation.

Method	Hedge	Risk(σ)	Risk Reduction	Cost	Hedging Portfolio
MC	None	36.0	0%	None	None
Static	Partial	4.4	88%	Low	Static, 2 calls
Dynamic	Perfect	0	100%	High	Dynamic, Delta-based

Table 1: The hedging performance of different methods are shown. The 'Risk' column provides the standard deviation of the net cashflows (whose mean is approximately null) at maturity. In the 'Risk Reduction' column we report the reduction in standard deviation with respect to the no hedging (MC) situation. The final two columns contain a rough description of the cost of the hedging strategy and of the hedging portfolio.

This example (together with many others we could select) shows that the lower variance obtained using MMC comes together with information that can be used to build a static portfolio that partially hedges the liability leaving a residual risk whose standard deviation is, up to a constant, the same of the sample price. Hence, if low variance is obtained in the enhanced Monte Carlo simulation, then low risk is associated to the static hedging portfolio. As far as we know, there is no other example of variance reduction technique that has a financial interpretation in terms of static hedging. In the following section we briefly review MMC and describe this static interpretation.

3 Mean Monte Carlo and approximate static hedging

In this section we describe the Mean Monte Carlo method extending its capabilities by optimally selecting some parameters that affect the variance. This procedure is equivalent to (partially) hedge the claim using a portfolio of simple assets. If the residual risk is deemed appropriate or can be reduced to a suitable level, the static portfolio can be employed in place of complex dynamic hedging procedures.

Referring the reader to [Pellizzari, 2001] for a detailed treatment, we describe the main ideas behind MMC. Assume we want to price at time $t = 0$ an european option on n stocks that pays the sum

$$f(S_{1T}, \ldots, S_{nT}), \tag{2}$$

at maturity T, where $S_{it}, 0 \leq t \leq T, 1 \leq i \leq n$ denotes the value of i-th asset at time t and the dynamic of each asset is described by

$$dS_i = rS_i dt + \sigma_i S_i dZ_i,$$

where r is the istantaneous riskless rate, σ_i is the volatility and Z_i is a standard Brownian motion such that $Cor(Z_i, Z_j) = \rho_{ij}$. The MMC method exploits a set of control variates (see [Rubinstein, 1981] or [Bratley et al., 1987]) obtained by plugging into (2) the known means $E[S_{iT}]$ to replace the random S_{iT} for all i's but one. The j-th control variate ($1 \leq j \leq n$) is given by:

$$M_T(j) = f(E[S_{1T}], \ldots, E[S_{j-1,T}], S_{jT}, E[S_{j+1,T}], \ldots, E[S_{nT}]). \tag{3}$$

The above definition matches the two requirements of a control variate, namely positive correlation with (2) and easy analytical valuation of the mean of (3) using the Black-Scholes formula. Technically, the replacement of $n-1$ expectations in the payoff is used to reduce the dependence of (2) to one risk source alone, allowing most often to apply standard pricing methods to the resulting univariate payoff. Observe that there is no guarantee that $E[M_T(j)]$ will be known in close form for every conceivable payoff profile f. However, even if numerical approximations should be used to solve the univariate integral for $E[M_T(j)$, there are extremely accurate methods (e.g., Gaussian quadrature) for univariate problems, see [Krommer and Ueberhuber, 1998] for a comprehensive account and [Schmeiser et al., 2000] for an application to control variates.

The estimate of the price is then obtained by taking the discounted average of many random payoffs

$$f(S_{1T}, \ldots, S_{nT}) - \sum_{i=1}^{n} \hat{b}_i (M_T(i) - E[M_T(i)]), \tag{4}$$

where $\hat{\mathbf{b}} = (\hat{b}_1, \ldots, \hat{b}_n)$ is a vector of estimated coefficients chosen to minimize the variance.

A moment of reflection shows that variance reduction is achieved using a combination of assets whose final payoff is given by the $M_T(j)$'s. Recall the example of Section 2, where we have

$$\begin{aligned} M_T(1) &= \max(S_{1T} + E[S_{2T}] - k, 0), \\ M_T(2) &= \max(E[S_{1T}] + S_{2T} - k, 0), \end{aligned}$$

noting that the basket payoff is mimicked by a portfolio made of one call option on asset S_1 (having strike price $k_1 = E[S_{2T}] - k$) and one call option on asset S_2 (having strike $k_2 = E[S_{1T}] - k$). Thus, if the variance reduction is effective, i.e. the standard deviation of (4) is small, then the payoff of the portfolio of the two calls is close to that of the basket option. This is tantamount to say that an approximate static hedging portfolio has been obtained selecting a combination of plain call options. This optimal selection of control variates can directly be interpreted in terms of approximate static hedging as discussed in the sequel.

The previous description of the MMC method can be used to propose a straightforward generalization based on the search of a hedging portfolio by minimization of the variance of the net difference of the option payoff and the value of the hedging portfolio. Assume we can build a hedge choosing among the assets in the set

$$H = \{H_1, \ldots, H_p\}, \tag{5}$$

where each $H_i, i = 1, \ldots, p$ is a payoff function (at maturity T), typically depending on some underlying stock $S_j, j = 1, \ldots, n$ and possibly on some other adjustable parameters collected in the vector $\mathbf{a}$. The set H should contain liquid assets (for low trading costs) that span a wide payoff space. Some reasonable examples of such set H will be described soon. Then we can approximately hedge the option (2) by solving the minimization problem

$$\min_{b_1,\ldots,b_p,\mathbf{a}} Var\left[f(S_{1T}, \ldots, S_{nT}) - b_1 H_1 - b_2 H_2 - \ldots - b_p H_p\right] \tag{6}$$

in the quantities $b_1, \ldots, b_p$ that are held in each H_i. We typically expect p to be much smaller than N, as the former is about of the same size as n, the number of risk sources, while the latter is at least in the thousands. This also ensures that few control variates are used with respect to the number of simulations, as suggested in [Nelson, 1990] or [Lavenberg and Welch, 1981].

