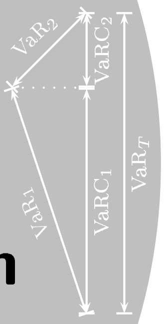
ose standard maturit
ich misses the 2 months and 18 mo
or the governments bonds, the standard risk f
bonds in the market have a *fixed maturity*. So the
observable in the market, and has to be calculated from a
smaller than 5 years and one longer.

Value-at-Risk:

The Delta-normal Approach

$\mathrm{R}_{\alpha,\tau}(p) = F^{-1}(\alpha)\sqrt{\tau p^T \Sigma p}$



Marc Henrard

Final version!

First version: 3 March 2001.
This version: 29 September 2005.

Preface

This book presents a simple model (the simplest?) for the computation of the value-at-risk: the delta-normal approach. It doesn't explain the shortcomings and advantages of the method nor compares it with other models. Even on this single topic, by no way it pretends to be complete or in the forefront.

Nowadays with the abundance of books and the facility to publish them it seems that every book is starting by justifying its existence. This book will continue this newly established tradition.

The main reason for this book existence is that ... I liked to write it! During the years I spent has risk manager I read a lot of books and articles and accumulated some experience. Unfortunately I have not found a source of information about delta-normal VaR that contained what I needed with enough details and a wide enough cover. I slowly started to write my experience in articles and notes. Those were useful to communicate on specific points. But the basics, those I was using almost every day were missing. I decided to write them and to organise my other notes in a coherent (to my mind) way. This took time with some (short) moments of advancement and some (long) moments of immobility. Now that I have reach a stage were I have the impression that the immobility will prevail for ever, I write the preface, thank the reader who read this paragraph and hope he will continue!

Any comments, suggestions, remarks, confidences, criticism,. . . about the content, the writing, the presentation, the typography,. . . are welcome.

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Chapter 1

Introduction

Value-at-risk (VaR) has become popular for limit settings, economic capital computation, regulatory reporting or for risk disclosure. This popularity comes from the simplicity of the concept and the fact that the results are written in term of value. A risk express in term of money you risk is more speaking than one express in year equivalent (duration), sensitivity or volatility.

Even if VaR is not a perfect risk measure (see for example the concept of coherent risk measure in [1]) it is widely used and it seems that it will continue to be used for a while.

VaR is a quantile approach to risk. The VaR of a portfolio is the forecast of a given (high) quantile of the distribution of the returns of the portfolio over a given period. In other words it is a money estimation of the level beyond which the lost will go only with a given (small) probability.

The difficulty with the concept, and what makes it interesting, is that all the risks of a portfolio are aggregated in a unique number. We not only have to estimate the distribution of the changes of each risk factor but also the way the different factors interact, how they are correlated. A typical approach is to restrict the number of risk factors to a given finite set of interest rates, currencies, commodities and equities and to suppose that their joint distribution is joint-normal. Then each particular instrument is split between the different risk factors using a first order approximation. This method, called delta-normal VaR, is the RiskMetrics one popularized by JP Morgan [15].

As said in the preface, this book doesn't explain why you should (or shouldn't) use the delta-normal approach to compute value-at-risk nor if this number is meaningful. We suppose that you have already decided to compute it!

If your goal is "just" to implement it, you can even skip several parts of this book. For a first reading, I suggest the following parts.

Chapter 2: Underlying principle. You can skip Chapter 1, but for you it is too late.

Chapter 3: Value-at-risk.

Chapter 4: Basic instruments. Only the instruments you need.

Chapter 5: Cash-flow maps. Sections 5.1 and 5.2.1 or 5.2.2.

Chapter 6: Variance-covariance matrix. If you don't have the matrix already.

Chapter 10: Reports. For ideas on how to present the results.

This book differs from the *RiskMetrics* technical document [15], general books on value at risk (like Jorion [10]): it contains only a description of "how to compute", not "why" and "when". Other books that contain similar aims are *Return to RiskMetrics* [14] and *Value-at-Risk* [3].

Chapter 2

Underlying principles

Entia non est multiplicanda praeter necessitatem[a].
Guillaume d'Ockham, 1285–1345

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[a] Principles should not be multiplied beyond the necessary. Personal translation.

2.1 Risk factors

The first principle in the computation of the value-at-risk using the delta-normal methodology is that the change of value of the portfolio can be explained by a *finite* number of risk factors. The type of risk factors depends of course of the portfolio itself. The common type of risk factors are:

- Currencies
- Equities
- Commodities
- Interest rates

The goal is to explain the *change* of value of the portfolio. So the risk factors that we use are the *change* of value of building blocks of financial products.

The first type is probably the simplest one. Usually the number of currencies is fairly limited. A total of 10 is usually enough. Some institutions with active forex desk in exotic currencies can have 30 currencies, but very seldom more. If the exchange rate at time i is denoted X_i the risk factor will be $(X_1 - X_0)/X_0$.

The second type requires already some choice. There are two usual approaches. The first one is the obvious one: each equity is a different risk factor. The drawback of this approach is that you need the information (time series) for each individual share. If the universe in which you can potentially invest is large, the required information is huge. For 10,000 different shares, the size of the variance-covariance matrix (see Chapter 6) is $10,000 \times 10,000 = 100,000,000$ different coefficients. In practice you are not forced to actually compute the matrix (see Section 6.1). This approach is realistic and meaningful if the position contains a limited number of different equities or this position represent a non-negligible part of the portfolio.

A second approach is to have only indexes and not individual equities as risk factors. This reduce dramatically the quantity of information required. In this case the β of the individual equities is required. The β is the slope of the best linear fit between the changes of the index and of the equity (see subsection 9.2). This approach is meaningful only if the equities are representative of the index. For example an equity desk which is allowed to keep some residual position in the component of an index resulting of transactions for clients. In Section 6.4 we also hint why it is dangerous to have both the index and the individual equities as risk factors. If the (equity or equity index) value at time i in its quotation currency is P_i, then the risk factor is $(P_1 - P_0)/P_0$.

The interest rate risk is represented in the following way. Each position affected by the interest rates (bonds, swaps,...) is associated to a series of cash-flows at standard maturities or vertices. Chapter 5 describes possible ways to do this association. For interest rate risk, the number of risk factors is very large. Suppose that one has 5 currencies (USD, EUR, GBP, JPY, CHF), for each currency, 2 types of curves are used (government and swap) and each curve contains 8 vertices (1m, 3m, 6m, 1Y, 2Y, 3Y, 5Y, 10Y). It is already 80 risk factors. Very often the number of currencies is closer to 10 and the number of points is closer to 15. This means 300 risk factors.

For the interest rate, the risk factor is the change of value of the zero-coupon bond of the given maturity. It is possible to use the rates as risk factors, but the change of value of a bond is not linear with respect to the change of rate. This means that in this last case an extra approximation is needed. In each currency a limited number of curves are used as risk factors (usually 2). This means that not all sector curves and not all spreads will be risk factors. Ways to reduce this problem are explain in Section 2.3 and 5.2.7. If the zero coupon bond value at time i is B_i, then the risk factor is $(B_1 - B_0)/B_0$.

Commodities are similar to equities and currencies in the sense that each of them is a risk factor by itself. But they are not only financial assets but also physical assets. So the future prices can not be deduced from commodity spot prices and interest rates. Like in the case of interest rate risk, new risk factors for the commodity futures with standard maturities (1m, 3m, 6m, 12m,...) are introduced.

2.2 Normality

The second principle is the joint-normality[1] of the risk factors with a null mean. It means that, if P_0 is the price at a certain moment and P_1 is the price after one period, we suppose that

$$P_1 - P_0 \sim \mathcal{N}(0, \sigma^2).$$

If several prices underlying the risk factors $(P_t^1, \ldots, P_t^N)$ are involved, we suppose that

$$\begin{pmatrix} P_1^1 - P_0^1 \\ \vdots \\ P_1^N - P_0^N \end{pmatrix} \sim \mathcal{N}(0, V)$$

where V is some (semi-)positively[2] defined matrix. The density function with respect to the Lesbegue measure is

$$\frac{1}{(2\pi)^{N/2}} \frac{1}{|V|^{N/2}} \exp\left(-\frac{1}{2} x^T V^{-1} x \right).$$

This normality hypothesis is not in line with usual hypothesis in finance. For example Black and Scholes pricing of derivatives suppose that the underlying assets (equities, bonds,...) are log-normal. In a lot of models of interest rate products, the rates (not the value of the bond) is supposed to be normal or log-normal.

The reason the delta-normal VaR is easy to compute is because it relies heavily on the nice computational properties of the normal distribution. So in order to achieve the goal of simplicity we have to be slightly inconsistent with other models. To the first order, we are not inconsistent and as the model is based on first order approximation (see next section) the normality hypothesis is in line with the other models.

2.3 Linear dependence - delta approximation

The simplicity of the delta-normal approach is coming not only from the hypothesis of the normality of the underlying risk factors but also from the fact that it is supposed that any portfolio is a linear combination of risk factors. As the linear transformation of a normal distribution is it-self normal, the computation of the VaR of a portfolio is reduced to the computation of a covariance matrix, the coefficient of the linear combination and a matrix multiplication.

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[1] For simplicity in the rest of the book *normality* will refer to joint-normality, except when otherwise stated.

[2] We'd like to have a positively defined matrix to ensure that no non-zero position has zero risk. But due to the way the matrix is usually constructed, with the length of the series smaller than the number of risk factors, we have to reduce our requirements to a semi-positively defined matrix.

To obtain the linear dependence to risk factors, each position is replaced by its first order approximation with respect to the different factors.

In particular for options, the position is replaced by the delta equivalent position in the underlying:

$$F(R^1, R^2, \ldots, R^N) \approx \sum_{i=1}^{N} D_i F(R_0^1, R_0^2, \ldots, R_0^N) R^i.$$

In this case the symbol $\approx$ means that the position will be replaced by the right-hand side term for the computation of the risk, where risk means *change* in value due to the risk factors. In particular this approximation does not contains the term of order 0 and so should not be used for valuation purposes.

2.4 Coherence!

Coherence is not an underlying principle of the approach! More exactly the model is coherent with himself but not necessarily with other models used around it. There are several cases where it can be seen as incoherent.

The first one is the treatment of currency risk. The change of value of one foreign currency with respect to the base currency is normal. In this case the change of value of the base currency with respect to the foreign currency, which is the inverse, can not be normal. So the model can be used in a coherent way only if it is used (worldwide) with only one base currency! Nevertheless in Section 6.3 we explain how to rebase a matrix to another base currency.

