Coalitions, Agreements and Efficiency*

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Abstract

If agents negotiate openly and form coalitions, can they reach efficient agreements? We address this issue within a class of coalition formation games with externalities where agents' preferences depend solely on the coalition structures they are associated with. We derive Ray and Vohra's (1997) notion of *equilibrium binding agreements* using von Neumann and Morgenstern abstract stable set and then extend it to allow for arbitrary coalitional deviations (as opposed to nested deviations assumed originally). We show that, while the new notion facilitates the attainment of efficient agreements, inefficient agreements can nevertheless arise.

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1 Introduction

If agents negotiate openly and can form coalitions, can they reach efficient agreements? We address this issue within a class of simple coalition formation games with externalities where each agent's preferences depend only on the coalition structure or partitions of the agents. Arguably, if binding agreements can be written without any informational imperfections, then all the gains from cooperation should be extracted. The resulting agreement must be Pareto-optimal; in particular, if utility is transferable, aggregate surplus must be maximized. Such an assertion encapsulates the Coase (1960) theorem. However, in Ray and Vohra (1997), it is shown that when coalitions can form, efficiency can no longer be guaranteed. Indeed, they define the notion of equilibrium binding agreements (henceforth EBA) for strategic form games, a more general framework than ours, and construct examples where the only agreements that can be reached are inefficient, even when utility is transferrable. This negative result casts a shadow on the validity of Coase theorem in environments where coalitions can form.

What is even more puzzling is that in Ray and Vohra's notion agents are sophisticated. In particular they are assumed to be *farsighted* in that when contemplating a deviation, a coalition takes into consideration that further deviations may occur and that other deviating coalitions also apply similar reasoning. For farsighted agents, it is the *final* agreement their deviations lead to that matters. Moreover, agents examine the *credibility* of the final outcome, thus, the notion is defined consistently. One feature of EBA is the assumption of internal deviations, that is, only a subset of an existing coalition can deviate. While this feature makes a recursive definition possible, it precludes the possibility of coalition merging and renegotiation. As Ray and Vohra (1997) wrote,

> We must state at the outset that our treatment is limited by the assumption that agreements can be written only between members of an existing coalition; once a coalition breaks away from a larger coalition it cannot forge an agreement with any member of its complement. Thus, deviations can only serve to make an existing coalition structure finer never coarser. This is also the assumption in the definition of a coalition proof Nash equilibrium. It must be emphasized that an extension of these notions to the case of arbitrary blocking is far from trivial. (p.33)

Within our framework, we extend the definition of EBA to allow for arbitrary coalitional deviations, thereby relaxing the assumption of internal (nested) deviations. We first reformulate Ray and Vohra's definition using von Neumann and Morgenstern (1944) stable set. Such a reformulation is akin to that of coalition proof Nash equilibrium[1] (Bernheim, Peleg, and Whinston, 1987) by Greenberg (1989) and it offers a simple definition of EBA in our framework. More importantly, the use of stable set enables us to deal with the circularity that results from allowing arbitrary coalitional deviations while maintaining consistency as in the original definition of EBA. Can inefficient outcomes emerge from an open, unrestricted negotiation as entailed by the new definition? We show that while the assumption of internal deviations hinders efficiency to some extent, inefficient agreements can still arise in open unrestricted negotiations. In particular, we identify a class of games where efficiency can be attained and construct a counter example where no agreement is efficient. The negative result reinforces the inefficiency puzzle posed by Ray and Vohra (1997).

In the open, unrestricted negotiation underlying our extension of EBA, it is *feasible* for any coalition to form and object to/deviate from any coalition structure. However, being farsighted, a coalition engages in a deviation if and only if it can ultimately benefit from doing so. Therefore, a coalition structure is "stable" if no coalition wishes to deviate, anticipating the final outcome its deviation may lead to and a coalition structure is "unstable" as long as one coalition wishes to deviate, again anticipating the final outcome of its deviation. While our approach captures the foresight of selfish agents, it differs from the "non-cooperative" approach of coalition formation (at least) in that we do not specify the exact order with which individual agents make proposals and counter proposals. Moreover, in the latter approach additional restrictions are often placed on the coalition formation process. For instance, Bloch (1996) studies the same class of games as ours in a noncooperative framework of sequential coalition formation, while Ray and Vohra (1999)[2] study a more general framework with endogenized payoff division. A common feature of their models is that once a coalition forms, the game is only played among the remaining players and established coalitions may not seek to attract new members nor break apart. The requirement for coalitions to commit plays an important role in determining the equilibrium coalition structures[3]. Inefficient coalition structures can arise in equilibrium as well.

[1] Note that the difference between EBA and coalition proof Nash equilibrium (CPNE) lies beyond the fact that the former considers binding agreements. In the definition of EBA, agents are farsighted in that each coalition considers the final outcome its deviation leads to, while in the definition of CPNE, a deviating coalition takes its complement's choice as given.

[2] See also Ray and Vohra (2001).

[3] Macho-Stadler, Pérez-Castrillo and Porteiro (2002) analyze a Cournot oligopoly, a

The organization of the paper is as follows: After the presentation of the preliminaries in the next Section we reformulate EBA in Section 3 and extend the notion in Section 4. Section 5 presents sufficient conditions for a game to admit efficient coalition structures and a counter example where no agreement is efficient. In Section 6 we discuss some directions that can be further pursued to achieve efficiency. Section 7 concludes the paper. Detailed analysis of the counter example is delineated in the appendix.

2 Preliminaries

We start with some basic notations:

- Let N be a finite set of players.
- A coalition S is a non-empty subset of N.
- A partition of $S \subset N$ is $P = \{S_1, S_2, ..., S_k\}$ such that $\bigcup_{j=1}^{k} S_j = S$ and for all $i \neq j$, $S_i \bigcap S_j = \emptyset$ and $\mathcal{P}(S)$ is the set of partitions of S. A partition of N is called a coalition structure and $\mathcal{P} \equiv \mathcal{P}(N)$ be the set of all coalition structures.

In the simple coalition formation games with externalities we study here, each player's preferences depend on the *entire* coalition structure.

A *simple coalition formation game with externalities* G is $(N, \{\succeq_i\}_{i \in N})$ or $(N, \{u_i\}_{i \in N})$ where

- N is the finite set of players;
- for all $i \in N$, $\succeq_i$ is a complete, reflexive, and transitive binary relation on $\mathcal{P}$, the set of coalition structures, $\succ_i$ denotes the asymmetric part of $\succeq_i$ (i.e., strict preferences) and $\sim_i$ the indifference relation;
- for all $i \in N$, $u_i : \mathcal{P} \to \Re$ is i's payoff function.