Note that the minimization variables could also be strike prices listed in $\mathbf{a}$ and are not necessarily limited to quantities. Without loss of generality, we assume for a moment that only the latter are to be estimated: in principle, in fact, we can always augment the number p of regressors and omit the vectorial parameter $\mathbf{a}$. If the payoff and the assets in H are $(p+1)$-dim multivariate normal, then the $\hat{b}_i$'s are obtained by regressing $f(S_{1T}, \ldots, S_{n,T})$ on the vectorial subspace $\mathcal{L}(H_1, \ldots, H_p)$, to get the theoretical minimizers $(b_1^*, \ldots, b_p^*)$. The normality assumption can be grossly violated in a pricing framework as the payoff density is likely to have positive mass at 0 (especially in the case of out-of-the-money options). Hovewer, theorem 3 in [Nelson, 1990] points out that, if the sample size $N \longrightarrow \infty$, then $\hat{b}_i \longrightarrow b_i^*$, hence justifying the use of least squares estimation in the case of large sample size. This approach, consistently used in [Pellizzari, 2001], provides good results for a variety of payoffs of different type and dimensionality.

Some examples of the set H are described below, together with the specification of the minimization problem to solve. We assume $n = 2$ and still refer for simplicity to the problem of approximately hedge the payoff of the example in Section 2.

1.
$$H = \{\max(S_{1T} + E[S_{2T}] - k, 0), \max(S_{2T} + E[S_{1T}] - k, 0)\}.$$

   This is a restrictive situation, corresponding to MMC, where we hedge using solely two call options with *given* strike prices. The minimization problem is

$$\min_{b_1, b_2} Var\left[f(S_{1T}, S_{2T}) - b_1 \max(S_{1T} + E[S_{2T}] - k, 0) - b_2 \max(S_{2T} + E[S_{1T}] - k, 0)\right]. \quad (7)$$

2. Assume we can hedge selecting the strike prices: then we have that

$$H = \{\max(S_{1T} - a_1, 0), \max(S_{2T} - a_2, 0)\}.$$

   It is obvious that more flexibility in the hedging strategy is allowed and the corresponding minimization problem is

$$\min_{u_1, u_2, \mathbf{a}=(a_1, a_2)} Var\left[f(S_{1T}, S_{2T}) - b_1 \max(S_{1T} - a_1, 0) - b_2 \max(S_{2T} - a_2, 0)\right].$$

3. Why should we hedge using options alone? If we set

$$H = \{\max(S_{1T} - a_1, 0), \max(S_{2T} - a_2, 0), S_{1T}, S_{2T}\}$$

   then better approximate replication can be obtained as stocks themselves can be used. The minimization problem becomes

$$\begin{aligned}\min_{b_1, b_2, b_3, b_4, \mathbf{a}=(a_1, a_2)} Var[f(S_{1T}, S_{2T}) \quad &- \quad b_1 S_{1T} - b_2 S_{2T} \\ &- \quad b_3 \max(S_{1T} - a_1, 0) - b_4 \max(S_{2T} - a_2, 0)\,].\end{aligned}$$

4. In realistic situations it might be that only some strikes are available in the option market. Assume for example that only call written at strikes 90, 100, 110 on S_1 and S_2 can be purchased, in such a way that just some out, at and in-the-money options are eligible. Then the set H is

$$H = \{\max(S_{1i} - 90, 0), \max(S_{1i} - 100, 0), \max(S_{1i} - 110, 0), S_{iT}\},$$

where $i = 1, 2$ and the variance is minimized with appropriate choice of the 8 variables $b_i^{(90)}, b_i^{(100)}, b_i^{(110)}, b_i, i = 1, 2$ denoting the quantities held in the call 90, call 100, call 110 (written on each assets) and in the two stocks S_1 and S_2, respectively. Formally, the optimization problem is

$$\begin{aligned}\min Var\,[f(S_{1T}, S_{2T}) &- b_1 S_{1T} - b_2 S_{2T} \\ &- b_1^{(90)} \max(S_{1T} - 90, 0) - b_2^{(90)} \max(S_{2T} - 90, 0) \\ &- b_1^{(100)} \max(S_{1T} - 100, 0) - b_2^{(100)} \max(S_{2T} - 100, 0) \\ &- b_1^{(110)} \max(S_{1T} - 110, 0) - b_2^{(110)} \max(S_{2T} - 110, 0)\,],\end{aligned}$$

where minimization is performed on the aforementioned 8 variables.

These examples are of course not exhaustive (for example, an exchange option on the two assets is indeed useful to further reduce risk) but show how approximate hedging can be built in different frameworks.

It is obvious that functionals other than the variance could be minimized and parameters in **b** and **a** selected, for example, maximizing the expected utility of the difference between the payoff and the hedge. There are however some reasons to prefer the simple quadratic approach of the variance. On one side this choice allows to interpret the procedure as a Monte Carlo method with control variates. On the other hand, the estimation of the parameters is done in the quadratic framework by least squares and, if **b** alone is to be estimated, only OLS is required to build the static portfolio. Note also that the variance minimization problem is in full generality non differentiable, due to the possible lack of smoothness of the payoff function f with respect to **a**.