The treatment of options suppose that we have a pricing model for the option. The one usually used is the Black and Scholes model. It suppose that the underlying assets (equities for examples) are log-normally distributed. This is not coherent with the normality principle of the model! Moreover the options are valued with the implied volatility where the risk is computed with the volatility of the risk model which is usually an historical volatility.

The mapping used to redistribute the interest rate risk on the different vertices (Section 5.2.1 and 5.2.2) are based on rates. The way those mappings are constructed suppose that the driven factor are the level of rates, through duration or through sensitivities. When this is done and we have the cash-flows at the standard vertices, we forget about rates and we use directly the change of value without linking it to rate!

Similarly if we suppose that interest rates are interpolated between standard tenor, we have

$$\begin{aligned} B(t) &= \exp(-r(t)t) = \exp(-(\alpha r(t_i) + (1-\alpha) r(t_{i+1}))t) \\ &= B(t_i)^{\frac{t_{i+1}-t}{t_{i+1}-ti}\frac{t}{t_i}} . B(t_{i+1})^{\frac{t-t_i}{t_{i+1}-ti}\frac{t}{t_{i+1}}}. \end{aligned}$$

where the t_i denote the tenors and $B(t)$ the discount factors. This means that if the $B(t_i)$ are normally distributed, the $B(t)$ can not be normally distributed. The choice of the tenors for the model excludes the choice of any other tenor if we want to have coherent implementations.

The coherence of the model has to be seen in a more local way. When the risk factors and the exposures to risk factors are given, if we suppose that the risk factors are normal and the covariance matrix is known then the computations done from there are coherent and give the value-at-risk numbers we are looking for!

Chapter 3

Value-at-risk

3.1 Definition

We have now all the ingredients to define the value at risk. We have N risk factors and any portfolio can be represented has a linear combination of those risk factors. We associate to any portfolio a N-uple $p = (p_1, \ldots, p_N)$ describing the decomposition of the portfolio in the factors. Each component p_i is the amount on that risk factor. The dimension of p is the currency unit in which the risk is computed. The risk factors are dimensionless, they are relative changes.

In particular if $c = (c_1, \ldots, c_N)$ represents the value of the risk factors in one period, the profit for the position p is

$$p.c = \sum_{i=1}^{N} p_i c_i.$$

The the risk factors are normally distributed. If we denote V the covariance matrix on one period, $c \sim \mathcal{N}(0, V)$. By a property of the normal distribution, the profit on the portfolio is also normally distributed and

$$p.c \sim \mathcal{N}(0, p^T V p).$$

The VaR, which is the α-quantile of the distribution is

$$\mathrm{VaR}_1^{\alpha,V}(p) = N^{-1}(\alpha)\sqrt{p^T V p}$$

where N^{-1} is the inverse of the cumulative normal distribution. Here we precise α, V and 1, the horizon, in the notation, but in the sequel, if those parameters are clear from the context, we omit them.

Before continuing, we need to be more precise on what VaR over one period exactly means. We compute the volatility of the risk factor without taking into account the time dimension. For example for the forex positions we look at the change of rate without looking at the interest rate paid overnight on the amount. By normality principle we take a mean 0, i.e. we ignore the carry. Over a one day period this is not important has the volatility is the largest component. Typically for a forex position, one day of volatility is $10\%/\sqrt{365}$ (see Section 3.3 for explanation on the square root) where one day of difference of interest is 5%/365. But on longer period this can be more important.

The VaR over one period should be interpreted as the loss when today the market is changing like it is changing usually over one period. The one-period volatility is applied in one instant. Speaking of *time horizon* is an abuse of language in this context.

3.2 Probability level

The choice of the probability level is a little bit arbitrary. The RiskMetrics standard data set is created for a 95% level (and 1 day horizon); the Basel committee recommend 99% (and 10 days horizon) ; 97.5% is also a popular number. Those levels are somehow arbitrary. From the point of view of the computation, a choice of 96.1234% is as relevant than the choice of 95%. If we had 6 fingers at each hand, standard choices would be B6% or BB% (resp. 95.84 and 99.31 of our percent).

To be meaningful, the number should be big enough. A 5% level means that 19 periods out of 20, you will lose more than the given number. It is not a risk measure anymore, it's a limit on the profit! It should not be too close to 100%. A probability of 99.99% means that the event appears 1 time every 10,000 periods. The estimation of the parameters (covariances) is done usually with 100 to 500 numbers. Such a high precision would not be statistically relevant. Moreover financial series tend to have "fat tails", which means that these extreme events are more frequent than the one of a normal distribution. Using a normal approximation to a distribution and then trying to obtain information about extreme event using the normal approximation will underestimate the results.

A probability between 95% and 99% is a good compromise between what we want to achieve and the precision of the tool.

Note also that the term *confidence interval* is often used to designate the probability level. This usage is inconsistent with the terminology of statistic. The VaR does not indicate any confidence. A number is computed using a model and estimated parameters of the model, but there is no computation of an interval in which the actual parameters and number we are looking for lies. Sentences like "We are α percent certain that we will not lose more than VaR in the next period" should be interpreted as "If our *model* is *correct* and our estimation of its *parameters* is *correct*, then we are α percent certain...".

3.3 Time scale

It is a tradition to consider that the VaR computed on a given time horizon can be rescaled to another time horizon using the *square root of time rule*. By this we mean that

$$\mathrm{VaR}_{t_2}^{\alpha}(p) = \frac{\sqrt{t_2}}{\sqrt{t_1}}\,\mathrm{VaR}_{t_1}^{\alpha}(p). \tag{3.1}$$

This scaling principle can not be deduces from the underlying principles (Chapter 2). We need to add a new one to obtain this property[1].

We suppose that the different risk factors over consecutive time periods are *independent and identically distributed*. Suppose that we have a position p and the covariance matrix for a time horizon of t is V. The profit on the position p for two consecutive time period is (see also Section 3.1 for the interpretation of VaR over one period)

$$p.c_1 + p.c_2$$

where c_i is the change of the risk factors of the i-th period. Note that we suppose that the position vector, which is a value, is unchanged. As $p.c_i \sim \mathcal{N}(0, p^T V p)$, and the distribution over the two period are independent, we have $p.c_1 + p.c_2 \sim \mathcal{N}(0, \Sigma)$ where

$$\Sigma = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} p^T V p & 0 \\ 0 & p^T V p \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2p^T V p.$$

So

$$\mathrm{VaR}_{2t}^{\alpha} = F^{-1}(\alpha)\sqrt{2p^T V p} = \sqrt{2}\,\mathrm{VaR}_{t}^{\alpha}.$$

The relation (3.1) can be obtained in a similar way for any t_1 and t_2 such that $t_2/t_1 \in \mathbb{Q}$. The time scale rule is valid only for rational numbers. But as the period are always in practice an integer number of days, this is not a limitation. We will not be able to rescale the VaR to a π day period, but this will not be a problem! As $\mathbb{Q}$ is dense in $\mathbb{R}$, we could extend the rule to non rational number by continuity, but this would add an extra principle to our list!

This derivation of the time scale rule differs from the one usually presented which start with risk factors satisfying the stochastic differential equation

$$dS_t = S_t\left(\mu\, dt + \sigma\, dW_t\right).$$

In this case the time scale property of the risk factors is a property of the log-normal solution of the equation. But this log-normal hypothesis is not coherent with the normal distribution of the risk factor. It is coherent only to the first order.

With our description of the time scale, the interpretation of the time rescaling change also. The VaR over 2 periods should not be interpreted as a position is fixed

---

[1] Note that if one wants to compute the VaR on only one time horizon, this extra principle is not needed.

at the beginning and kept unchanged for 2 periods and the VaR refers to the profit after those two periods. The description is that a position is chosen, kept unchanged for one period and after this period, a new position equal to the initial one (in term of value) is kept for the second period. The profit considered is the sum of the two one-period profits.

Suppose our base currency is EUR. The 2 days VaR of a USD position with a value of 1000 EUR is obtain by holding USD equivalent to 1000 EUR for 1 day, then selling or buying USD to have again 1000 EUR equivalent in USD for the second day[2]. The difference between this interpretation and the log-normal one is negligible for currencies over two days. But if one considers interest rate positions over a longer period, it is the only way to have a meaningful interpretation of the numbers. Consider the VaR with a one month time-horizon for a one month deposit. If only market risk is considered, the risk (measured by the VaR) is null for the position. The value after one month will be exactly the nominal plus the interest, not less not more. The one month VaR of a one month deposit should be interpreted has the one of a one month deposit held for one day, at the end of the day the position is sold and one enters in a new one month depo at the beginning of the next day (with a maturity of one month and one day with respect to the original date). This new one month deposit is sold at the end of the day for another one and so on. With this interpretation, the total profit and loss is not known and the one month VaR of a one month position acquires a meaning.

3.4 Geometric interpretation

The VaR is defined by

$$\mathrm{VaR}(p) = \sqrt{p^T V p}$$

where V is a positively defined matrix (for simplicity we have included the scaling factor $N^{-1}(\alpha)$ into V).

To start with suppose that there is only one risk factor and its volatility is σ. Then $\mathrm{VaR} = p\sigma$. If the risk factor is represented by a vector $\vec{r}$ of length σ, a position equivalent to p risk factors has a VaR equal to the length of the vector $p\vec{r}$.

In general by its positive definitiveness the matrix V defines a scalar product

$$(p\,|q) = p^T V q.$$

The length of the base vector $\vec{r_i} = (0, \ldots, 0, 1, 0, \ldots, 0)$ is $\sqrt{r_i^T V r_i} = \sqrt{V_{ii}} = \sigma_i$. The angles between the vectors are given by

$$\cos(\alpha_{i,j}) = \frac{r_i^T V r_i}{\|r_i\| \|r_j\|} = \frac{V_{i,j}}{\sigma_i \sigma_j} = \rho_{i,j}.$$

---

[2] As indicated at the end of Section 3.1 we do not take into account the carry or natural growth of the position

So the VaR can be interpreted as the length of the vector of component p_i in the base $\vec{r_i}$. The vector $\vec{r_i}$ have a length equal to the volatility of the risk factors. The cosines of the angle between them is given by the correlation.

Example: Let $V = \begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}$ and $p = (\frac{1}{2}, \frac{1}{2})$. Note that the position p is written in the base $\{\vec{r_i}\}$. We have $\sigma_1 = 2$, $\sigma_2 = 3$ and $\rho_{1,2} = -\frac{1}{6}$.