The simple class of games we study here extends the class of "hedonic games" [see, e.g., Banerjee, Konishi and Sönmez (2001), Bogomolnaia and Jackson (2002), Barberà and Gerber (1999), and Diamantoudi and Xue

special case of the class of games we study here. They assume that coalitions form by merging bilaterally, thereby relaxing the commitment of a formed coalition. They show that if the number of firms is sufficiently large then the monopoly (grand coalition) is the equilibrium outcome.

(2001)], where each agent's preferences depend only on the coalition he belongs to. The class of games studied in this paper can also be viewed as a special class (with fixed payoff division) of partition function games introduced by Thrall and Lucas (1963), which, in turn, is a special case of normal form TU games studied by Zhao (1992). Yi (1997) studies a more restricted class of games than ours; in particular, he studies symmetric games and examines three models of coalition formation: simultaneous coalition formation model of Yi and Shin (1995), sequential coalition formation model of Bloch (1996), and EBA of Ray and Vohra (1997). Moreover, Yi classifies symmetric games into two categories, one with positive externalities (e.g., Cournot oligopoly, a class of public good economies) and one with negative externalities (e.g., customs unions).

3 EBA and Inefficiency

As discussed in the introduction, Ray and Vohra's (1997) notion of EBA is defined for strategic form games, a more general framework than ours. We shall adapt their definition to our setting. The negotiation process underlying EBA is as follows. Suppose the grand coalition N is under consideration. A coalition $S \subsetneq N$ can break away from N and in doing so, it induces the coalition structure $\{S, N \setminus S\}$. This coalition structure is likely to be a temporary one since S or $N \setminus S$ may further break apart. More generally, given a coalition structure $P \in \mathcal{P}$, any coalition $T \subsetneq S$, for some $S \in P$, can break away from S. In addition, once T breaks away from S, it cannot forge an agreement with any member of $N \setminus S$; thus, deviations can only lead to finer coalition structures. Such an "internal" or "nested" deviation can be formalized as follows.

Internal Coalitional Deviation Given a coalition structure $P \in \mathcal{P}$ and some $S \in P$, coalition $T \subsetneq S$, by breaking away from S, induces a temporary coalition structure given by $P' = P \setminus \{S\} \cup \{T, S \setminus T\}$. We shall write $P \overset{T}{\rightarrowtail} P'$ in this case. Call Q a refinement of P if Q can be reached from P through a sequence of nested coalitional deviations.

Agents are assumed to be farsighted: Each deviating coalition is aware that further deviations may occur; thus, in contemplating a deviation, a coalition considers the ultimate consequence of its deviation. Given the nature of the negotiation process, EBA can be defined recursively.

- Start with the finest coalition structure, P^*, of singleton coalitions. Since no further deviations are possible, P^* is the finest EBA.

- Now consider a coalition structure P such that $P \overset{T}{\rightarrowtail} P^*$; thus, P comprises all singleton coalitions but one coalition of size 2. Let $\{ij\} \in P$. Then P is an EBA if and only if neither i nor j has an incentive to break away from the coalition and induce P^*.

- Consider $Q \in \mathcal{P}$ such that $Q \overset{S}{\rightarrowtail} P$ for some $S \subset N$ and P comprises all singleton coalitions but one coalition of size 2. For Q to be an EBA it must be the case that all members of S do not benefit by inducing P. Note that S is farsighted in contemplating its deviation to P: If P is an EBA itself, S compares P with Q; otherwise, it compares P^* with Q since once it deviates to P, there will be a further deviation to P^*.

  ...

- Consider $Q \in \mathcal{P}$. Suppose all EBAs have been defined for all refinements of Q. Then Q is an EBA if and only if there do not exist a sequence of nested coalitional deviations that lead to an EBA Q', benefiting every deviating coalition (who compares Q' with the temporary coalition structure from which it deviates)[4].

The solution of a game is considered to be the set of coarsest EBAs. In the above recursive definition, coalition structures are compared via the following dominance relation.

Sequential (Nested) Dominance P' sequentially dominates P, or $P' \gg^{R\&V} P$, if there exist a sequence of coalition structures $P^1, P^2, \cdots, P^k \in \mathcal{P}$, where $P^1 = P$, $P' = P^k$, and a sequence of coalitions $T^1, T^2, \cdots, T^{k-1}$ such that for all $j = 1, \cdots k-1$

(i) $P^j \overset{T^j}{\rightarrowtail} P^{j+1}$ and

(ii) $P^j \prec_{T^j} P'$.

Thus, each coalition looks ahead and compares the final coalition structure with the temporary one from which it deviates. The fact that a coalition

[4]Within our framework of simple coalition formation games with externalities, each coalition structure is associated with a unique payoff vector. As a result, if there exists a sequence of nested coalitional deviations that lead to an EBA Q', benefiting every deviating coalition, then each temporary coalition structure in the sequence cannot be an EBA. This simplifies the definition of EBA. See Ray and Vohra (1997, p. 38) for the definition in the more general framework of strategic games.

structure is an equilibrium one if and only if it is not "defeated" or sequentially dominated by another equilibrium coalition structure signifies the consistency embedded in the definition of EBA. In our simple framework, such a consistency can be captured by "von Neumann and Morgenstern (vN-M) stable set". A vN-M stable set is a set of agreements, called a solution set, that is free of inner contradiction and that accounts for every elements it excludes; in particular, no agreement in the solution set is defeated by another agreement in the same solution set and if an agreement is excluded from the solution set, it must be defeated by an agreement in the solution set. More formally,

vN-M stable set of $(\mathcal{P}, >)$**:** Let $>$ be a binary relation on $\mathcal{P}$ and $\mathcal{R} \subset \mathcal{P}$. Then,

- $\mathcal{R}$ is *vN-M internally stable* for $(\mathcal{P}, >)$ if there do not exist $P, P' \in \mathcal{R}$ such that $P' > P$;
- $\mathcal{R}$ is *vN-M externally stable* for $(\mathcal{P}, >)$ if for all $P \in \mathcal{P} \setminus \mathcal{R}$, there exists $P' \in \mathcal{R}$ such that $P' > P$.
- $\mathcal{R}$ is a *vN-M stable set* for $(\mathcal{P}, >)$ if it is both internally and externally stable.

Within our framework, Ray and Vohra's EBA can be reformulated as a vN-M stable set[5].

EBA (*A Reformulation*) Let Ω be the vN-M stable set of $(\mathcal{P}, \gg^{R\&V})$. Then P is an EBA if and only if $P \in \Omega$. The coarsest coalition structures in Ω are referred to as the solution of the game.

Such a reformulation does not only offer a simple definition of EBA but also enables us to extend it in various directions. One of the motivations for extending it was the inefficiency puzzle presented in Ray and Vohra: although binding agreements are possible, inefficient outcomes can, nevertheless, emerge. The following example illustrates the inability of players to reach efficient binding agreements[6]. In the table below are the payoff vectors associated with ten partitions and all other partitions are assumed to yield 0 payoff vectors.

[5] This reformulation of EBA using a vN-M stable set cannot be directly generalized to strategic form games.