Finally, though not used in the following, we note that problem (6) can be generalized as follows:

$$\begin{aligned}\min_{b_1,\ldots,b_p,\mathbf{a}} \quad & Var\,[f(S_{1T}, \ldots, S_{nT}) - b_1 H_1 - b_2 H_2 - \ldots - b_p H_p] \\ \text{subject to:} \quad & g_i(b_1, \ \ldots b_p, \mathbf{a}) \geq 0, i = 1, \ldots, m. \end{aligned} \tag{8}$$

Costrained Monte Carlo is a recent research area: both [Szechtman and Glynn, 2001] and [Avellaneda et al., 2000] are recent papers that deal with similar frameworks, though their ideas are somewhat different. In the simplest case (normal random variables and linear costraints), the optimal b_i's can be estimated by constrained least squares but more theoretical and applied research is needed to clarify the potential of the generalization.

The m constraints could take into account some important financial features: no short selling, for example, can be enforced by setting $m = p$ and

$$g_i(b_1, \dots b_p, \mathbf{a}) = b_i, i = 1, \dots, p.$$

Another interesting possibility is to require the static hedging to be approximately self-financing: assume that the p assets in H can be priced by Black–Scholes formulas, like in all the cases shown previously, or by numerical methods. Let now $\mathbf{C} = (C_1, \dots, C_{p)}$ be the vector of the prices of the simple assets in H and let $\hat{C}_0$ be a pilot estimate of the price of the complex derivative, for example computed by a preliminary simulation. Then the constraint

$$g_i(b_1, \dots b_p, \mathbf{a}) = \hat{C}_0 - \sum_{i=1}^{p} b_i C_i \tag{9}$$

will produce a hedge that costs at $t = 0$ approximately the same amount that is cashed to write the derivative. This (partial, static) hedge can then be compared with a standard delta-based dynamic hedge in a more rigorous sense.

4 Some static hedges

In the following we present some practical application of the method by pricing and hedging options on the maximum of two stocks and portfolio options written on multiple assets.

We first price some european call and put options on the maximum of two stocks and compare against the results obtained in [Boyle, 1988] using a bivariate lattice approach with 50 steps. This is an indirect way to assess the quality of the replication of the static hedge at maturity: if in fact the static portfolio closely matches the derivative final payoff, then its price should be extremely close to the theoretical one to avoid arbitrage opportunities. See [Rebonato and Cooper, 1998] for another application of the same idea.

The parameters are the following: $S_{01} = S_{02} = 40, \sigma_1 = 0.2, \sigma_2 = 0.3, \rho_{12} = 0.5, r = 0.04879$ continuously compounded, $T = 0.58333$ and exercise prices of 35, 40 and 45.

Table 2 reports the result of the lattice method, analytical evaluation formulas, [Stulz, 1982], the cost of the static hedging, namely $\sum_i b_i C_i$, composed of plain options on each of the assets (two call options for the maximum and two put options for the minimum) and of cash, as exemplified in the second example of Section 2. We report the mean and the standard deviation (in brackets) of the cost on 100 tries. The hedging parameters are estimated using 1000 simulated samples. The value of the static portfolio is close to both the theoretical price and the lattice approximation, always showing pricing errors smaller than 1 cent. Observe, in the rightmost column, that the replication error has still a sizeable standard deviation thus pointing to residual non-hedged risk.

Exercise Price	[Boyle, 1988]	[Stulz, 1982]	Static	$\mathrm{SD}[f - \sum_i \hat{b}_i H_i]$
		European Call on the Maximum of Two Assets		
35	9.419	9.420	9.414 (0.063)	2.033
40	5.483	5.488	5.484 (0.051)	1.615
45	2.792	2.795	2.792 (0.034)	1.072
		European Put on the Minimum of Two Assets		
35	1.392	1.387	1.383 (0.016)	0.525
40	3.795	3.798	3.792 (0.035)	1.119
45	7.499	7.500	7.491 (0.050)	1.657

Table 2: Comparison of max/min option prices. The results of a lattice method ([Boyle, 1988]), analytical evaluation formula ([Stulz, 1982]) and cost of static hedging are shown. The last column shows the average standard deviation of the replication error.

The next examples will compare the well known dynamic delta-hedging with the performance of a static hedging approach on portfolio options written on 2 and 7 assets. As we are mainly interested in quantifying the risk exposure, we report two popular risk measures, the Value at Risk (VaR) and the Expected Shortfall (ES). The VaR at confidence level α is such that

$$Pr(X \leq \mathrm{VaR}_\alpha) = \alpha,$$

where X is the random replication error at maturity. We estimate VaR_α by the empirical 100α percentile. The ES is defined as

$$ES_\alpha = E[X | X \leq \mathrm{VaR}_\alpha].$$

We empirically obtain ES_α by sorting the realizations of X and taking the mean of the smallest α percent of the sample. It is standard practice to compute VaR's and ES's at confidence levels $\alpha = 0.01$ and $\alpha = 0.05$ and we do not depart from this custom.

Consider first an European portfolio option written on two risky stocks S_1, S_2 with final payoff $\max(S_{1T} + S_{2T} - k, 0)$ and select the parameters as in the example of Section 2. We simulate[1] 1000 stock paths, and approximatively weekly and monthly rebalancings of the delta-based hedging

[1] In the following, all simulated paths are obtained using the risk neutral probability densities. It is true that hedging errors depend on the drift of the assets, but we do not provide a full set of simulations for different drift values, that are left to future research. Some experimentation performed by us shows that while the risk figures are obviously changing with drifts, the comparative picture of dynamic and static hedging is not and the final considerations apply to a broad range of drifts.

portfolios (hence 60 and 12 revisions were allowed, respectively). Of course no rebalancing is performed for the static hedging strategy. There is no close formula to compute the deltas for a portfolio option and they were accurately[2] evaluated by Monte Carlo simulation. The parameters of the static hedge (again composed by call options in each asset and cash) are calculated as in the above example making use of 1000 simulations and are shown in Table 3. We can see in the table that the optimal static hedging portfolio is made of 1.00 call option on S_1, with strike 98.06, and 1.01 call options on S_2, with strike 89.17. Both options have maturity 1 year and borrowing 3.87 is necessary at $t = 0$. This amount, as described in [Avellaneda et al., 2000], is the constant term in the regression used to compute the weights, discounted back at time $t = 0$.