The risk factors can be represented by the vectors $\vec{r_1} = (0, 2)$ and $\vec{r_2} = (\frac{\sqrt{35}}{2}, -\frac{1}{2})$. Here the vectors $\vec{r_i}$ are written in the canonical base.

The VaR of the position is the length of the vector $\frac{1}{2}\vec{r_1} + \frac{1}{2}\vec{r_2}$ and is equal to $\sqrt{11}/2$.

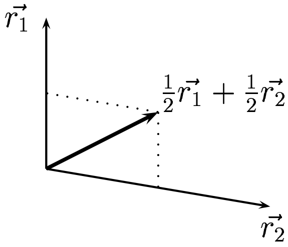


Figure 3.1: Geometric interpretation of the VaR of the position (0.5, 0.5).

Chapter 4

Basic instruments

To simplify the notations, we suppose in this section that the risk factors for the currency XXX_i is the i-th in the list of risk factors and that its exchange rate against the base currency is FX_i ($1\,\mathrm{XXX}_i = \mathrm{FX}_i\,\texttt{base}$).

4.1 Cash currency position

This is the easiest instrument. The vector for a position of $y\,\mathrm{XXX}_i$ is $p_i = y\,\mathrm{FX}_i$ and $p_j = 0$ for $j \neq i$.

4.2 Equity

Suppose the equity is among the risk factors (say the j-th). If the position is x shares which have a value of $y\,\mathrm{XXX}_i$, then the position is $p_i = xy\,\mathrm{FX}_i$, $p_j = xy\,\mathrm{FX}_i$ and $p_k = 0$ for $k \neq i, j$.

If the equity is not a risk factor, see Section 9.2.

Note: For now on we will not repeat the currency position for the other instruments. For each of them, there is a currency position equal to the value of the instrument. Note also that, as explain in Section 5 in case of interest rate risk, the sum of the present value of the mapped position can be different of the value of the instrument. So the value to be used is the original one, not the one obtain after the mapping process.

The only *exception* to this currency exposure is the *forex options* that are presented in Section 9.

4.3 Commodity

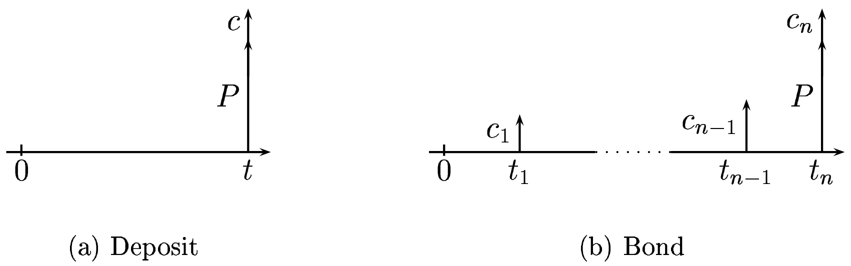
Like for equity, for a position of x unit of commodity (risk factor j), each of which has a value of $y\,\mathrm{XXX}_i$, the position is $p_j = xy\,\mathrm{FX}_i$ (plus the FX position $p_i = xy\,\mathrm{FX}_i$).

4.4 Loan, deposits and money market instruments

The mappings of a cash-flow between the different standard maturities is presented in Section 5. The only thing we do now is to obtain a set of cash-flows that represents adequately the (risk of the) instruments. Note that the cash-flows we use are not necessarily actual payments. The figures obtain by this process can not be used to manage the liquidity position. They are cash-flows interest rate risk equivalent to the instrument.

A loan is simply a short position in a deposit. The cash-flow for the loan is the opposite of the one of a deposit. We will present only the long position for each instrument. The short one has simply the opposite cash-flows.

The deposit is the simplest fixed income instrument. The cash-flow at maturity[1] is the principal plus the interest (if any) (Figure 4.1(a)). The only important thing is the cash-flow at maturity. The actual market convention (discounted instrument, quoted with clean or dirty price,...) doesn't matter for the cash-flow representation.



(a) Deposit

(b) Bond

Figure 4.1: Cash-flows of deposits and bonds.

Note: In the different figures representing the instruments we denote today by 0.

---

[1] To be precise, if the deposit has not yet settled, there are two cash-flows. The payment of the principal at the start date and the reception of the principal and interest at maturity. From the point of view of the interest rate risk, the initial payment is usually negligible. It can be taken into account by considering the deposit as a forward deposit (see Section 4.9). If the cash-flows are also used to compute the forex risk, it is important to take this initial payment into account.

4.5 Forward currency position

A FX (outright) forward is the payment of one currency against another one at a fixed rate at a fixed date in the future. From a market risk point of view it is equal to a deposit (with cash-flow equal to the amount received at maturity) in the currency that will be received and a loan in the currency that will be paid.

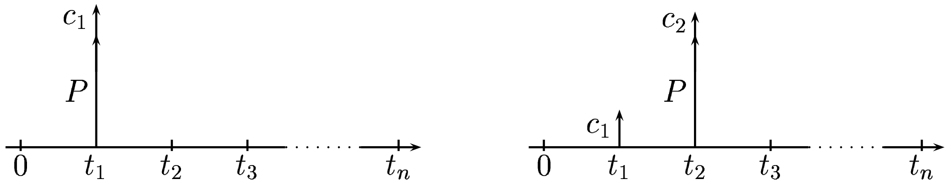
4.6 Bond with fixed coupon

The cash-flows representing the bond are simply the different coupons and the principal at maturity (Figure 4.1(b)). The coupons can be unequal ($c_i \neq c_j$) and/or not equally spaced ($t_{i+1} - t_i \neq t_{j+1} - t_j$). The next coupon ($c_1$) is the full coupon without any deduction for accrued interest.

Like for deposits, if the bond has not yet settled or is forward, the settlement amount should be considered as a cash-flow at the settlement date.

4.7 Floating rate notes

The cash-flow representing the FRN is the next coupon[2] plus the nominal at the payment date of the next coupon (Figure 4.2(a)). In the case where no coupon has fixed yet (newly issued note, before the first fixing), there is no interest rate risk (except spread risk).



(a) FRN with one coupon fixed

(b) FRN with two coupons fixed

Figure 4.2: Cash-flows of FRN.

This representation of a FRN do not take into consideration the spreads: the floating coupon is fixed on a regular basis based on a reset index (usually LIBOR)

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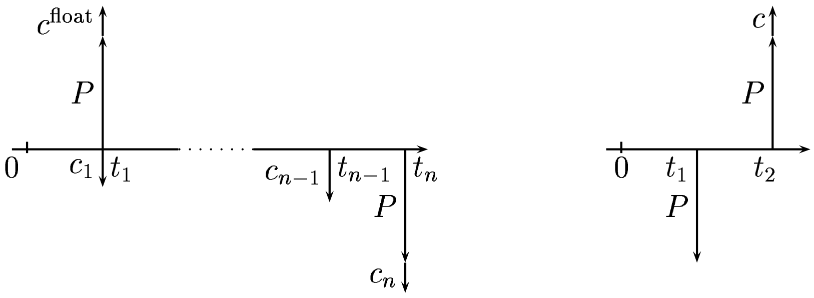
[2] In the case where the second next coupon has already fixed, the cash-flows representation is made of the next coupon plus the nominal and the second next coupon at the payment date of the second next coupon (Figure 4.2(b))

plus a spread and then depending on the credit standing of the issuer the FRN is valued with a spread above or below the reference index. It is possible to differentiate between the index used for the fixing of the coupon (we call it the *projection index*) and the index used for the valuation (we call it the *discounting index*). This is done in Section 9.1.

4.8 Interest Rate Swap

A swap for which we pay fixed can be represented by a combination of a long position is a FRN[3] and a short position in a fixed coupon bond. The cash-flows that represent the swap is just the sum of the two sets of cash-flows (Figure 4.3(a)).



(a) Swap

(b) Forward deposits and FRA

Figure 4.3: Cash-flows of swaps and forward deposits/FRA.

For forward starting swaps, the two legs should be considered forward. The fixed coupon bond has not yet settled (settlement date at the starting date of the swap) and the FRN has no interest rate risk.

4.9 Forward deposit - Forward Rate Agreement

A forward deposit where we receive fixed is a contract by which a deposit with a starting date in the future is agreed with a rate fixed at the moment of the agreement. A Forward Rate Agreement (FRA) is equivalent except that the value of the agreement is settled in cash at the starting date and there is no actual deposit. For

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[3] The same remark that applies to the FRN concerning the fixings applies also to the swap

the purpose of computing the market risk of the instruments, there is no difference between a FRA and a forward deposit. Again this representation can not be used for liquidity or credit risk analysis. The cash-flows are an outflow of the principal at the start date and the payment of the principal and the fixed interest at maturity (Figure 4.3(b)).

4.10 Interest rate futures

Even is interest rate futures are not exactly like FRA as there is a daily margining, the difference is small enough (in term of market risk, not necessarily pricing or credit risk) that we treat them in the same way.

4.11 Bond futures

Like for interest rate futures, even if there is a daily margining on futures, we treat them as forwards (payment of a fixed proceed at the delivery date of the future against the bond). The difference for bonds futures is the notion of *conversion factor*. At the delivery date the amount of cash paid for the bond is

$$C = (\text{future price} \times \text{conversion factor} + \text{accrued interest}) \times \text{notional}.$$

The conversion factor is a factor associated to each deliverable bond. The *accrued interest* is the one on the bond delivered from the last coupon date to the delivery date. The exact mechanism of the conversion factor can be found in many textbooks on fixed income instruments and in particular in [9, Section 4.11].

At delivery we have that *future price* × *conversion factor* = *bond price.* The margining is done on the base of the future price. So the profit coming from the future is similar to the one of the bond on a notional of the notional of the future divided by the conversion factor.

In any case the nominal of the bond delivered is the nominal of the future contract. This implies that the delivery of the underlying change the risk from a risk on the notional/conversion factor to a risk on the notional.

The cash-flow representation of the bond future is given in Figure 4.4. The coupons $c_1, ...c_n$ and P the principal are those of the underlying bond for a nominal equal to the notional of the future divided by the conversion factor.

The exact value of a bond future does also depend on the fact that several bonds can be delivered. We take the current cheapest-to-deliver as the underlying bond. The possible change of cheapest-to-deliver in case of rate movement should be taken into account for a precise pricing. As we look only at the first order approximation we disregard this possibility.