[6] Ray and Vohra have a 3-player example of strategic form game where the only equilibrium binding agreements are inefficient. Because of the simple framework we use, we need a 5-player game to illustrate the inefficiency of EBA.

a	b	c	d	e
$\{123, 45\}$	$\{234, 51\}$	$\{345, 12\}$	$\{451, 23\}$	$\{512, 34\}$
$5, 5, 5, 9, 9$	$5, 5, 5, 9, 9$	$5, 5, 5, 9, 9$	$5, 5, 5, 9, 9$	$5, 5, 5, 9, 9$
f	g	h	i	j
$\{12, 34, 5\}$	$\{23, 45, 1\}$	$\{34, 51, 2\}$	$\{45, 12, 3\}$	$\{51, 23, 4\}$
$4, 4, 4, 8, 8$	$4, 4, 4, 8, 8$	$4, 4, 4, 8, 8$	$4, 4, 4, 8, 8$	$4, 4, 4, 8, 8$

Table 1

It is easy to see that f, g, h, i, and j are all EBAs. So are those for which f, g, h, i, or j is not a refinement (for example, $\{135, 24\}$). On the other hand, a, b, c, d and e are not EBAs. Take a, for example. a is not an EBA because $a \overset{1}{\rightarrowtail} g$ and $a \prec_1 g$ or $a \ll^{R\&V} g$. Therefore, the only EBAs for this game are f, g, h, i and j, which are all inefficient. Indeed, f, g, h, i, and j are Pareto dominated by a, b, c, d and e, respectively. One might ask, why are matters not renegotiated at this stage to the dominating outcome? The assumption of nested deviations rules out the possibility of such renegotiation. As Ray and Vohra wrote,

> This is a serious issue that is neglected in our model, because we only permit "internal" deviations. (p.51)

This is precisely the reason why we extend the definition of EBA to allow for arbitrary coalitional deviations. Equipped with the notion of vN-M stable set, we can define a notion consistently (as the original definition) while allowing for arbitrary coalitional deviations.

4 Extended EBA

In this section we extend the notion of EBA by relaxing the assumption of internal deviations and allowing arbitrary coalitions to deviate. Moreover, a deviating coalition is not constrained to stay together; that is, we empower the deviating coalition with the ability to restructure itself. These features (of open negotiation) are introduced to facilitate the attainment of Pareto efficient coalition structures.

Given a coalition structure, when a coalition of players, $T \subset N$, deviates by partitioning itself in a certain way, the new coalition structure is the one consisting of the partition of T, all the unaffected coalitions, as well as all the disrupted coalitions. Formally, a coalitional deviation is defined as follows:

Coalitional Deviation Given a coalition structure $P = \{S_1, \cdots, S_k\} \in \mathcal{P}$, a coalition $T \subset N$ can reorganize itself to some partition $\{T_1, \cdots, T_\ell\} \in \mathcal{P}(T)$. The resulting coalition structure, before any further regrouping and restructuring, is $P' \in \mathcal{P}$ such that

(i) $\{T_1, \cdots, T_\ell\} \subset P'$, that is, the new partitioning of T is included in the new coalition structure.

(ii) $\forall j = 1, \cdots, k$, $S_j \cap T \neq \emptyset \Longrightarrow S_j \setminus T \in P'$, that is, the residuals of all coalitions affected by the deviation of T are also included in the new coalition structure.

(iii) $\forall j = 1, \cdots, k$, $S_j \cap T = \emptyset \Longrightarrow S_j \in P'$, that is, all those coalitions that were unaffected by the deviation of T remain members of the new coalition structure.[7]

We will write $P \xrightarrow{T} P'$ to denote that "T induces P' **temporarily** from P".

Once it is precisely defined what a coalition can (directly) induce we proceed to define our dominance relation. While myopic agents look only at the next step, farsighted players consider the ultimate outcomes of their actions. Thus, a coalition may choose to "deviate" to a coalition structure, which does not necessarily make its members better off, as long as its deviation leads to a final coalition structure that benefits all its members; similarly, a coalition may choose not to deviate to a coalition structure it prefers if its deviation *eventually* leads to coalition structures that make its members worse off. Similar to the sequential (nested) dominance defined in the previous section the following "indirect dominance[8]" captures foresight when arbitrary coalitions can deviate.

Indirect dominance: P' sequentially (or indirectly) dominates P, or $P' \gg P$, if there exist a sequence of coalition structures $P^1, P^2, \cdots, P^k \in \mathcal{P}$, where $P^1 = P$ and $P' = P^k$, and a sequence of coalitions $T^1, T^2, \cdots, T^{k-1}$ such that for all $j = 1, \cdots k-1$

(i) $P^j \xrightarrow{T^j} P^{j+1}$ and

(ii) $P^j \prec_{T^j} P'$.

[7]Note that points (ii) and (iii) can be written more concisely as follows: $\forall j = 1, ..., k$, $S_j \backslash T \neq \emptyset \Longrightarrow S_j \backslash T \in P'$.

[8]See also Harsanyi (1974), Chwe (1994) and Xue (1998).

An extended notion of EBA with unrestricted coalitional deviations can be defined consistently by using the vN-M stable set of $(\mathcal{P}, \gg)$, where the sequential nested dominance, $\gg^{R\&V}$, is replaced with indirect dominance, $\gg$.

Extended EBA (EEBA) Let $\mathcal{Q} \subset \mathcal{P}$ be a vN-M stable set of $(\mathcal{P}, \gg)$. $P \in \mathcal{P}$ is an extended EBA or EEBA if $P \in \mathcal{Q}$.

Revisiting the example presented in the previous section we can see that efficiency can be restored. We will argue that a is an EEBA. Indeed, $Q = \{a\}$ is a vN-M stable set of $(\mathcal{P}, \gg)$. The following table summarizes how all other coalition structures are (indirectly) dominated by a. Coalition structure p denotes any arbitrary coalition structure that is not listed in Table 1.

$a \gg f:$	$f \xrightarrow{N} a$	$a \gg j:$	$j \xrightarrow{5} \{23,1,4,5\} \xrightarrow{N} a$
$a \gg e:$	$e \xrightarrow{5} f \xrightarrow{N} a$	$a \gg g:$	$g \xrightarrow{5} \{23,1,4,5\} \xrightarrow{N} a$
$a \gg b:$	$b \xrightarrow{4} j \xrightarrow{5} \{23,1,4,5\} \xrightarrow{N} a$	$a \gg h:$	$h \xrightarrow{4} \{15,2,3,4\} \xrightarrow{N} a$
$a \gg c:$	$c \xrightarrow{4} \{12,35,4\} \xrightarrow{N} a$	$a \gg i$:	$i \xrightarrow{4} \{12,3,4,5\} \xrightarrow{N} a$
$a \gg d:$	$d \xrightarrow{5} \{14,23,5\} \xrightarrow{N} a$	$a \gg p:$	$p \xrightarrow{N} a$

Table 2

Due to the symmetry of the game, b, c, d and e are also EEBAs. Moreover, none of f, g, h, i and j is an EEBA. Assume in negation that there exists a vN-M stable set $\mathcal{Q}$ of $(\mathcal{P}, \gg)$ that supports an inefficient coalition structure. Then, given the indirect dominance relation depicted in Table 2, $\mathcal{Q}$ cannot contain any of the efficient outcomes due to internal stability. If, however, all a, b, c, d and e are excluded $\mathcal{Q}$ must contain more than one coalition structures to account for their exclusion. But, coalition structures f, g, h, i and j dominate each other and hence cannot co-exist in $\mathcal{Q}$. The following table illustrates how j indirectly dominates every other inefficient outcome. Similar arguments can be developed for the rest of the outcomes due to the symmetry of the game.