	Quantity	Strike
Call on S_1	1.00	98.06
	(0.010)	(0.38)
Call on S_2	1.01	89.17
	(0.006)	(0.36)
Cash	-3.87	-
	(0.27)	-

Table 3: Parameters of the static hedging portfolio, with standard deviations in brackets.

Assuming a perfect market with no transaction costs, Table 4 compares the replication error of static and dynamic hedging portfolios. A look at the last rows shows that approximate null mean is achieved as expected, though standard deviation of the replication error is still sizeable. This is true in particular for dynamic hedging, that should (in continuos time) offer perfect risk coverage. However, due to discrete rebalancings (and possibly inaccurate delta calculations) the residual risk is far from being negligible. The $\mathrm{VaR}_{0.05}$ of the of dynamically revised hedges are -4.01 and -8.21 for weekly and monthly revisions, respectively. In comparison the static hedging portfolio $\mathrm{VaR}_{0.05}$ is -4.28. It is interesting to note that the maximum loss cannot exceed 4.28 because in the worst possible scenario (when both call options in the static portfolio expire out-of-the-money) $4.28 = 3.87 \exp(rT)$ must be paid for the initial borrowing. This bounded loss might be an additional nice feature of the static hedging for portfolio options, though the same property does not hold in the previous example for maximum options. Figure 1 shows the distribution of the hedging errors in the monthly case: the peak corresponding to -4.28 is clearly visible as it is

[2] In detail, central differences and correlated Monte Carlo is used, see [Rubinstein, 1981] for an introduction. This means that we evaluate the greeks using the *same* sequence of random numbers, thus limiting the sample noise that can deteriorate numerical derivative calculations. The resulting deltas appear to be accurate and stable.

the heavy left tail for the dynamic hedge. The amount of residual risk inherent to the different kinds of hedging is largely dependent on the chosen risk measure: if $\mathrm{VaR}_{0.01}$ or expected shortfalls are examined then the static portfolio appears to be even less risky than weekly revised dynamic delta-hedging, as shown in the appropriate rows of Table 4.

	Dynamic (monthly)	Dynamic (weekly)	Static
$\mathrm{VaR}_{0.01}$	-18.92	-5.30	-4.28
$\mathrm{VaR}_{0.05}$	-8.21	-4.01	-4.28
$\mathrm{ES}_{0.01}$	-26.79	-6.34	-4.28
$\mathrm{ES}_{0.05}$	-14.39	-4.83	-4.28
Mean	-0.26	0.36	-0.05
SD	5.60	3.20	4.58

Table 4: Sample statistics of the replication error of different hedging strategies, based on 1000 simulations. In detail, we provide sample VaR_α and ES_α for $\alpha = 0.01, 0.05$.

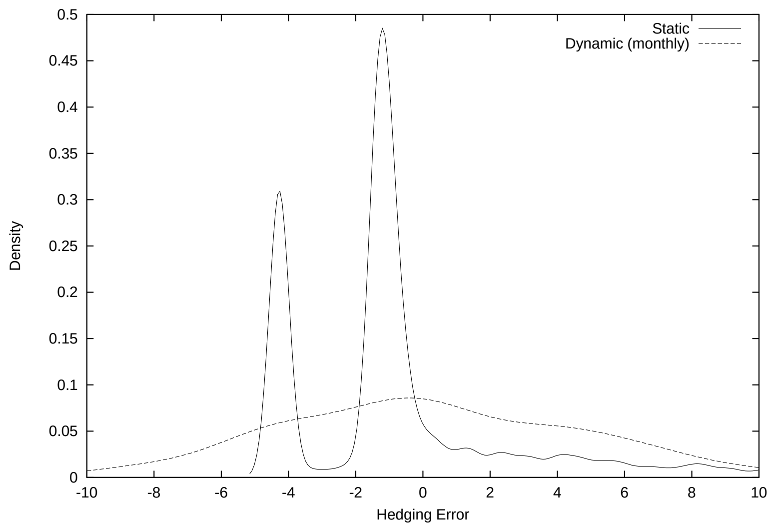

Figure 1: Hedging error density (kernel estimated) for static and monthly revised delta-based portfolio.

We finally examine a 7 assets portfolio option, first presented in [Milevsky and Posner, 1998a], and priced by MMC in [Pellizzari, 2001]. This option, embedded in a an Index-Linked Guaranteed Investment Certificate (ILGIC), is written on a weighted sum of several major stock indexes, namely TSE100 (Canada), CAC40 (France), DAX (Germany), MIB30 (Italy), Nikkei225 (Japan), FTSE100 (U.K.) and S&P500 (U.S.A.). An ILGIC is sold with various maturities (1, 3, 5 and 10 years) and we refer the reader to the above mentioned papers and to [Milevsky and Posner, 1998b] for more details and full parameter list. We feel that this example is well suited to explore the practical problems in hedging long-lived derivatives and to explore if static hedging methods can be useful for hedging purposes.