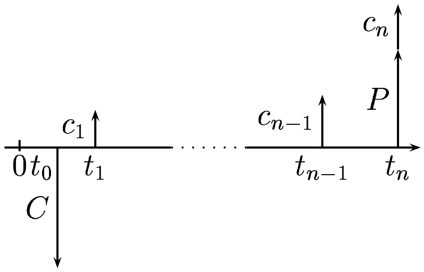


Figure 4.4: Cash-flows of a bond future.

4.12 Example

We give the cash flow equivalent of several portfolios. Those portfolios will be used as example for mapping in the next chapter and for reports. Today is April 2, 2001.

Portfolio 1(Money market in EUR)

The positions are:

Name	Start date	End date	Interest (%)	Nominal	Currency
Depo	01-Dec-2000	03-Dec-2001	5	5,000,000	EUR
Loan 1	01-Dec-2000	03-Dec-2001	4.9	-4,900,000	EUR
Loan 2	03-Apr-2001	03-Jul-2001	4.1	-7,000,000	EUR
Govt bill Germany	20-Mar-2001	20-Mar-2002	–	3,500,000	EUR
Govt bond Germany	15-Aug-1991	15-Aug-2001	10	2,000,000	EUR
FRN	10-Nov-1999	10-Nov-2004	Libor	2,000,000	EUR
FRA	10-Sep-2001	11-Mar-2002	6.5	-3,300,000	EUR

The cash-flows are:

Date	Currency	Curve	Amount	Remark
03-Dec-2001	EUR	Swap EUR	5,254,861.11	Depo 1 - ACT/360
03-Dec-2001	EUR	Swap EUR	-5,144,768.61	Loan 1 - ACT/360
03-Apr-2001	EUR	Swap EUR	7,000,000.00	Loan 2 - Settlement
03-Jul-2001	EUR	Swap EUR	-7,072,547.22	Loan 2 - ACT/360
20-Mar-2002	EUR	Govt DEM	3,500,000.00	Govt bill - discount
15-Aug-2001	EUR	Govt DEM	2,200,000.00	Govt bond
10-May-2001	EUR	Swap EUR	2,070,722.22	FRN - Libor = 4.75
10-Sep-2001	EUR	Swap EUR	3,300,000.00	FRA - leg 1
11-Mar-2001	EUR	Swap EUR	-3,533,566.67	FRA - leg 2

Portfolio 2(Bonds in EUR)

The positions are:

Name	Settle date	End date	Rate	Nominal	Currency
Govt bond DEM	25-Jun-2000	17-Jun-2005	7	10,000,000	EUR
Govt bond DEM	10-Jul-1999	10-Jul-2005	6	15,000,000	EUR
Swap	05-May-2000	05-May-2005	5.45	-5,000,000	EUR
Rate future	20-Sep-2001	20-Dec-2001	4.52	15,000,000	EUR
Rate future	20-Dec-2001	20-Mar-2002	4.78	15,000,000	EUR
Bond future	20-Sep-2001	25-Apr-2005	8	7,000,000	EUR
Depo	03-Apr-2001	17-Apr-2001	4.12	8,000,000	EUR
Bond (corp)	01-Apr-2001	01-Apr-2003	8	5,000,000	USD
FX swap	01-Apr-2001	15-Apr-2001	0.91	5,000,000	USD

The cash-flows are:

Date	Currency	Curve	Amount	Remark
17-Jun-2005	EUR	Govt DEM	10,700,000	Bond 1
17-Jun-2004	EUR	Govt DEM	700,000	Bond 1
17-Jun-2003	EUR	Govt DEM	700,000	Bond 1
17-Jun-2002	EUR	Govt DEM	700,000	Bond 1
17-Jun-2001	EUR	Govt DEM	700,000	Bond 1
10-Jul-2005	EUR	Govt DEM	15,900,000	Bond 2
10-Jul-2004	EUR	Govt DEM	900,000	Bond 2
10-Jul-2003	EUR	Govt DEM	900,000	Bond 2
10-Jul-2002	EUR	Govt DEM	900,000	Bond 2
10-Jul-2001	EUR	Govt DEM	900,000	Bond 2
05-May-2005	EUR	Swap EUR	-5,272,500	Swap
05-May-2004	EUR	Swap EUR	-272,500	Swap
05-May-2003	EUR	Swap EUR	-272,500	Swap
05-May-2002	EUR	Swap EUR	-272,500	Swap
05-May-2001	EUR	Swap EUR	-272,500	Swap
05-May-2001	EUR	Swap EUR	5,056,250	Swap
20-Sep-2001	EUR	Swap EUR	-15,000,000	Rate future (leg 1)
20-Dec-2001	EUR	Swap EUR	15,169,500	Rate future (leg 2)
20-Dec-2001	EUR	Swap EUR	-15,000,000	Rate future (leg 1)
20-Mar-2002	EUR	Swap EUR	15,179,250	Rate future (leg 2)
25-Apr-2005	EUR	Govt DEM	7,560,000	Bond future
25-Apr-2004	EUR	Govt DEM	560,000	Bond future
25-Apr-2003	EUR	Govt DEM	560,000	Bond future
25-Apr-2002	EUR	Govt DEM	560,000	Bond future
20-Sep-2001	EUR	Govt DEM	7,297,068	Bond future (cf 1.01)
01-Apr-2003	USD	Swap USD	5,200,000	Bond (corp)
01-Oct-2002	USD	Swap USD	200,000	Bond (corp)
01-Apr-2002	USD	Swap USD	200,000	Bond (corp)

01-Oct-2001	USD	Swap USD	200,000	Bond (corp)
15-Apr-2001	USD	Swap USD	-5,000,000	FX swap
15-Apr-2001	EUR	Swap EUR	+5,494,505	FX swap

Chapter 5

Cash-flow maps

The devil lies in the details.

This chapter is a modified version of a paper [6] that appeared is *The Journal of Risk*.

5.1 Introduction

The next step in the process is to map a cash-flow (or delta equivalent cash-flow) which has a maturity between two standard maturities to the appropriate (standard) maturities. This section tries to answer the following questions concerning this mapping: What are the possible methods? What are their properties? Which one is the best? How are those methods coherent with other hypothesis or methodologies?

Before looking at the mapping itself, we can perhaps think to reduce its importance by adding more points to the curve to have smaller intervals between those points. However, some intermediate points are not liquid, so it is difficult to obtain good rates. If a data series is not of good quality, the statistics deduced from the series will be of poor quality also. Moreover, adding too many points can lead to some singularities in the methodology. If one uses more risk factors than the number of historical data in the computation of the covariance matrices, the matrix obtained is singular. So some positions have an estimated risk of zero. Moreover the number of elements of the variance-covariance matrix grows with the *square* of the dimension. So if one adds too many points, it becomes very difficult to manipulate.

With a good mapping, we can reduce the number of points and solve some of the problems describe above. A good mapping can be built if the points selected correspond to liquid, standard maturities. But often there are points which are not completely explained by the standard ones. This can be the case if we choose standard maturities of 1, 3, 6, 12 months and 2 years for example, which misses the 2 months and 18 months rates that can be quite liquid. Also for the governments bonds, the standard risk factors have a *fixed term* but the bonds in the market have a *fixed maturity.* So the 5-years rate is not directly observable in the market, and has to be calculated from a bond with maturity smaller than 5 years and one longer. This is why it is of interest to estimate the predictive quality of the mapping used.

As this chapter shows, mapping can be performed with a variety of methods. We consider cash flows with a fixed *present value.* Once the present value is computed, none of the mappings we describe uses the rates. It means that the *valuation* of the cash flows and their *mapping* can be done separately with different curves. This further emphasizes that risk management methods can be quite different from pricing methods[1].

5.2 Description of the cash flow maps

The notations for this section are the following. We will always consider a cash flow with present value 1. The issue is to allocate our cash flow to positions X_1 and X_2 on two standard points. This allocation doesn't necessarily preserve the present value, i.e. $X_1 + X_2$ can be different of 1. The vector of risk we want to estimate is denoted v. Its term is t. The two risk factors v_i on which the mapping is done have norms (variance) σ_i, their terms are t_i and the correlation between them is ρ. We will approximate v by some $\bar{v} = X_1 v_1 + X_2 v_2$.

5.2.1 Elementary map

The elementary mapping, also called duration mapping, conserves the present value ($X_1 + X_2 = 1$) and the duration (which is a way to measure the risk, see [10, p. 217]).

For this elementary mapping we have

$$X_1 = \frac{t_2 - t}{t_2 - t_1} \qquad \text{and} \qquad X_2 = \frac{t - t_1}{t_2 - t_1}. \tag{5.1}$$

In the plane v_1-v_2, the vector $\bar{v}$ is a convex combination (convex homotopy) of v_1 and v_2 with the duration as parameter.

The main advantages of this mapping is that it is simple (to understand, to implement and to compute) and continuous in the input. Some advantages are presented in [13] where it is called linear mapping.

Another way to derive this mapping is the following. Suppose that interest rates are compounded continuously and *forward rates* are constant between two standard terms. We obtain (after some easy calculations) a rate for the term t between t_1 and t_2 of

$$r_t = \alpha \frac{t_1}{t} r_1 + (1 - \alpha) \frac{t_2}{t} r_2 \tag{5.2}$$

---

[1] see Section 5.2.7 for more on that

where $\alpha = (t_2 - t)/(t_2 - t_1)$. The price of a cash flow of 1 at the end of the term t has a present value

$$P_t = e^{-r_t t}.$$

We will denote by $\hat{r}_t$ and $\hat{P}_t$ the new rates and prices after the changes of the market. With those notations, we have that the gain on a position of term t and present value 1 is

$$\begin{aligned}\frac{\hat{P}_t - P_t}{P_t} &\sim \left(-\alpha\frac{t_1}{t}t(\hat{r}_1 - r_1) - (1-\alpha)\frac{t_2}{t}t(\hat{r}_2 - r_2)\right)\\ &\sim \alpha\frac{\hat{P}_1 - P_1}{P_1} + (1-\alpha)\frac{\hat{P}_2 - P_2}{P_2}. \qquad (5.3)\end{aligned}$$

This means that the investment in a security of present value 1 and term t generates the same gain as the investment of α in a security of term t_1 and $(1-\alpha)$ in a security of term t_2.

5.2.2 Rates map

For this map, the result is obtained by interpolating interest rates. For this reason, we call it the "rates map". This map is presented in [13].