$j \gg f$	$: f \xrightarrow{3} \{12,3,4,5\} \xrightarrow{N} j$
$j \gg g$	$: g \xrightarrow{4} \{23,1,5,4\} \xrightarrow{N} j$
$j \gg h$	$: h \xrightarrow{3} \{3,4,51,2\} \xrightarrow{N} j$
$j \gg i$	$: i \xrightarrow{4} \{4,5,12,3\} \xrightarrow{N} j$
$j \gg p$	$: p \xrightarrow{N} j$

Table 3

Existence of vN-M stable set of $(\mathcal{P}, \gg)$ is not guaranteed. Case in point is a version of the roommate problem where $\{ij,k\} \succ_i \{ik,j\} \succ_i \{ijk\} \succ_i \{i,j,k\} \succ_i \{i,jk\}$ and i prefers to have j as a roommate, while j prefers to have k as a roommate and lastly k prefers to have i as a roommate. It is easy to see that no stable set exists for this game since $\{ij,k\} \ll \{jk,i\} \ll \{ik,j\} \ll \{ij,k\}$: while no two coalition structures can coexist in a stable set because of internally stability, a single coalition structure is externally stable. In contrast, all structures involving pairs and the singletons structure are equilibrium binding agreements since the cyclicality is assumed away through the assumption of internal deviations: once $\{ij\}$ are formed j is not allowed to collude with k. Fortunately, the presence of cycles is not always a problem as can be seen in the example in Table 1. In the following section we identify classes of games where both existence and efficiency are resolved.

Implicit in the definition of vN-M stable set is the optimism on the part of deviating coalitions: a coalition engages in a deviation as long as its members benefit form one of the final outcomes its deviation may lead to. An alternative behavioral assumption on deviating coalitions is caution: a coalition engages in a deviation if its members benefit form all the final outcomes its deviation may lead to. Under the assumption of caution, a more inclusive notion (than vN-M stable set) can be defined and existence is guaranteed in our framework. See Chwe (1994), Xue (1998) and Diamantoudi and Xue (2001), among others, for notions that are built on cautious behavior of deviating coalitions.

5 (In)Efficiency

5.1 Positive Results

To proceed with the study of efficiency we need to distinguish between the classic notion of Pareto efficiency and strong efficiency. Strong efficiency compares the aggregate payoff of all the players across coalition structures and it implies Pareto efficiency. When transfer payments are allowed, strong efficiency and Pareto efficiency are equivalent.

Pareto Efficiency P efficient if there does not exist P' such that $P \prec_N P'$.

Strong Efficiency P is strongly efficient if there does not exist P' such that $\sum_{i\in N} u_i(P') > \sum_{i\in N} u_i(P)$.

The following simple example from Ray and Vohra (1997) illustrates the inability of players to reach strongly efficient binding agreements.

Inefficiency Puzzle – A Cournot Oligopoly

The market demand is given by $p = a - by$, where p is the market price and y is aggregate demand. We assume symmetric firms with constant marginal cost c. Consider a coalition structure $P = \{S_1, S_2, \cdots, S_m\} \in \mathcal{P}$. Then, the profit of each firm in S_i is

$$\pi_i = \frac{1}{s_i(m+1)^2}\frac{(a-c)^2}{b}$$

where $s_i = |S_i|$ for all $i = 1, ..., m$.

The following table displays the per firm profits for the simple case of $n = 5$ and $\frac{(a-c)^2}{b} = 1$. The last two columns indicate which coalition structures survive the notion of Ray and Vohra (1997). In symmetric games, coalitions need to be identified only by their sizes. For example, $\langle 4, 1\rangle$ denotes coalition structures with a coalition of size 4 and a coalition of size 1.

Structure size-wise	Perpetrator size-wise	Per Firm Profit	EBA
$\langle 5\rangle$	——	$\frac{1}{20}$	$\times$
$\downarrow$	1		
$\langle 4, 1\rangle$	——	$\frac{1}{36}, \frac{1}{9}$	$\times$
$\downarrow$	2		
$\langle 2, 2, 1\rangle$	——	$\frac{1}{32}, \frac{1}{32}, \frac{1}{16}$	$\checkmark$
$\downarrow$	1		
$\langle 2, 1, 1, 1\rangle$	——	$\frac{1}{50}, \frac{1}{25}, \frac{1}{25}, \frac{1}{25}$	$\times$
$\downarrow$	1		
$\langle 1, 1, 1, 1, 1\rangle$	——	$\frac{1}{36}$	$\checkmark$

Table 4

$\langle 2, 2, 1\rangle$ associated with per firm profit of $\frac{1}{32}, \frac{1}{32}$ and $\frac{1}{16}$ is the coarsest EBA[9]. $\langle 2, 1, 1, 1\rangle$, on the other hand is not an equilibrium coalition structure because the size-2 coalition will have incentive to break apart. $\langle 2, 2, 1\rangle$ is an EBA because if a coalition of size 2 breaks apart and induces $\langle 2, 1, 1, 1\rangle$, the other coalition of size-2 will also break apart as we already argued and they will end up at the finest structure $\langle 1, 1, 1, 1, 1\rangle$ which makes the members of the doubleton worse off. Coalition structure $\langle 4, 1\rangle$ is not an EBA since a coalition of size-2 will break away to induce $\langle 2, 2, 1\rangle$ which is an EBA and hence a credible deviation. Similarly, coalition structure $\langle 5\rangle$ is not an EBA

[9] To be more precise, all coalition structures with 2 size-2 coalitions and 1 singleton are EBAs.

since one member will break away to temporarily induce $\langle 4,1\rangle$ and then a coalition of size 2 will break away to induce $\langle 2,2,1\rangle$. Observe that the leading perpetrator 1 receives $\frac{1}{16}$ under $\langle 2,2,1\rangle$ and $\frac{1}{20}$ under $\langle 5\rangle$. Similarly, $\langle 3,2\rangle$ and $\langle 3,1,1\rangle$, that are omitted from the above table for simplicity, are not EBAs.