In order to estimate the replication error of a hedging strategy we simulate one thousand 7-dimensional paths and look at VaR and ES at confidence level 1% and 5%. We consider 6, 12, 60 and 120 rebalancings per year, roughly equivalent to one revision every two months (2M), every month (M), every week (W) and twice a week (W/2). We also take into account proportional transaction costs: in order to rebalance at time $t+1$ the portfolio quotas from $\mathbf{b}_t$ to $\mathbf{b}_{t+1}$, the transaction costs

$$\nu \sum_{i=1}^{7} |b_{it} - b_{i(t+1)}| S_{i(t+1)}$$

are to be paid. We assume there are no costs to manage the riskfree account and the commission rates $\nu = 0, 0.001, 0.002, 0.004$ to be somehow representative of typical costs (many online brokers, for example, offer $\nu = 0.2\%$). The deltas for the dynamic hedging portfolios are calculated using 1024 correlated simulations, as in the previous example. The static hedging portfolio is build in $t = 0$ and is composed of 7 at-the-money call options.

To save space we focus only on the two extreme maturities, 1 and 10 years: the results relative to 3 and 5 years maturities are available on request and do not alter in any substantial way the following considerations. Tables 5 and 6 show values at risk and expected shortfalls of 1 and 10 years maturities if dynamic (with various rebalancing frequencies and commission rates) or static hedges are used. For example, the lower left part of Table 5 shows that if $\nu = 0.2\%$ then the expected shortfalls (at 1% level) of the dynamic hedging strategies are respectively -0.0630, -0.0429, -0.0267, and -0.0248 for different rebalancing intervals. We also see that the static hedging produces an ES of -0.041.

Observe preliminarily that many of the entries in the tables are a sizeable fraction of the price of the derivatives and sometimes even exceed them, thus pointing to massive residual risk *even* if hedging was carefully performed. This is in substantial agreement, for example, to figures reported in [Green and Figlewsky, 1999] for the univariate case.

The analysis of Table 5 is eased looking at Figure 2 that graphically depicts the upper right and lower left parts of the table. It can be seen in the first exhibit of Figure 2 that $\mathrm{VaR}_{0.05}$ is generally increasing with the rebalancing frequency, with the exception of the $\nu = 0.004$ case

Maturity: $T = 1$, price: 0.0590

	$VaR_{0.01}$ Rebalancing freq.				$VaR_{0.05}$ Rebalancing freq.			
Trans. Cost	2M	M	W	W/2	2M	M	W	W/2
$\nu = 0$	-0.0460	-0.0295	-0.0122	-0.0085	-0.0247	-0.0189	-0.0079	-0.0045
$\nu = 0.001$	-0.0474	-0.0316	-0.0172	-0.0155	-0.0261	-0.0209	-0.0120	-0.0104
$\nu = 0.002$	-0.0489	-0.0338	-0.0223	-0.0225	-0.0276	-0.0227	-0.0165	-0.0165
$\nu = 0.004$	-0.0518	-0.0382	-0.0311	-0.0366	-0.0312	-0.0268	-0.0253	-0.0293
Static	-0.0366				-0.0287			

	$ES_{0.01}$ Rebalancing freq.				$ES_{0.05}$ Rebalancing freq.			
Trans. Cost	2M	M	W	W/2	2M	M	W	W/2
$\nu = 0$	-0.0596	-0.0380	-0.0172	-0.0109	-0.0382	-0.0259	-0.0112	-0.0070
$\nu = 0.001$	-0.0613	-0.0404	-0.0218	-0.0178	-0.0399	-0.0282	-0.0157	-0.0133
$\nu = 0.002$	-0.0630	-0.0429	-0.0267	-0.0248	-0.0415	-0.0304	-0.0204	-0.0200
$\nu = 0.004$	-0.0663	-0.0479	-0.0366	-0.0392	-0.0448	-0.0350	-0.0299	-0.0335
Static	-0.0410				-0.0329			

Table 5: Replication error risk measures for a 1 year maturity ILGIC. The rebalancing frequences reported are two months (2M), one month (M), one weekk (W) and half-week (W/2).

that shows that it is not convenient to revise too often the hedge in the presence of relatively high transaction costs. A visible flattening occurs for $\nu = 0.002$ too and there is not much difference between weekly and bi-weekly revisions with this commission rate. The static hedging is represented by the horizontal line in the graph and we observe that it is roughly equivalent to dynamic hedging (no matter of ν) for 2M revisions, while it is generally riskier for other frequencies (but again this does not hold when $\nu = 0.004$).

The previous consideration are strengthened observing the second exhibit of Figure 2, depicting expected shortfall at 1% confidence level. The static hedging portfolio achieves roughly the same performances of the dynamic hedges with monthly revisions and, in particular, it is at least as good as the dynamic strategy when $\nu = 0.004$.

As pointed out by a referee, it is interesting and of practical importance to compare the

Maturity: $T = 10$, price: 0.3113

	$\mathrm{VaR}_{0.01}$ Rebalancing freq.				$\mathrm{VaR}_{0.05}$ Rebalancing freq.			
Trans. Cost	2M	M	W	W/2	2M	M	W	W/2
$\nu = 0$	-0.0246	-0.0226	-0.0135	-0.0145	-0.0198	-0.0146	-0.0093	-0.0099
$\nu = 0.001$	-0.0288	-0.0295	-0.0295	-0.0471	-0.0232	-0.0191	-0.0253	-0.0403
$\nu = 0.002$	-0.0339	-0.0355	-0.0469	-0.0830	-0.0265	-0.0238	-0.0417	-0.0725
$\nu = 0.004$	-0.0439	-0.0486	-0.0851	-0.1560	-0.0338	-0.0341	-0.0763	-0.1383
Static	-0.0859				-0.0709			