The rate r_t for the term t is interpolated linearly from r_1 and r_2, the rates for the terms t_1 and t_2

$$r_t = \alpha r_1 + (1-\alpha) r_2$$

where $\alpha = (t_2 - t)/(t_2 - t_1)$. We use continuously compounding rates. The price of a cash flow of 1 at the end of the term t has a present value

$$P_t = e^{-r_t t}.$$

We will denote by $\hat{r}_t$ and $\hat{P}_t$ the new rates and prices after a change in the market. With those notations, we have that the profit on a position of term t and present value 1 is

$$\begin{aligned}\frac{\hat{P}_t - P_t}{P_t} &\sim \left(-t\alpha e^{-r_t t}(\hat{r}_1 - r_1) - t(1-\alpha)e^{-r_t t}(\hat{r}_2 - r_2)\right)/P_t\\ &\sim -\frac{t}{t_1}t_1\alpha(\hat{r}_1 - r_1) - \frac{t}{t_2}t_2(1-\alpha)(\hat{r}_2 - r_2)\\ &\sim \alpha\frac{t}{t_1}\frac{\hat{P}_1 - P_1}{P_1} + (1-\alpha)\frac{t}{t_2}\frac{\hat{P}_2 - P_2}{P_2}.\end{aligned}$$

This means that the investment in a security of present value 1 and term t generate the same profit as an investment of $\alpha\frac{t}{t_1}$ in a security of term t_1 and $(1-\alpha)\frac{t}{t_2}$ in a security of term t_2.

Thus we obtain,

$$X_1 = \frac{t}{t_1} \frac{t_2 - t}{t_2 - t_1}$$

and

$$X_2 = \frac{t}{t_2} \frac{t - t_1}{t_2 - t_1}.$$

Another way to see this mapping is the following. One calculates the result of the move of the rate at a standard term on the move of the rate at the term of the cash-flow (using the interpolated rates between two standard terms). The equivalent cash-flow at the vertex t_1 is the cash flow that has the same change of value if r_1 is moved (and not r_2).

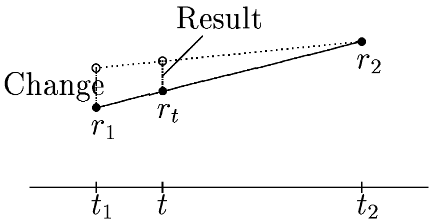


Figure 5.1: Result of the change of the rate at the standard term on an intermediate term.

Note that this mapping preserves the sensitivities with respect to the standard terms but not the present value of the cash flows ($X_1 + X_2 \neq 1$).

Note also that this map is singular in $t_1 = 0$ (or $t_2 = 0$). Cash-flows have all to be on or after the first standard date.

5.2.3 RiskMetrics map

We call RiskMetrics map the one described in the RiskMetrics Technical Document [15]. It is the one originally used for the computation of the value at risk. In *Return to RiskMetrics* [14] this method is not used anymore and the rates map (see previous section) is suggested. This mapping conserves the present value, the sign of the present value and the volatility obtained from a linear interpolation.

The conservation of the present value gives

$$X_1 + X_2 = 1. \tag{5.4}$$

We estimate the volatility of v by a linear interpolation of volatilities

$$\sigma = \sigma_1 + \frac{t - t_1}{t_2 - t_1}(\sigma_2 - \sigma_1). \tag{5.5}$$

On the other hand, we also have

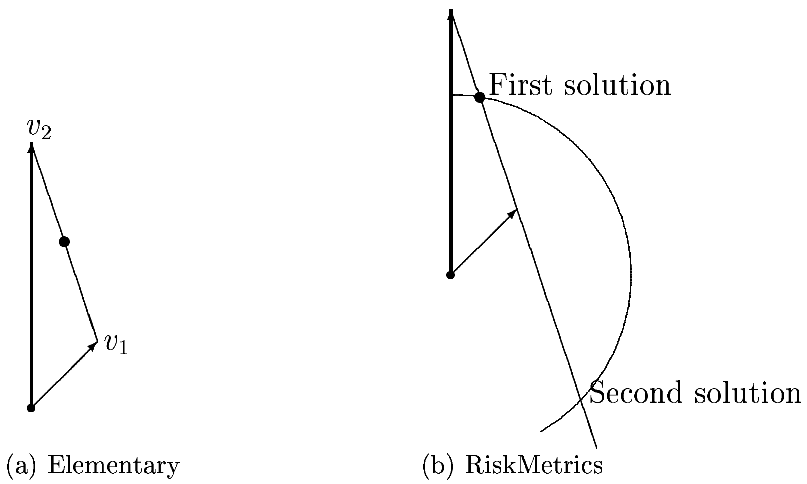
$$\sigma^2 = (X_1\sigma_1, X_2\sigma_2)\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\begin{pmatrix} X_1\sigma_1 \\ X_2\sigma_2 \end{pmatrix} = X_1^2\sigma_1^2 + 2\rho\sigma_1\sigma_2X_1X_2 + X_2^2\sigma_2^2.$$

Replacing X_2 by $1 - X_1$, solving the equation with respect to X_1, we have that

$$X_1 = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

where $a = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$, $b = \rho\sigma_1\sigma_2 - \sigma_2^2$ and $c = \sigma_2^2 - \sigma^2$. Between the two possible solutions we choose the one such that X_1 and X_2 are between 0 and 1.

The pictures of the geometrical representations of the elementary mapping and the RiskMetrics one are given in figure 5.2.



(a) Elementary (b) RiskMetrics

Figure 5.2: Geometrical representation of the elementary and the RiskMetrics mappings

In the plane v_1-v_2, the vector v is on the convex hull of v_1 and v_2 and the norm is given by the interpolation between the one of v_1 and the one of v_2 with the duration as parameter.

A problem with the RiskMetrics map is that the map may be non-continuous in the sense that one does not necessarily have $X_1 \to 1$ and $X_2 \to 0$ when $t \to t_1$. This happens when

$$(v_2 - v_1 \,|v_1) < 0 \quad \text{and} \quad \sigma_1 < \sigma_2 \tag{5.6}$$

$((.|.)$ is the scalar product between two vectors), i.e. when v_2 is in the half plane with boundary perpendicular to and passing through v_1 (see figure 5.3). In this case for $t = t_1$, the two solutions satisfy the conditions and the mapping is ambiguous.

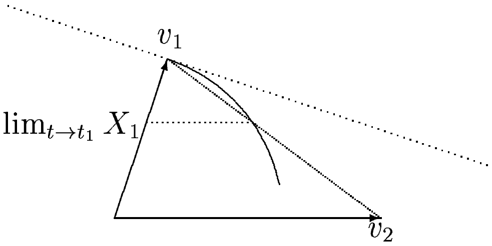


Figure 5.3: Case where the RiskMetrics map is not continuous

We prove that under the conditions (5.6), there is two solutions. We have $v = (1-X_2)v_1 + X_2v_2$. It is obvious that $X_2 = 0$ and $X_1 = 1$ satisfy (5.4) and (5.5). But we also have

$$\begin{aligned} |v|^2 &= |(1-X_2)v_1 + X_2v_2|^2 \\ &= |v_1|^2 + {X_2}^2|v_2 - v_1|^2 + X_2\,(v_2 - v_1\,|v_1)\,. \end{aligned}$$

So there exists $\epsilon > 0$ such that for all $0 < X_2 < \epsilon$, $|v| < |v_1|$. As on the other hand $\lim_{X_2\to 1} |v| = |v_2| > |v_1|$, by the intermediate value theorem, there exists $X_2 > 0$ such that $|v| = |v_1|$. This proves that there exists a second, distinct solution.

This can happen for example when the correlation is very small (and is always the case when it is negative) or the volatilities of the two factors of risks are similar. The mapping is made for terms between two consecutive standard terms, as the correlations are usually large for those risks and the volatilities are usually different (larger for the longer term), the phenomenon is not so frequent (see also [13] for more explanations about this phenomenon).

For example with the covariance matrix of March 15, 1999, this was the case for 5 combinations of risk factors. These are shown in table 5.1. The first line of this table is a case where the volatility (of the price) for the risk with the longer term is smaller that the one of the risk with shorter term. In the table 5.2, the risk factors at which the mapping is discontinuous for different matrices is given.

In the figure 5.4, we give a picture of the proportion of 1 ZAR mapped to the different risk factors as function of the term with the term between 9 and 15 years. In this extreme example, the discontinuity of the mapping appears clearly.

The elementary and the RiskMetrics mappings are characterized by the conservation of the present value. From a geometrical point of view, this means that the

Risk 1	Risk 2	$\langle v_2 - v_1 \vert v_1 \rangle$	σ_1	σ_2	ρ
CAD.Z15	CAD.Z20	-0.00002278	0.0097	0.0094	0.79
EUR.Z09	EUR.Z10	-0.00000033	0.0096	0.0098	0.98
ZAR.Z02	ZAR.Z03	-0.00000106	0.0067	0.0084	0.78
ZAR.Z07	ZAR.Z09	-0.00001688	0.0160	0.0197	0.76
ZAR.Z10	ZAR.Z15	-0.00095859	0.0276	0.0635	-0.11
ZAR.Z15	ZAR.Z20	-0.00002744	0.0635	0.0657	0.96

Table 5.1: Risk factors for which the RiskMetrics mapping is discontinuous (matrices of March 15, 1999).

Date	Risk factors					
15/3/1999	CAD.Z15	EUR.Z09	ZAR.Z02	ZAR.Z07	ZAR.Z10	ZAR.Z15
15/4/1999	CAD.Z15	DKK.Z09	ZAR.Z10			
14/5/1999	BEF.Z03	BEF.Z07	CAD.Z15	DKK.Z09	FRF.Z04	ITL.Z04
15/6/1999	BEF.Z09					
14/7/1999						
10/8/1999	FRF.Z05	FRF.Z15	GBP.Z09	GBP.Z10	ITL.Z09	JPY.Z09

Table 5.2: Risk factors for which the RiskMetrics mapping is discontinuous.

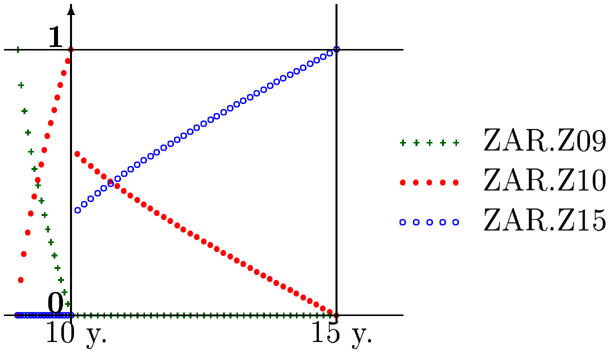


Figure 5.4: Graph of the mapping of 1 ZAR on different risk factors for terms between 9 and 15 years

estimation of v is in the convex hull of v_1 and v_2. Other mappings not having this property are possible.