Observe that none of the EBAs of the above game is strongly efficient[10]. However, there exists a strongly efficient EEBA. Indeed, the grand coalition alone, with payoff of $\frac{1}{20}$ per firm, constitutes a vN-M stable set of $(\mathcal{P}, \gg)$. Nevertheless, this example admits other EEBAs that are not strongly efficient: Each permutation of the $\langle 2,2,1\rangle$ coalition structure constitutes an EEBA as well, since $\langle 2,2,1\rangle$ is a Pareto efficient partition that satisfies both conditions (a) and (b) of Proposition 1 below.

Proposition 1 *Let $P^* \in \mathcal{P}$ be Pareto efficient. P^* is an EEBA if*

(a) $\{1,2,\cdots,n\} \prec_N P^*$ and

(b) for all $P \in \mathcal{P}$ such that $P \neq P^*$ and $P \neq \{1,2,\cdots,n\}$, there is a coalition $S \in P$ such that $|S| > 1$ and $P \prec_i P^*$ for some $i \in S$.

Proof. We shall show that $\{P^*\}$ is a vN-M stable set of $(\mathcal{P}, \gg)$. Obviously, $\{P^*\}$ is internally stable. We now need to show that $P \ll P^*$ for all $P \in \mathcal{P} \setminus P^*$. First, note that $\{1,2,\cdots,n\} \xrightarrow{N} P^*$. Since $\{1,2,\cdots,n\} \prec_N P^*$, we have $\{1,2,\cdots,n\} \ll P^*$. Let $P^1 \in \mathcal{P}$ be such that $P^1 \neq P^*$ and $P \neq \{1,2,\cdots,n\}$. By assumption (ii), there is a coalition $S_1 \in P$ such that $|S_1| > 1$ and $P \prec_{i_1} P^*$ for some $i_1 \in S_1$. Thus, $P^1 \xrightarrow{i_1} P^2 \equiv \{i_1, S_1 \setminus \{i_1\}, P^1 \setminus S_1\}$. Again by assumption (ii), there is a coalition $S_2 \in P^2$ such that $|S_2| > 1$ and $P^2 \prec_{i_2} P^*$ for some $i_2 \in S_2$. Thus, $P^2 \xrightarrow{i_2} P^3 \equiv \{i_2, S_2 \setminus \{i_2\}, P^2 \setminus S_2\}$. Continuing in this fashion, we can identify a sequence of agents $i_1, i_2, \ldots, i_k$ such that $P^1 \xrightarrow{i_1} P^2 \xrightarrow{i_2} P^3 \xrightarrow{i_3} \cdots \xrightarrow{i_k} \{1,2,\cdots,n\}$ and $P^{i_\ell} \prec_{i_\ell} P^*$ for all $\ell = 1,\ldots,k$. Given that $\{1,2,\cdots,n\} \xrightarrow{N} P^*$ and $\{1,2,\cdots,n\} \prec_N P^*$, we have $P^1 \ll P^*$. ■

In a symmetric game, the grand coalition constitutes an EEBA under similar conditions.

Corollary 2 *In a symmetric game where $\{N\}$ is an EEBA if*

(i) $\{N\}$ is Pareto efficient,

(ii) $\langle 1,1,\cdots,1\rangle$ is not Pareto efficient and

[10] They are, however, Pareto efficient.

(iii) *for all* $P \in \mathcal{P}$ *such that* $P \neq \{N\}$ *and* $P \neq \langle 1, 1, \cdots, 1 \rangle$ *there is a coalition* $S \in P$ *such that* $|S| > 1$ *and* $P \prec_i \{N\}$ *for some* $i \in S$.

An alternative corollary applies when agents' payoffs are comparable.

Corollary 3 *Consider a symmetric game* $(N, \{u_i\}_{i \in N})$ *where agents' payoffs are comparable.* $\{N\}$ *is an EEBA if*

(i) *for all* $P, P' \in \mathcal{P}$*, if* $|P| > |P'|$ *then* $\sum_{i \in N} u_i(P) < \sum_{i \in N} u_i(P')$ *and*

(ii) *for all* $P \in \mathcal{P}$*, if* $S, T \in P$ *and* $|S| > |T|$*, then* $u_i(P) < u_j(P)$ *for* $i \in S$ *and* $j \in T$.

Condition (i) states that as coalition structures become coarser aggregate payoff increases. Condition (ii) states that in a given coalition structure smaller coalitions yield higher per member payoffs, implying thus, that coalition formation has positive externalities. Games with positive externalities were defined in Yi (1997) and one property is that when any two coalitions merge, others benefit. Note that Yi's definition of positive externalities is stronger than conditions (i) and (ii) above.

The above results apply to the symmetric oligopoly model studied earlier in this section as well as to a public good economy [see Ray and Vohra (1997) and Yi (1997)].

5.2 Counter Example

In the previous section, we identified sufficient conditions for a game to admit an efficient EEBA. However, as the example in Table 5 illustrates, it is possible for a game to have only inefficient EEBAs. This result reinforces the inefficiency puzzle posed by Ray and Vohra (1997).

Observe that the first row is the Pareto efficient cycle while every other cell is inefficient. In particular, the first entry in each column Pareto dominates all other entries in the same column.

Table A in the appendix shows how A (indirectly) dominates all other outcomes except D and d. Therefore, we cannot construct a stable set containing A alone since it cannot account for the exclusion of D and d. If A is included in an stable set, according to internal stability none of the outcomes it dominates can be included in the same stable set. Moreover, D and d cannot be included in the same stable set either, since they (indirectly) dominate A as illustrated in Table A. By the symmetry of the game the same arguments extend to all the other efficient outcomes B, C, D and E. Hence no stable set can support (contain) efficient outcomes.

A $\{512, 34\}$ 19,27,32,17,22	B $\{123, 45\}$ 19,27,32,17,22	C $\{234, 51\}$ 19,27,32,17,22	D $\{345, 12\}$ 19,27,32,17,22	E $\{451, 23\}$ 19,27,32,17,22
a $\{512, 3, 4\}$ 18,26,31,16,21	b $\{123, 4, 5\}$ 18,26,31,16,21	c $\{234, 5, 1\}$ 18,26,31,16,21	d $\{345, 1, 2\}$ 18,26,31,16,21	e $\{451, 2, 3\}$ 18,26,31,16,21
f $\{5, 12, 3, 4\}$ 15,25,30,15,20	g $\{1, 23, 4, 5\}$ 15,25,30,15,20	h $\{2, 34, 5, 1\}$ 15,25,30,15,20	i $\{3, 45, 1, 2\}$ 15,25,30,15,20	j $\{4, 51, 2, 3\}$ 15,25,30,15,20
k $\{51, 23, 4\}$ 15,25,30,15,20	ℓ $\{12, 34, 5\}$ 15,25,30,15,20	m $\{23, 45, 1\}$ 15,25,30,15,20	n $\{34, 51, 2\}$ 15,25,30,15,20	o $\{45, 12, 3\}$ 15,25,30,15,20
p $\{5, 14, 2, 3\}$ 15,25,20,30,15	q $\{1, 25, 3, 4\}$ 15,25,20,30,15	r $\{2, 31, 4, 5\}$ 15,25,20,30,15	s $\{3, 42, 5, 1\}$ 15,25,20,30,15	t $\{4, 53, 1, 2\}$ 15,25,20,30,15
u $\{5, 142, 3\}$ 15,25,20,30,15	v $\{1, 253, 4\}$ 15,25,20,30,15	w $\{2, 314, 5\}$ 15,25,20,30,15	x $\{3, 425, 1\}$ 15,25,20,30,15	y $\{4, 531, 2\}$ 15,25,20,30,15
		z $\{1, 2, 3, 4, 5\}$ 15,15,15,15,15		