	$\mathrm{ES}_{0.01}$ Rebalancing freq.				$\mathrm{ES}_{0.05}$ Rebalancing freq.			
Trans. Cost	2M	M	W	W/2	2M	M	W	W/2
$\nu = 0$	-0.0299	-0.0331	-0.0153	-0.0170	-0.0238	-0.0201	-0.0121	-0.0128
$\nu = 0.00$	-0.0342	-0.0404	-0.0319	-0.0505	-0.0277	-0.0257	-0.0280	-0.0445
$\nu = 0.002$	-0.0386	-0.0478	-0.0501	-0.0868	-0.0316	-0.0315	-0.0452	-0.0784
$\nu = 0.004$	-0.0480	-0.0626	-0.0892	-0.1632	-0.0397	-0.0434	-0.0821	-0.1487
Static	-0.0935				-0.0802			

Table 6: Replication error risk measures for a 10 years maturity ILGIC. The rebalancing frequences reported are two months (2M), one month (M), one weekk (W) and half-week (W/2).

sensitivity of the performance of dynamic and static hedges if volatilities move. There are countless ways to perturb the volatility structure of a multivarate claim (think, for example, to one-time shocks or changes in the level of some σ's). Hence, with no hope to be exhaustive, the upper (lower) panel of Figure 3 shows the $\mathrm{VaR}_{0.05}$ of static and dynamic hedges for the shortest maturity ILGIC if all the volatilities $\sigma_1, \ldots, \sigma_7$ increase (decrease) linearly by 25% over the lifespan of the claim (1 year). We feel that this resembles a typical situation where the overall volatility of the market starts increasing (decreasing) just after the option has been written, altering the *ex-ante* expectations of the issuer that is neverthelees binded to the contract. The important case, if downside risk protection is seeked for, is the increasing volatility situation, because obviously much less risk is incurred if the the σ's are shrinking, as can be seen in the lower part of Figure 3. A look at upper panel of the picture and comparison to Figure 2 shows that the VaR of the static hedging strategy is remarkably insensitive to moving volatilities (indeed, exact numbers in the two figures differ by 10^{-3}). On the contrary, a gradual increase in volatilities inflates considerably the risk of a dynamic hedging strategy, that is to be preferred only if frequent revisions are performed with very low transaction costs.

Figure 4 shows the $\mathrm{VaR}_{0.01}$ and $\mathrm{ES}_{0.05}$ for the long maturity (10 years) ILGIC, depicting the upper left and lower right parts of Table 6. The plots are quite similar and illustrate that dynamic hedging reduces risk much more than static hedging if low revision frequencies are selected. However, there is no real advantage to rebalance more often than every two months, perhaps with the exception of the case of no transaction costs. This can be understood in view of the fact that the long maturity comes together with many revisions that might be imprecise due to discretization and/or numeric greeks calculation. Hence the replication error at maturity is adversely influenced by revision frequency. As far as the weak results of static hedging are concerned, note that this is an extreme case in which one portfolio is held for 10 years with no modification. It can be conjectured (and the examples in this paper support this point) that static portfolios can better cope with short to medium maturities, say less than a year.

5 Conclusion

This paper presents an alternative hedging method, based on an approximate replication of a multivariate derivative using a portfolio of simple options. The risk resulting from writing an option is not totally removed, but the same can be said for standard BS dynamic hedging, that is perfect only in the limit of continuous rebalancings and no friction. Many papers have shown that practical implementations of BS hedging strategy can expose the writer to substantial residual risk. Among the advantages of the static hedging procedure there are conceptual simplicity and easy implementation, negligible transaction costs and flexibility: the method can for example be tailored to the assets available in the market and provides different approximate hedging portfolio depending on various situations (like budget constraints, or no short selling requirements).

At a higher level of abstraction, the method generalizes and casts further light on a variance reduction MC scheme recently proposed and known as Mean Monte Carlo. This interpretation ensures that increasing the number of 'simulations' produces more accurate pricing results (though the residual risk obviously remains bounded away from zero).

The potential of the proposed technique is assessed pricing some bivariate max options studied in [Boyle, 1988]: the results are fairly accurate using only 1000 simulations and each price is supported by a viable static hedging strategy. Analysis of portfolio options written on 2 and 7 assets shows that the risk of static hedging is in some cases comparable to that produced by dynamic hedging, especially for short maturities and in the presence of transaction costs.

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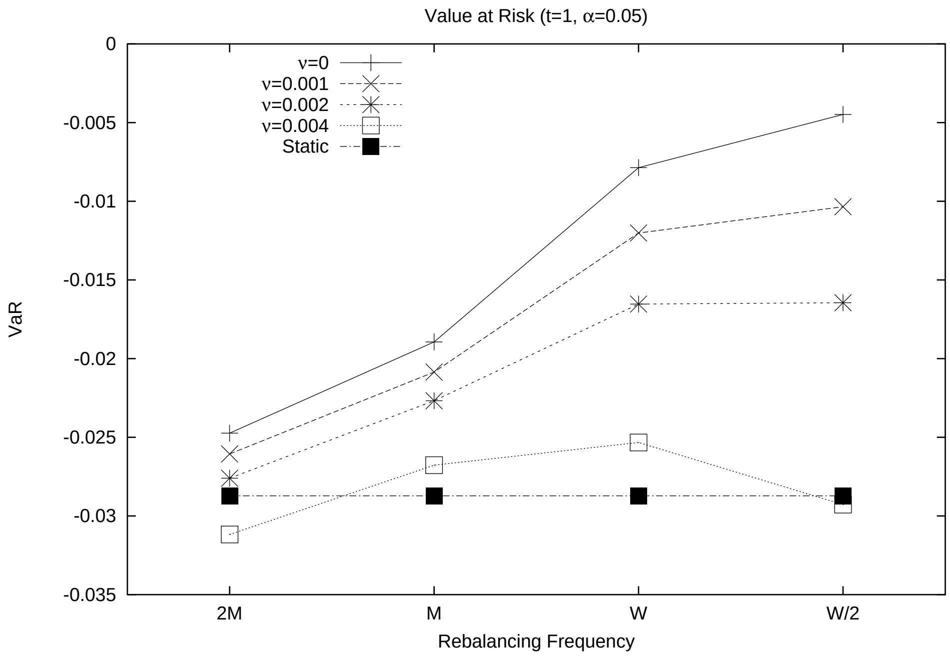