5.2.4 Schaller's map

Schaller's mapping, presented in [17], is based also on the conservation of the estimated volatility. The way the risk is distributed on the two other vectors is chosen to avoid discontinuities in the parameters that can appears in the RiskMetrics mapping.

The proportion $X_1/(X_1+X_2)$ attributed to the vector v_1 varies linearly with the duration as parameter from 1 in t_1 to 0 in t_2. So we have

$$\frac{X_1}{X_1+X_2}=\frac{t_2-t}{t_2-t_1}$$

and the conservation of the risk

$$\sigma^2=|X_1v_1+X_2v_2|^2=X_1^2\sigma_1^2+2\rho\sigma_1\sigma_2X_1X_2+X_2^2\sigma_2^2.$$

This gives as solution, where $\tau=(t-t_1)/(t_2-t)$,

$$X_1=\frac{\sigma}{\sqrt{\sigma_1^2+\sigma_2^2\tau^2+2\sigma_1\sigma_2\rho\tau}}$$

and

$$X_2=\frac{\sigma}{\sqrt{\sigma_1^2/\tau^2+\sigma_2^2+2\sigma_1\sigma_2\rho/\tau}}.$$

(a) Schaller (b) Polar coordinates

Figure 5.5: Geometrical representation of the Schaller and the polar coordinates mappings

5.2.5 Polar coordinates mapping

We describe now a new mapping system. The idea is as follows. The vector is constructed in the plane of the two vectors on which the mapping is done by interpolating linearly the norm (the VaR) of the vectors and the angles between them using the term as parameter.

So we estimate the norm of v by

$$\sigma = \sigma_1 + \frac{t - t_1}{t_2 - t_1}(\sigma_2 - \sigma_1).$$

We note $\alpha = \arccos \rho$. The angle between v_1 and v is then estimated by $\beta = \frac{t-t_1}{t_2-t_1}\alpha$. So

$$X_1 = \frac{\sin(\alpha - \beta)}{\sqrt{1-\rho^2}} \frac{\sigma}{\sigma_1}$$

and

$$X_2 = \frac{\sin(\beta)}{\sqrt{1-\rho^2}} \frac{\sigma}{\sigma_2}$$

The general idea of those two last mappings is the same. The estimation of v "move" from v_1 to v_2 rotating in the plane with a length given by the interpolated norm. The type of rotation is different for the two mappings.

5.2.6 Three dimensional map

This is also a new approach. The vector is not in the plane formed by the two vectors on which the mapping is done, so we have to add a third dimension. The position of this vector is obtained by estimating the covariance (correlations) with the other two vectors. This is done by using a linear interpolation of the covariance between the vectors with the term as parameter. Then this vector is projected orthogonally (to minimize the distance (the VaR)).

The picture of the construction of X_i is given in the figure 5.6.

The estimation of the covariance between v and v_i is

$$\rho_i = 1 + \frac{t - t_i}{t_{1-i} - t_i}(\rho - 1).$$

Similarly the estimation of the norm is

$$\sigma = \sigma_1 + \frac{t - t_1}{t_2 - t_1}(\sigma_2 - \sigma_1).$$

Let $x_i = \sigma_i X_i$. We denote by y_i the length of the orthogonal projection of v on the line of v_i. So $y_i = \sigma \rho_i$. We denote by z_1 the length of the projection of the vector

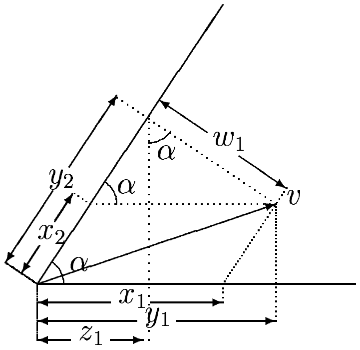


Figure 5.6: Construction used in the three dimensional mapping

of length y_2 and of direction v_2 on the line of v_1, so $z_1 = y_2\rho$. Then $y_1 - z_1 = w_1\rho$ where $w_1 = x_1\rho$. Combining all those relations, we obtain

$$X_1 = \frac{x_1}{\sigma_1} = \frac{\sigma}{\sigma_1}\left(\frac{\rho_1 - \rho_2\rho}{1-\rho^2}\right)$$

and

$$X_2 = \frac{\sigma}{\sigma_2}\left(\frac{\rho_2 - \rho_1\rho}{1-\rho^2}\right).$$

Note that the last four mappings do not conserve the present value of the securities. So the sum of the mapped positions is not equal to the real present value of the positions (see also end of section 5.3).

5.2.7 Unused interesting property

Before comparing the different maps, it worths mentioning an interesting property of all of them that seems unused. If we have the present value of the cash-flows, the actual rates are not used any more in the different mapping formulas. This feature is interesting if different types of products are priced on the same curve but with different spreads. Suppose you use the following algorithm to compute the VaR:

1. Decompose the book into equivalent cash-flows.
2. Do the mapping to the risk factors using the curves of the risk factors.
3. Compute the VaR.

Then you lose any information about the spread. An algorithm suggested by the above property is the following:

1. Decompose the book into equivalent cash-flows

2. Discount them with the correct curve (even if it is not the one used in the risk factors).
3. Do the mapping to the risk factors using the present value of the cash-flows.
4. Compute the VaR.

This approach is more coherent with the linear approximation. A cash-flow of 1 valued with a spread of s has a value of $e^{-(r+s)t} = e^{-rt}e^{-st}$. So it can be viewed as e^{-st} times the standard risk factor.

The difference between the two can be of some importance in estimating the risks. To see this we compare two situations for which the two methods give an estimated risk of 0. For the first method we take two positions with *opposite cash-flows* priced on the same curve but with a fixed spread (for example a product sold to a customer and its hedging). For the second one we take two positions with *opposite present values* also priced on the same curve with a spread. The two situations have an estimated risk of 0 with their respective VaR computation algorithm. But what happens in the two cases if the rates change? Suppose we have a nominal of 100,000,000 for the first method and a present value of the same amount for the second method (which is a larger position) with a term of one year. If the rate is 5%, the spread of 20 bps and the change of rates of 20 bps, the change of present value is 691 in the first case and 362 in the second, almost the half.

If we look at the first order approximation in the change of rates of the change of value in the two cases, for a term of 1 year, a rate of r, a spread of s and a rate movement of ϵ we have respectively

$$\left(\frac{-1}{(1+r+s)^2} + \frac{1}{(1+r)^2}\right)\epsilon \quad \text{and} \left(\frac{-1}{1+r+s} + \frac{1}{1+r}\right)\epsilon.$$

If we take now the first order approximation in the spread, we have respectively

$$2(1+r)^{-3}s\epsilon \quad \text{and} \quad (1+r)^{-2}s\epsilon$$

As $1+r$ is close to 1, it means that the risk due to the spread hidden by the method is approximatively twice bigger with the first method.

We can see the same result through a practical case. As previously, suppose that we have a product sold to a customer with a margin and hedged. The hedge is perfect in term of sensitivities (not in term of cash-flows), which means that if the cash-flow of the one year product sold is C_1, the cash-flow of the hedging instrument is the value C_2 such that

$$-\frac{C_2}{(1+r+s)^2} = -\frac{C_1}{(1+r)^2}.$$

Using the first methodology to evaluate the VaR, we obtain a present value of the cash-flow that is used in the VaR computation of

$$\frac{C_2}{(1+r+s)^2}\left(2s+\frac{s^2}{(1+r)^2}\right).$$

By using the second methodology, we have a present value of

$$\frac{C_2}{(1+r+s)^2}s.$$

Once more, the proposed improvements reduce the error by a factor 2.

We quantify this for our example of the hedging of a liability with a one year cash-flow of 100,000,000. If the sensitivity of the book is zero, the position of the book will be seen through the VaR as being long of $361,778$ in the first case and $180,717$ in the second.

Moreover if the cash-flows are used to compute the FX risk it is certainly necessary to discount them with the correct curve. If not an asset and a liability of the same value in the same currency but with an asset yielding more will create a fake positive currency exposure.

5.2.8 Mapping of mapping

We describe now, for the elementary and rates mappings, a property that we call "the mapping of a mapping is a mapping". This property is only valid for those two maps.

Suppose that we have five times $t_1 \leq t_2 \leq t \leq t_3 \leq t_4$. If we map a cash flow at t to t_2 and t_3 and then the results to t_1 and t_4, then we obtain the same results that the mapping of the cash flow directly to t_1 and t_4.

To prove this, we use the following notations: the direct mapping on t_2 and t_3 are denoted X_2 and X_3, the mapping on t_1 and t_4 by X_1 and X_4 and the composed mappings of X_i on t_1 and t_4 by $Y_{i,1}$ and $Y_{i,4}$.

For the elementary map, we have the following equations.

$$X_2 + X_3 = 1 \quad X_2 t_2 + X_3 t_3 = t$$

and

$$Y_{i,1} + Y_{i,4} = X_i \quad Y_{i,1} t_1 + Y_{i,4} t_4 = X_i t_i.$$

Combining those equations, we have

$$(Y_{2,1} + Y_{3,1}) + (Y_{2,4} + Y_{3,4}) = X_2 + X_3 = 1$$

and

$$(Y_{2,1} + Y_{3,1})\, t_1 + (Y_{2,4} + Y_{3,4})\, t_2 = X_2 t_2 + X_3 t_3 = t.$$

This proves that $X_j = Y_{2,j} + Y_{3,j}$, as announced.

On the other hand, for the rates map, we have

$$X_2 = \frac{t}{t_2}\frac{t_3 - t}{t_3 - t_2} \quad X_3 = \frac{t}{t_3}\frac{t - t_2}{t_3 - t_2}$$

and

$$Y_{i,1} = \frac{t_i}{t_1}\frac{t_4 - t_i}{t_4 - t_1} \quad Y_{i,4} = \frac{t_i}{t_4}\frac{t_i - t_1}{t_4 - t_1}.$$

Combining those equations, we have

$$\begin{aligned} Y_{2,1} + Y_{3,1} &= \frac{1}{t_1}\frac{1}{t_4 - t_1}\left(t_2(t_4 - t_2)X_2 + t_3(t_4 - t_3)X_3\right) \\ &= \frac{t}{t_1}\frac{t_4 - t}{t_4 - t_1}. \end{aligned}$$

This proves that $X_j = Y_{2,j} + Y_{3,j}$, as announced.