Table 5

Next we argue that $\mathcal{Q} = \{k, \ell, m, n, o, z\}$ is stable. For external stability we need to show that all outcomes not in $\mathcal{Q}$ are (indirectly) dominated by some outcome in $\mathcal{Q}$. Table B in the appendix lists all such paths of dominance. To prove the internal stability of $\mathcal{Q}$ we have to show that no element of $\mathcal{Q}$ indirectly dominates another element of $\mathcal{Q}$. We start with k and ℓ. First we argue that $k \not\gg \ell$. Note that $k \succ_{124} \ell$ and from ℓ coalition 124 and its subsets can induce u, f, p, h, s and z. But, u, f and p have the same payoffs for all players as k does, thus, once at u, f or p, no sequence will initiate with destination k. Note that although $k \succ_{12} h$ from h coalition 12 can only induce ℓ. The following table summarizes the above information:

$$
\begin{array}{llll}
\ell \xrightarrow{124} & u \underset{N}{\sim} k & & \\
\ell \xrightarrow{124} & f \underset{N}{\sim} k & & \\
\ell \xrightarrow{124} & p \underset{N}{\sim} k & & \\
\ell \xrightarrow{12} & h \underset{12}{\prec} k & h \xrightarrow{12} & \ell
\end{array}
$$

Table 6

We still have to show that s and z will not lead to k. Observe that $k \succ_{12} s$ and from s coalition 12 can induce f and z. While f does not lead to k, z is still to be checked. But $k \succ_{124} z$ and from z coalition 124 can induce u, f, p and s which are all examined already. The following table summarizes this information:

$$
\begin{array}{llll}
\ell \xrightarrow{124} & s \underset{12}{\prec} k & s \xrightarrow{12} & f \underset{N}{\sim} k \\
 & {}_{12}\swarrow & & \\
\ell \xrightarrow{124} & z \underset{124}{\prec} k & z \xrightarrow{124} & u \underset{N}{\sim} k \\
 & & z \xrightarrow{124} & f \underset{N}{\sim} k \\
 & & z \xrightarrow{124} & p \underset{N}{\sim} k \\
 & & z \xrightarrow{124} & s \not\ll k
\end{array}
$$

Table 7

Next we show that $\ell \not\gg k$ in Table C in the appendix. Similarly, we show that k does not dominate and is not dominated by o, m, and n, respectively, by outlining the blocked paths in Tables D, E and F in the appendix. Moreover, it is already shown, in Table 7 above, that $k \not\gg z$. It is also easy to see that z cannot indirectly dominate k, ℓ, m, n and o since no players strictly prefers z to them. The rest of the proof of internal stability follows from symmetry.

6 Alternative Routes

In this section we discuss some alternatives that can potentially facilitate the attainment of efficient agreements.

6.1 Transfers

It is easy to construct examples where transfer payments within coalitions are necessary to achieve strongly efficient agreements. Consider a simple example with $N = \{1, 2\}$ and the following payoffs.

$\{12\}$ $10, 5$
$\{1, 2\}$ $6, 6$

$\{1, 2\}$ is the only EBA and EEBA and it is efficient but not strongly efficient. If transfers are possible, then there exist strongly efficient EEBAs. For example, $\{12\}$ with $(8, 7)$ payoff allocation constitutes an EEBA.

However, the following example, a perturbed roommate problem, illustrates how transfers *across* coalitions may be necessary to achieve strong efficiency.

Structure	Per-person Payoff	EEBA
$\langle 3 \rangle$	4	$\times$
$\langle 2, 1 \rangle$	$5, 11$	$\times$
$\langle 1, 1, 1 \rangle$	6	✓

$\langle 1, 1, 1 \rangle$ is the only EBA and EEBA; it is Pareto efficient but not strongly efficient. Inter-coalition transfers are necessary to achieve strong efficiency in this case. A similar phenomenon was identified as beneficial altruism in Greenberg (1980) and Dréze and Greenberg (1980) .

6.2 Uncovered Set and Efficiency

Another venue worth investigating is the adoption of an alternative solution concept where outcomes are not ruled out as "easily" as in the vN-M stable set. One such notion is the uncovered set, originally defined for majority voting tournaments by Miller (1980). It requires that an outcome is excluded only by another outcome that can fully replace all its "functions". More formally,

Covering Relation Let $>$ be a binary relation on $\mathcal{P}$. For any $P, Q \in \mathcal{P}$, P covers Q in $\mathcal{P}$, written $P \mathrel{\widehat{>}} Q$, if (1) $P > Q$ and (2) for all $R \in \mathcal{P}$, $Q > R \Longrightarrow P > R$.

That is, P covers Q if P not only dominates Q, but also dominates whatever Q dominates.

Uncovered Set Uncovered set is the core of the abstract system $(\mathcal{P}, \widehat{>})$.

$$UC(\mathcal{P}, >) = \left\{ Q \in \mathcal{P} \mid \nexists P \in \mathcal{P} \text{ s.t. } P \mathrel{\widehat{>}} Q \right\}.$$

In the original definition by Miller (1980) the binary relation $>$ used is the direct (myopic) dominance relation[11]. By replacing it with the indirect dominance relation[12] introduced in Section 4 we obtain a farsighted uncovered set.

[11] In our framework, $P \in \mathcal{P}$ directly dominates $Q \in \mathcal{P}$ if $\exists S \subset N$ such that $Q \overset{S}{\longrightarrow} P$ and $Q \underset{S}{\prec} P$.

[12] Unlike in a tournament, the binary relation $\gg$ is neither complete nor asymmetric.

Farsighted Uncovered Set $UC(\mathcal{P}, \gg) = \left\{Q \in \mathcal{P} \mid \nexists P \in \mathcal{P} \text{ s.t. } P \mathrel{\widehat{\gg}} Q\right\}$

Not surprisingly, the farsighted uncovered set is always non-empty.