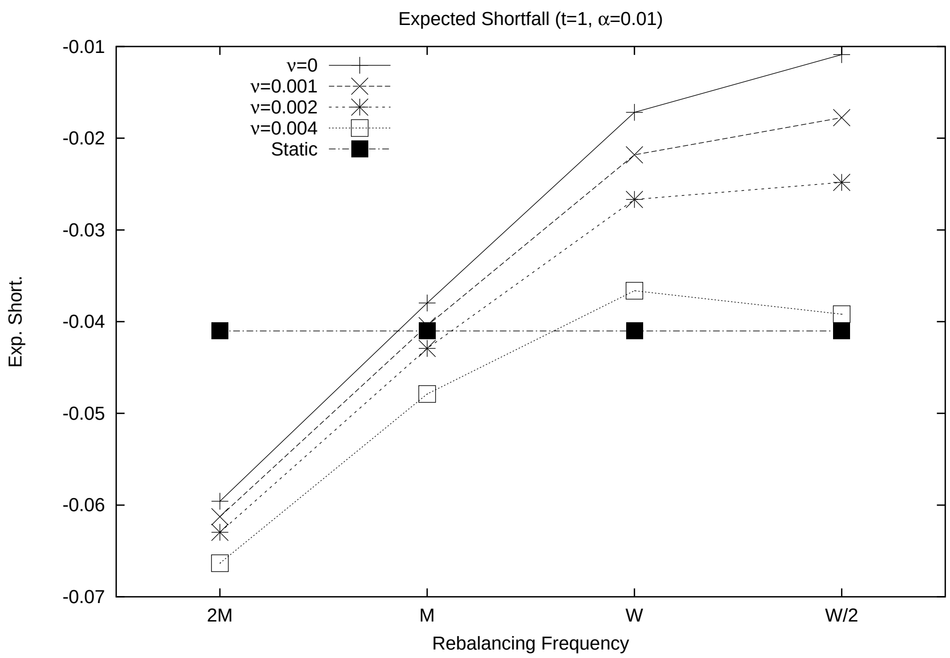
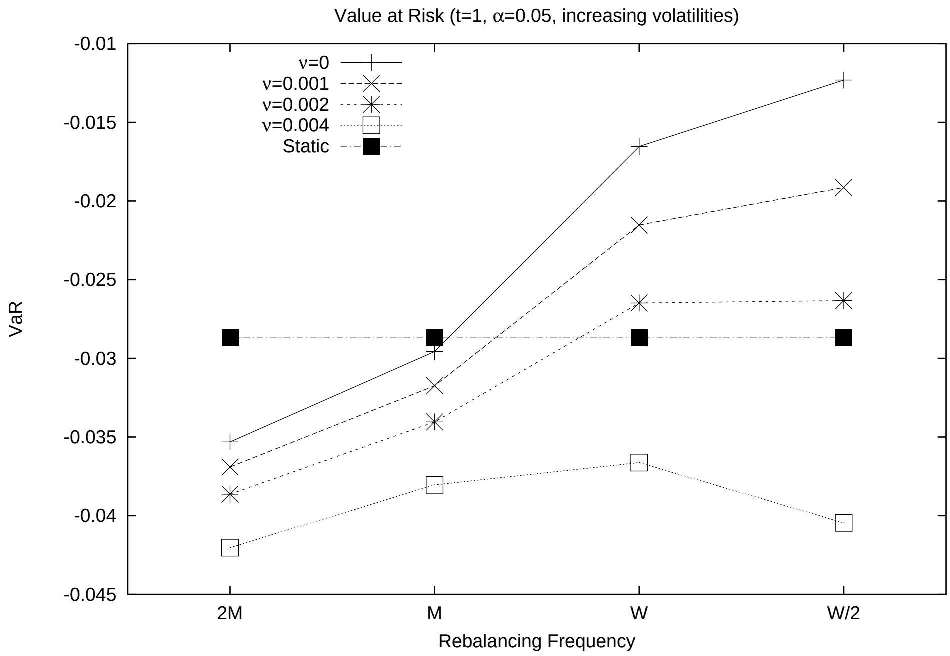


Figure 2: VaR and ES measures for a 1 year maturity ILGIC. The panels show $\mathrm{VaR}_{0.05}$ and $\mathrm{ES}_{0.01}$ for dynamic and static hedging corresponding to various transaction rates.