5.3 Comparisons

For the comparison between the mappings, we use the following technique. We hedge a cash-flow of present value 1 by mapping this position to the preceding and the following terms. The VaR of this hedged position is used as an accuracy measure. It represents the residual VaR due to the use of the mapped cash-flow instead of the true cash-flow.

As all the data are known we have the precise value of the residual risk, i.e. the error in the computation due to the mapping.

We measure the error by the VaR of the difference. The measure of the error by the VaR of the difference is a better one than the difference of the VaR's. We are of course interested in having a VaR as close as possible of the real one. But for a portfolio of cash flows (p), if one adds a new cash flow (v), the error of the total will be small if the distance between the true position of the new cash flow and the estimated one (v_1) is small. Adding a cash flow with the same norm that the true one (v_2) but at a very distant position will give a larger error on the estimate of the total of the VaR (see Figure 5.7).

We did this comparison for zero-coupon government rates. For each maturity (3, 4, 5, 7, 9 years and 10, 15, 20 years when possible), we have to compute for the different mappings the residual risk of a position which is short 1 million on that maturity and long the mapping of this same million on the two maturities surrounding it. The figures obtained represent the risk induced by the *mapping* on this particular position.

We did this computation with the matrix published by RiskMetrics [16] (for March 15, April 15, May 14, June 15, July 14 and August 10, 1999). We compared

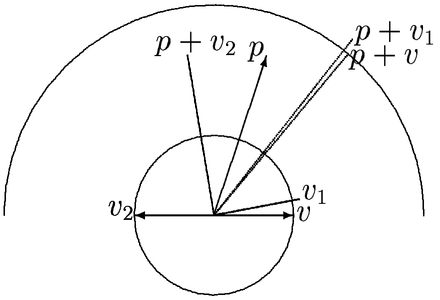


Figure 5.7: Different estimates of the VaR of a portfolio

	Improvements				
Dates	*RiskMetrics*	Schaller	Rates	Polar	3 D
15/3/1999	38	45	62	64	65
15/4/1999	38	39	61	54	56
14/5/1999	41	41	64	63	65
15/6/1999	37	48	72	64	67
14/7/1999	44	46	64	62	61
10/8/1999	42	58	70	70	68

Table 5.3: Number of improvements for the different mappings with respect to the elementary mapping. Number of risk factors = 119.

the results of the different mappings with the elementary one, that we use as a "benchmark" (we count the number of risk factors for which the residual risk is less than the one of the elementary map). The results are in Table 5.3.

The detailed results for three of the main currencies (DEM, JPY, USD) and matrix of July 14, 1999 are given in Tables 5.4, 5.5 and 5.6.

We see that globally the best maps seems to be the elementary, the rates, the polar coordinates and the three dimensional one. For the main currencies the results are better for the polar coordinates, the rates and the three dimensional one (16/23). Moreover the improvements can be substantial, with maximum of improvement of 94 % for the rates map, and of 75 % for the three dimensional one for the 3-year USD, and 76 % for the rates map for the 15-year USD.

It is also worth noticing that the quality of the maps differs across the different currencies. For the same term (15-year), the figures are around 1000 for the DEM, 6000 for the JPY and 500 for the USD.

As said before, two of the mappings conserve the cash-flows present value, the others do not. In Table 5.7, we give the decomposition for the USD for a cash-flow

Induced risk

	Elem.	*RiskMetrics*	Schaller	Rates	Polar	3 dim.
Govt 3Y	667	724	720	576	750	684
Govt 4Y	811	794	782	847	741	778
Govt 5Y	1492	1547	1492	1373	1385	1385
Govt 7Y	1782	1795	1838	1994	2047	1987
Govt 9Y	1213	1211	1203	1211	1194	1206
Govt 10Y	1384	1382	1387	1444	1444	1442
Govt 15Y	1342	1487	1417	726	880	750
Govt 20Y	7022	7376	7177	5676	5678	5521

Differences (with respect to elementary mapping)

	RiskMetrics		Schaller		Rates		Polar		3 dim.	
	Abs.	%	Abs.	%	Abs.	%	Abs.	%	Abs.	%
Govt 3Y	-57	-9	-53	-8	91	14	-84	-13	-17	-3
Govt 4Y	17	2	29	4	-36	-4	70	9	33	4
Govt 5Y	-55	-4	1	0	120	8	108	7	107	7
Govt 7Y	-13	-1	-56	-3	-212	-12	-265	-15	-206	-12
Govt 9Y	2	0	10	1	2	0	19	2	7	1
Govt 10Y	2	0	-3	-0	-61	-4	-60	-4	-58	-4
Govt 15Y	-146	-11	-76	-6	615	46	461	34	591	44
Govt 20Y	-354	-5	-155	-2	1346	19	1344	19	1501	21
Improv.	3 / 8		3 / 8		5 / 8		5 / 8		5 / 8	

Table 5.4: The risk induced by a position of 1 million long the risk factor and short the million mapped on the adjacent risk factors, e.g. 3 year mapped on the 2 and 4 years. The value of the risk is given in the first part of the table and the improvement with respect to the elementary map in the second part. Figures for the DEM with the matrices of July 14, 1999.

Induced risk

	Elem.	*RiskMetrics*	Schaller	Rates	Polar	3 dim.
Govt 3Y	362	359	357	421	295	304
Govt 4Y	346	353	349	334	335	332
Govt 5Y	588	600	586	546	526	529
Govt 7Y	1550	1522	1496	1444	1420	1482
Govt 9Y	1846	2003	1856	1716	1777	1772
Govt 10Y	2640	2748	2709	2641	2703	2652
Govt 15Y	6385	6492	6488	6242	6599	6474

Differences (with respect to elementary mapping)

	RiskMetrics		Schaller		Rates		Polar		3 dim.	
	Abs.	%	Abs.	%	Abs.	%	Abs.	%	Abs.	%
Govt 3Y	3	1	5	1	-59	-16	67	18	57	16
Govt 4Y	-7	-2	-3	-1	12	3	11	3	14	4
Govt 5Y	-12	-2	3	0	42	7	62	11	59	10
Govt 7Y	28	2	54	3	106	7	130	8	68	4
Govt 9Y	-157	-9	-9	-1	131	7	69	4	74	4
Govt 10Y	-108	-4	-70	-3	-1	-0	-64	-2	-13	-0
Govt 15Y	-107	-2	-103	-2	143	2	-214	-3	-89	-1
Improv.	2 / 7		3 / 7		5 / 7		5 / 7		5 / 7	

Table 5.5: The risk induced by a position of 1 million long the risk factor and short the million mapped on the adjacent risk factors. The value of the risk is given in the first part of the table and the improvement with respect to the elementary map in the second part. Figures for the JPY with the matrices of July 14, 1999.

Induced risk

	Elem.	*RiskMetrics*	Schaller	Rates	Polar	3 dim.
Govt 3Y	112	122	118	6	45	28
Govt 4Y	160	173	167	91	106	95
Govt 5Y	383	375	376	419	409	415
Govt 7Y	312	318	312	250	250	250
Govt 9Y	188	196	191	159	162	158
Govt 10Y	486	483	484	489	502	504
Govt 15Y	572	558	549	136	361	399
Govt 20Y	1639	1778	1524	460	871	1043

Differences (with respect to elementary mapping)

	RiskMetrics		Schaller		Rates		Polar		3 dim.	
	Abs.	%	Abs.	%	Abs.	%	Abs.	%	Abs.	%
Govt 3Y	-9	-8	-5	-5	106	94	67	60	85	75
Govt 4Y	-13	-8	-7	-4	69	43	54	34	65	41
Govt 5Y	7	2	7	2	-37	-10	-27	-7	-32	-8
Govt 7Y	-6	-2	-0	-0	62	20	61	20	61	20
Govt 9Y	-7	-4	-3	-1	30	16	27	14	30	16
Govt 10Y	3	1	3	1	-3	-1	-16	-3	-18	-4
Govt 15Y	14	3	23	4	436	76	210	37	173	30
Govt 20Y	-139	-8	115	7	1179	72	768	47	596	36
Improv.	3 / 8		4 / 8		6 / 8		6 / 8		6 / 8	

Table 5.6: The risk induced by a position of 1 million long the risk factor and short the million mapped on the adjacent risk factors. The value of the risk is given in the first part of the table and the improvement with respect to the elementary map in the second part. Figures for the USD with the matrices of July 14, 1999.

Term	Elementary			*RiskMetrics*			Schaller		
	X_1	X_2	X_1+X_2	X_1	X_2	X_1+X_2	X_1	X_2	X_1+X_2
Govt 3Y	500	500	1000	492	508	1000	503	503	1005
Govt 4Y	500	500	1000	490	510	1000	503	503	1006
Govt 5Y	667	333	1000	661	339	1000	669	335	1004
Govt 7Y	500	500	1000	492	508	1000	502	502	1004
Govt 9Y	333	667	1000	327	673	1000	334	668	1002
Govt 10Y	833	167	1000	832	168	1000	834	167	1001
Govt 15Y	500	500	1000	495	505	1000	501	501	1003
Govt 20Y	667	333	1000	630	370	1000	679	340	1019

Term	Rates			Polar coordinates			3 dim.		
	X_1	X_2	X_1+X_2	X_1	X_2	X_1+X_2	X_1	X_2	X_1+X_2
Govt 3Y	750	375	1125	778	372	1150	774	369	1143
Govt 4Y	667	400	1067	702	392	1094	698	390	1087
Govt 5Y	833	238	1071	846	236	1082	844	235	1079
Govt 7Y	700	389	1089	694	394	1088	691	392	1083
Govt 9Y	429	600	1029	429	602	1031	428	600	1028
Govt 10Y	926	111	1037	911	117	1028	911	117	1028
Govt 15Y	750	375	1125	686	395	1081	683	394	1078
Govt 20Y	889	222	1111	814	254	1069	803	249	1052

Table 5.7: Present value of the mapped cash flows for the different maps with an unmapped cash flow of present value 1000. Figures for the USD with the matrices of July 14, 1999

with present value 1000 across maps. The table shows X_1, X_2 and $X_1 + X_2$.

5.4 Conclusions

We now summarize the characteristics of the various cash flow maps.

From a numerical point of view, the elementary, the rates and the three dimensional maps are the fastest (they use only simple arithmetic operations). The RiskMetrics and the Schaller one use one (or two) square roots and the polar coordinates one use trigonometric and inverse trigonometric functions.