Proposition 4 $UC(\mathcal{P}, \gg) \neq \emptyset$

Proof. First, assume that utility is not transferable. We shall show that the covering relation $\widehat{>}$ is acyclic. That is, there do not exist $P^1, P^2, \ldots, P^k \in \mathcal{P}$ such that $P^j \mathrel{\widehat{>}} P^{j-1}$ for $j = 2, 3, \ldots, k$ and $P^1 = P^k$. Otherwise, given that $\blacktriangleright$ is transitive, we have $P^k \mathrel{\widehat{>}} P^1 = P^k$, implying $P^1 \gg P^1$. A contradiction. Since $\blacktriangleright$ is acyclic, $\mathcal{P}$ admits a maximal element with respect to $\widehat{>}$. Thus, $UC(\mathcal{P}, \gg) \neq \emptyset$. $\blacksquare$

More importantly, the farsighted uncovered set identifies only Pareto efficient coalition structures, as the following proposition asserts. The efficiency result here relies on the property of indirect dominance. It is easy to see that the uncovered set with direct (myopic) dominance does not have such an efficiency property.

Proposition 5 *Every* $P \in UC(\mathcal{P}, \gg)$ *is efficient.*

Proof. To show the first part of the proposition, assume in negation that $P \in UC(\mathcal{P}, \gg)$ and there exists $Q \in \mathcal{P}$ such that $P \prec_N Q$. Obviously, $Q \gg P$ (trivially). We proceed to show that $Q \mathrel{\widehat{>}} P$. Let $P' \in \mathcal{P}$ be such that $P \gg P'$. Then there exist a sequence of coalition structures $P^1, P^2, \cdots, P^k \in \mathcal{P}$, where $P^1 = P'$ and $P^k = P$, and a sequence of coalitions $T^1, T^2, \cdots, T^{k-1}$ such that for all $j = 1, \cdots k-1$ $P^j \xrightarrow{T^j} P^{j+1}$ and $P^j \prec_{T^j} P$ for all $j = 1, \cdots k-1$. Since $P \xrightarrow{N} Q$ and $P \prec_N Q$, we have $P^j \prec_{T^j} Q$ for all $j = 1, \cdots k-1$. Thus, $Q \gg P'$, implying that $Q \mathrel{\widehat{>}} P$. This contradicts that $P \in UC(\mathcal{P}, \gg)$. $\blacksquare$

If utility is transferable across and within coalitions the definition of the farsighted uncovered set can be extended on pairs, each consisting of a coalition structure and a payoff allocation associated with it. It is easy to see that in this case the farsighted uncovered set identifies only strongly efficient pairs.

7 Conclusion

In this paper we extend Ray and Vohra's (1997) notion of EBA to allow for arbitrary coalitional deviations. We show that despite this extension, the new notion, EEBA, although it facilitates the attainment of efficient agreements, does not guarantee efficiency. In particular, we identify sufficient conditions for a game to admit an efficient EEBA and construct a counter example where none of the EEBAs is efficient. The negative result strengthens Ray and Vohra's (1997) inefficiency puzzle. Therefore, this subject warrants further study to explore solution concepts and negotiation processes, among agents who exercise their decision power freely, that lead to efficient agreements.

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9 Appendix

Table A

$A \gg a$	$: a \xrightarrow{N} A$	$A \gg B$	$: B \xrightarrow{2} \phi_1 \xrightarrow{N} A$
$A \gg f$	$: f \xrightarrow{N} A$	$A \gg b$	$: b \xrightarrow{12} f \xrightarrow{N} A$
$A \gg k$	$: k \xrightarrow{N} A$	$A \gg g$	$: g \xrightarrow{12} f \xrightarrow{N} A$
$A \gg p$	$: p \xrightarrow{N} A$	$A \gg \ell$	$: \ell \xrightarrow{4} f \xrightarrow{N} A$
$A \gg u$	$: u \xrightarrow{N} A$	$A \gg q$	$: q \xrightarrow{12} f \xrightarrow{N} A$
$A \gg z$	$: z \xrightarrow{N} A$	$A \gg v$	$: v \xrightarrow{124} \phi_2 \xrightarrow{N} A$
$A \gg C$	$: C \xrightarrow{125} A$	$A \gg E$	$: E \xrightarrow{4} k \xrightarrow{N} A$
$A \gg c$	$: c \xrightarrow{125} A$	$A \gg e$	$: e \xrightarrow{24} \phi_3 \xrightarrow{N} A$
$A \gg h$	$: h \xrightarrow{125} A$	$A \gg j$	$: j \xrightarrow{24} \phi_3 \xrightarrow{N} A$
$A \gg m$	$: m \xrightarrow{125} a \xrightarrow{N} A$	$A \gg o$	$: o \xrightarrow{4} f \xrightarrow{N} A$
$A \gg r$	$: r \xrightarrow{125} a \xrightarrow{N} A$	$A \gg t$	$: t \xrightarrow{24} \phi_8 \xrightarrow{N} A$
$A \gg w$	$: w \xrightarrow{125} A$	$A \gg y$	$: y \xrightarrow{24} \phi_4 \xrightarrow{N} A$
$A \gg i$	$: i \xrightarrow{13} \phi_1 \xrightarrow{N} A$	$A \not\gg D$	while
$A \gg n$	$: n \xrightarrow{123} f \xrightarrow{N} A$	$D \gg A$	$: A \xrightarrow{345} D$
$A \gg s$	$: s \xrightarrow{12} f \xrightarrow{N} A$	$A \not\gg d$	while
$A \gg x$	$: x \xrightarrow{13} \phi_5 \xrightarrow{N} A$	$d \gg A$	$: A \xrightarrow{45} o \xrightarrow{2} i \xrightarrow{N} d$

Table B

$A \ll \ell$	$: A \xrightarrow{5} \ell$	$B \ll m$	$: B \xrightarrow{1} m$
$a \ll \ell$	$: a \xrightarrow{35} \phi_6 \xrightarrow{N} \ell$	$b \ll m$	$: b \xrightarrow{14} \phi_9 \xrightarrow{N} m$
$f \ll \ell$	$: f \xrightarrow{35} \phi_6 \xrightarrow{N} \ell$	$g \ll m$	$: g \xrightarrow{14} \phi_9 \xrightarrow{N} m$
$p \ll \ell$	$: p \xrightarrow{35} \phi_7 \xrightarrow{N} \ell$	$q \ll m$	$: q \xrightarrow{14} \phi_{10} \xrightarrow{N} m$
$u \ll \ell$	$: u \xrightarrow{35} \phi_2 \xrightarrow{N} \ell$	$v \ll m$	$: v \xrightarrow{14} \phi_{11} \xrightarrow{N} m$
$C \ll n$	$: C \xrightarrow{2} n$	$D \ll o$	$: D \xrightarrow{3} o$
$c \ll n$	$: c \xrightarrow{25} \phi_{12} \xrightarrow{N} n$	$d \ll o$	$: d \xrightarrow{13} \phi_1 \xrightarrow{N} o$
$h \ll n$	$: h \xrightarrow{25} \phi_{12} \xrightarrow{N} n$	$i \ll o$	$: i \xrightarrow{13} \phi_1 \xrightarrow{N} o$
$r \ll n$	$: r \xrightarrow{25} \phi_{13} \xrightarrow{N} n$	$s \ll o$	$: s \xrightarrow{13} \phi_{15} \xrightarrow{N} o$
$w \ll n$	$: w \xrightarrow{25} \phi_{14} \xrightarrow{N} n$	$x \ll o$	$: x \xrightarrow{13} \phi_5 \xrightarrow{N} o$
$E \ll k$	$: E \xrightarrow{4} k$	$t \ll k$	$: t \xrightarrow{24} \phi_8 \xrightarrow{N} k$
$e \ll k$	$: e \xrightarrow{24} \phi_3 \xrightarrow{N} k$	$y \ll k$	$: y \xrightarrow{24} \phi_4 \xrightarrow{N} k$
$j \ll k$	$: j \xrightarrow{24} \phi_3 \xrightarrow{N} k$		