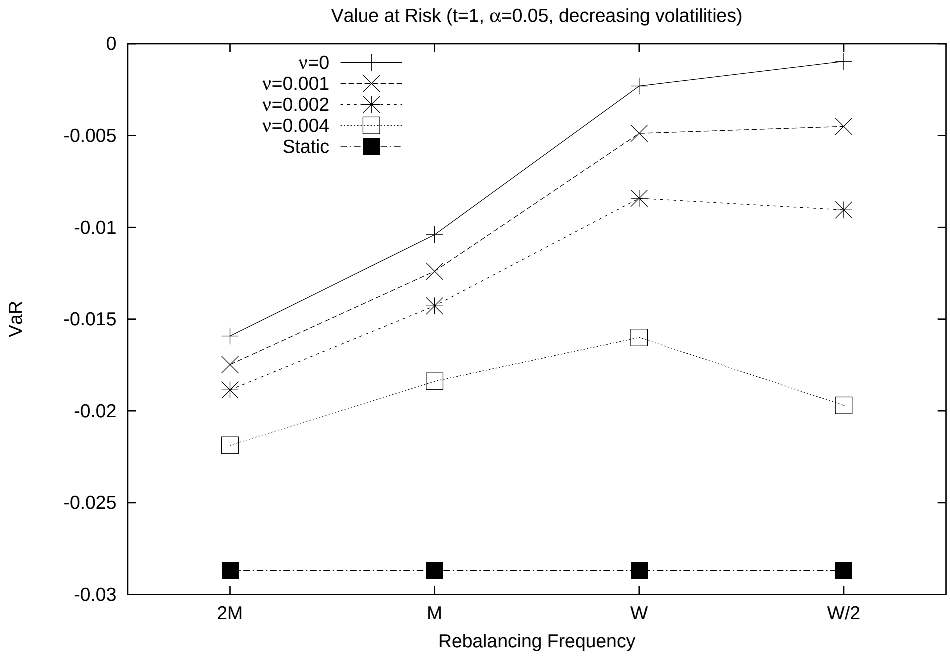


Figure 3: VaR of a 1 year maturity ILGIC, for static and dynamic hedging corresponding to different transaction rates. The upper (lower) panel shows the case of linearly increasing (decreasing) volatilities over the lifespan of the option. The constant volatility case is depicted in the upper part of Figure 2.

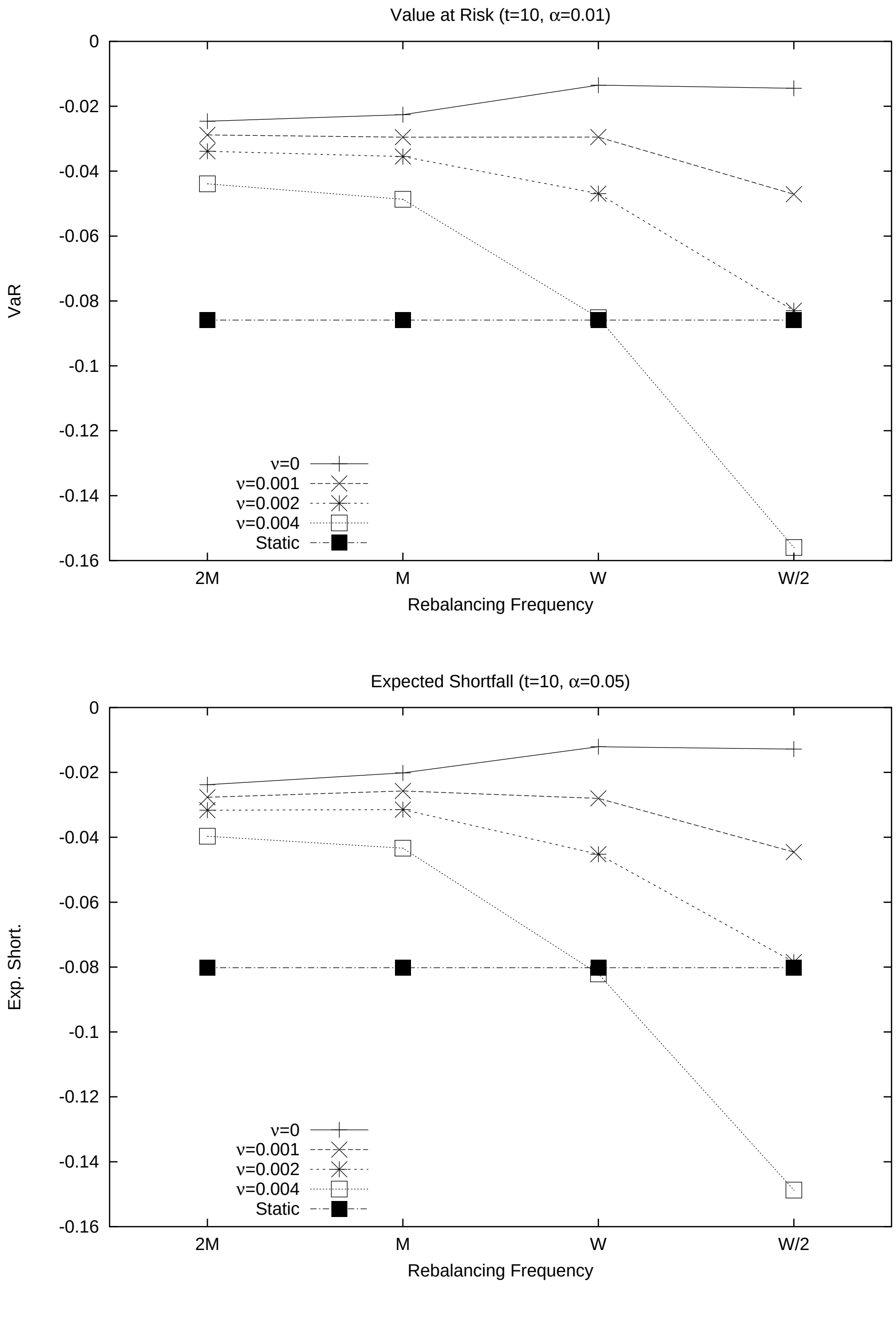


Figure 4: VaR and ES measures for a 10 years maturity ILGIC. The panels show $\mathrm{VaR}_{0.01}$ and $\mathrm{ES}_{0.05}$ for dynamic and static hedging corresponding to various transaction rates.