Let ϕ_i, where $i = 1, ..., 15$, represent coalition structures that do not appear in Table 5 and whose payoff is 0 for all the players. Obviously all ϕ_i's are directly dominated by any element of $\mathcal{Q}$. The ϕ_i's are provided in the table below.

ϕ_1 ={13,45,2}	ϕ_4 ={135,24}	ϕ_7 ={14,35,2}	ϕ_{10} ={14,25,3}	ϕ_{13} ={13,25,4}
ϕ_2 ={124,53}	ϕ_5 ={245,13}	ϕ_8 ={24,35,1}	ϕ_{11} ={235,14}	ϕ_{14} ={134,25}
ϕ_3 ={24,51,3}	ϕ_6 ={12,35,4}	ϕ_9 ={14,23,5}	ϕ_{12} ={34,25,1}	ϕ_{15} ={13,24,5}

Table C

$\ell \not\gg k$ while $\ell \succ_{35} k$	$k \xrightarrow{5}$	$g \underset{N}{\sim} \ell$		
	$k \xrightarrow{3}$	$j \underset{23}{\prec} \ell$	$j \xrightarrow{23}$	k
	$k \xrightarrow{35}$	$t \underset{23}{\prec} \ell$	$t \xrightarrow{23}$	$g \underset{N}{\sim} \ell$
		${}_{23}\swarrow$		
	$k \xrightarrow{35}$	$z \underset{235}{\prec} \ell$	$z \xrightarrow{235}$	$g \underset{N}{\sim} \ell$
				$q \underset{N}{\sim} \ell$
				$v \underset{N}{\sim} \ell$
				$t \not\ll \ell$

Table D

$k \not\gg o$ while $k \succ_{24} o$	$o \xrightarrow{4}$	$f \underset{N}{\sim} k$		
	$o \xrightarrow{24}$	$z \not\ll k$	see Table 7	
	$o \xrightarrow{24}$	$s \not\ll k$	see Table 7	
	$o \xrightarrow{2}$	$i \underset{12}{\prec} k$	$i \xrightarrow{12}$	o
$o \not\gg k$ while $o \succ_{135} k$	$k \xrightarrow{3}$	$j \underset{N}{\sim} o$		
	$k \xrightarrow{135}$	$y \underset{N}{\sim} o$		
	$k \xrightarrow{135}$	$t \underset{N}{\sim} o$		
	$k \xrightarrow{15}$	$g \underset{15}{\prec} o$	$g \xrightarrow{15}$	k
	$k \xrightarrow{135}$	$r \underset{15}{\prec} o$	$r \xrightarrow{15}$	$j \underset{N}{\sim} o$
		${}_{15}\swarrow$		
	$k \xrightarrow{135}$	$z \underset{135}{\prec} o$	$z \xrightarrow{135}$	$y \underset{N}{\sim} o$
			$z \xrightarrow{135}$	$j \underset{N}{\sim} o$
			$z \xrightarrow{135}$	$t \underset{N}{\sim} o$
			$z \xrightarrow{135}$	$r \not\ll o$

Table E

$k \not\gg m$	$m \xrightarrow{12}$	$o \underset{24}{\prec} k$	$o \not\ll k$	see Table D		
while $k \succ_{12} m$	$m \xrightarrow{12}$	$i \underset{12}{\prec} k$	$i \xrightarrow{12}$	o		
$m \not\gg k$	$k \xrightarrow{34}$	$n \underset{134}{\prec} m$	$n \xrightarrow{134}$	$w \underset{N}{\sim} m$		
while $m \succ_{34} k$			$n \xrightarrow{134}$	$r \underset{N}{\sim} m$		
			$n \xrightarrow{134}$	$h \underset{N}{\sim} m$		
			$n \xrightarrow{34}$	$j \underset{34}{\prec} m$	$j \xrightarrow{34}$	n
			$n \xrightarrow{134}$	$p \underset{34}{\prec} m$	$p \xrightarrow{34}$	$h \underset{N}{\sim} m$
				${}_{34}\swarrow$		
			$n \xrightarrow{134}$	$z \underset{134}{\prec} m$	$z \xrightarrow{134}$	$w \underset{N}{\sim} m$
						$r \underset{N}{\sim} m$
						$h \underset{N}{\sim} m$
						$p \not\ll m$
	$k \xrightarrow{34}$	$j \underset{34}{\prec} m$	$j \xrightarrow{34}$	$n \not\ll m$		

Table F

$k \not\gg n$	$n \xrightarrow{12}$	$\ell \underset{124}{\prec} k$	$\ell \not\ll k$	Tables 6 & 7		
while $k \succ_{12} n$	$n \xrightarrow{12}$	$h \underset{12}{\prec} k$	$h \xrightarrow{12}$	$\ell \not\ll k$	Tables 6 & 7	
$n \not\gg k$	$k \xrightarrow{45}$	$m \underset{25}{\prec} n$	$m \xrightarrow{25}$	$i \underset{N}{\sim} n$		
while $n \succ_{45} k$			$m \xrightarrow{25}$	$g \underset{45}{\prec} n$	$g \xrightarrow{45}$	m
			$m \xrightarrow{25}$	$q \underset{45}{\prec} n$	$q \xrightarrow{45}$	$i \underset{N}{\sim} n$
				${}_{45}\swarrow$		
			$m \xrightarrow{25}$	$z \underset{245}{\prec} n$	$z \xrightarrow{245}$	$x \underset{N}{\sim} n$
					$z \xrightarrow{245}$	$s \underset{N}{\sim} n$
					$z \xrightarrow{245}$	$i \underset{N}{\sim} n$
					$z \xrightarrow{245}$	$q \not\ll n$
	$k \xrightarrow{45}$	$g \underset{45}{\prec} n$	$g \not\ll n$			