From the point of view of the data used for the computation, no external information is needed for the elementary and the rates maps. For all the others, the volatilities of the standard terms and their correlations are used. It is worth to note that once the present value of the cash flow that we want to map is obtained, the rates are not used any more.

From a financial point of view, the elementary mapping and the RiskMetrics one conserve the present value of the securities. So the figures obtained in the analysis of the VaR are easier to present.

The rates mapping is coherent with linear interpolation of continuously compounded rates and the elementary mapping is coherent with constant forward rates between two standard terms. So if VaR is used in parallel with some other methodologies for marked to market calculation and risk measures, it is better to use a mapping with the same methodology.

To summarize, the various methods could be ranked as follows, with the best at the top.

1. Rates mapping
2. Elementary mapping
3. Three dimensional mapping
4. Polar coordinates mapping
5. Schaller mapping
6. RiskMetrics mapping

This order is of course very subjective. The map using the rates has a lot of advantages. Its algorithm is fast and the quality of the result is very good. Moreover it is compatible with the linear interpolation of the rates between standard terms. The elementary one is similar except that the financial underlying hypothesis is probably less used. A part of the (my) interest for the three dimensional map is probably due to its nice geometrical construction.

Chapter 6

Variance-covariance matrix

6.1 Computing the matrix

To compute the covariance matrix, we suppose that we have series of the prices underlying the risk factors long enough. We denote them $r_{i,t}$ where i is the number of the risk factor and t is the date of the value. As we are interested by the profit, we need the series of the changes $c_{i,t} = (r_{i,t} - r_{i,t-1})/r_{i,t-1}$. Let

$$R = \operatorname{diag}(\beta_1, \ldots, \beta_n) \begin{pmatrix} c_{1,T_2} & \cdots & c_{N,T_2} \\ \vdots & \ddots & \vdots \\ & \cdots & c_{N,t} \\ \vdots & \ddots & \vdots \\ c_{1,T_1+1} & \cdots & c_{N,T_1+1} \end{pmatrix}$$

where T_2 is the last data used for the computation and T_1 is the most ancient data used. Usually T_1 is one year before T_2 and T_2 is the last data available and are expressed in days. The numbers $\beta_1, \ldots, \beta_n$ are weighting scheme ($\sum \beta_i = 1$). On popular scheme is the EWMA (Exponentially Weighted Moving Average). A parameter λ is chosen ($0 < \lambda < 1$). The weights are defined by

$$\beta_i = \frac{1 - \lambda}{1 - \lambda^{T_2 - T_1}} \lambda^{T_2 - t}.$$

If the parameter is 1, it means that all the element in the series have the same weight, if it is smaller that 1, more weight is put on the last occurrences. The usual values are 0.94 for a daily horizon and 0.97 for a monthly horizon (see [15] for more details on the choice of those numbers).

The covariance matrix is estimated by

$$V = R^T R$$

Other ways to estimate the matrix are possible.

Moving average (unweighted) The previous method with $\lambda = 1$.

ARCH or GARCH approach

Log-return Instead of using the return one can use the log-return $c_{i,t} = \ln(r_{i,t}/r_{i,t-1})$. This approach is consistent with the log-normal hypothesis for the risk factors (see Section 3.3).

Note that the numbers $r_{i,t}$ are the values of the risk factors for currencies, equities and commodities. For the interest rate, it is the value of the zero coupon bonds, so

$$c_t = \frac{\exp(-s_t T) - \exp(-s_{t-1} T)}{\exp(-s_{t-1} T)}.$$

where s_t is the interest rate in t for the maturity T. The interest is quoted with a ACT/365 continuous compounding.

If the standard tenor for interest rate are in months, the numbers of days can be between 28 and 31 without taking the week-ends into consideration and between 26 and 33 with the week-ends. This can create a change of 1 to 2% due to the way we measure the months. To avoid this extra volatility, we propose to fix the length of the month at 30 days and to compute the discount factor by interpolating between the market quoted prices. This interpolation can be done by linearly interpolating the zero coupon continuously compounded rate. This is coherent with the rates mapping described in Chapter 5. For the vertices defined in term of year, it is also better to avoid to take into account the week-ends as it can change the year from 363 days to 368. The only period change left is the leap years, it means a 1 day change twice every 4 years. This slight variation can be ignored.

6.2 Not computing the matrix

The explicit computation of the matrix can be avoided. In certain circumstances not computing it explicitly can be numerically more efficient.

The computation of the VaR is done through the matrix multiplication $p^T V p$. If we use the notation of the previous section we have

$$p^T V p = p^T (R^T R) p = (Rp)^T Rp.$$

So if we have R we do not need to compute V explicitly.

We first look at the dimensions of the matrix and the number of operations required. Let N be the number of risk factors and T the size of the series of risk factors data ($T_2 - T_1$ in the previous section).

The dimensions of V is $N \times N$ and the dimension of R is $T \times N$. If the number of risk factors is larger than the size of the data series, it requires less space to store the risk factors series than the covariance matrix. A smart storage of the matrix could save almost half of the space (the matrix is symmetric) but if there are a lot of risk factors this is only a small improvement. Note also that when the number of risk factors is larger than the size of the data series, the covariance matrix is singular.

For the number of operations, we compute only the number of multiplication (additions are almost free in computer world). To compute $V = R^T R$, we need $N \times N \times T$ multiplications. The computation of Vp take $N \times N$ operations and the one of $p^T(Vp)$ takes another N. On the other side, Rp takes $T \times N$ operation and $(Rp)^t(Rp)$ takes another T.

So even if N and T are the same we have on one side $N^3 + N^2 + N$ and on the other only $N^2 + N$. When $N = T = 250$ (this is 1 year of data and about 8 currencies with their interest rates), the numbers of operations are 15,687,750 and 62,750. This is about 250 times more.

So when you have the data series and not the covariance matrix, *do not compute* the covariance matrix. Use directly the historical changes c_i applied to the current position p. This correspond to the computation of the historical simulations and the computation of the volatility of the position from there.

Note also that the decomposition $V = R^T R$ can be used to simulate the normal distribution (for Monte Carlo simulation for example). Usually this is done by generating independent normals of means 0 and variance 1 and multiplying the result by a matrix C such that $V = C^T C$. This last matrix is not unique but is usually the Cholesky decomposition of V. But the Cholesky decomposition is defined only if V is positive definite (i.e. when V is not singular) (see [14, Section 2.2, p. 19]). The use of R avoids the computation of V and works in all cases.

6.3 Change of base currency

As said in Section 2.4 on coherence, it is not possible to have a currency and its inverse to be both normal (except if the exchange rate is constant). The change of base currency can be done only by changing somehow the hypothesis. For this section we suppose that the logarithmic return (not the usual return) of the FX rates are normally distributed. So we will be interested by the series $c_{i,t} = \ln(x_{i,t}/x_{i,t-1})$ where $x_{i,t}$ denotes the exchange rate of the i-th currency with respect to the initial base currency.

The risk factors are ordered with the currencies first (1 to N_c). The new base currency is the k-th. For simplicity purposes we decide to numbered the current base currency with 0 and we extend the original matrix $v_{i,j}$ with a 0-th line and column. They are made of 0 except.

The series of exchange rate for the currencies have to be changed to the new base. The new exchange rate is $x_{i,t}/x_{k,t}$. The return in the new base is

$$d_{i,t} = \ln\left(\frac{x_{i,t}/x_{k,t}}{x_{i,t-1}/x_{k,t-1}}\right) = \ln(x_{i,t}/x_{i,t-1}) - \ln(x_{k,t}/x_{k,t-1}) = c_{i,t} - c_{k,t}.$$

The new covariance matrix is

$$\Sigma = S^T S$$

with S equal to R with $c_{i,t}$ replaced by $d_{i,t}$. We can rewrite S as $R - X$ with

$$X = \begin{pmatrix} c_{k,t_2} \\ \vdots \\ \lambda^{T_2-t} c_{k,t} \\ \vdots \\ \lambda^{T_2-T_1-1} c_{k,T_1+1} \end{pmatrix} (1, \cdots, 1, 0, \cdots, 0).$$

Then we have that $\Sigma = R^T R - X^T R - R^T X + X^T X$.

It is easy to verify that

$$X^T R = \begin{pmatrix} v_{k,0} & \cdots & v_{k,N} \\ \vdots & \ddots & \vdots \\ v_{k,0} & \cdots & v_{k,N} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \qquad X^T X = v_{k,k} \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$R^T X = \begin{pmatrix} v_{0,k} & \cdots & v_{0,k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ v_{N,k} & \cdots & v_{N,k} & 0 & \cdots & 0 \end{pmatrix}$$

where the matrix is $N \times N$ and its upper-left part is $N_c \times N_c$.

This leads to the following rebasing formula:

Two currencies $(0 \le i, j \le N_c)$ $\sigma_{i,j} = v_{i,j} - v_{k,j} - v_{i,k} + v_{k,k}$

One currency $(0 \le i \le N_c < j \le N)$ $\sigma_{i,j} = v_{i,j} - v_{k,j}$

No currency $(N_c < i, j \le N)$ $\sigma_{i,j} = v_{i,j}$

The VaR of a portfolio with this way to change the base currency can be computed easily. The VaR in the new base currency is equal to the VaR in the original currency of the position plus a short position in the new base currency equal to the total value of the portfolio.

In particular if the total value of the portfolio is 0, the change of base currency doesn't change the risk of the portfolio.

Example: We take the first 7 risk factors of our example (see Appendix A) to illustrate this rebasing. The original base currency is USD and the new one is EUR.

		(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1)	Gold	69.9732	26.6462	27.5898	0	0.013	-0.0284	-0.1822
(2)	DKK	26.6462	54.8785	54.9751	0	0.0574	0.17	0.6454
(3)	EUR	27.5898	54.9751	56.19	0	0.0604	0.1657	0.5986
(4)	USD	0	0	0	0	0	0	0
(5)	DEM govt-1m	0.013	0.0574	0.0604	0	0.001	0.0012	0.0056
(6)	DEM govt-3m	-0.0284	0.17	0.1657	0	0.0012	0.0072	0.0222
(7)	DEM govt-12m	-0.1822	0.6454	0.5986	0	0.0056	0.0222	0.1318

Table 6.1: Original variance-covariance matrix with USD as base currency

		(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1)	Gold	69.9732	-0.9436	0	-27.5898	0.013	-0.0284	-0.1822
(2)	DKK	-0.9436	1.1184	0	1.215	-0.003	0.0044	0.0468
(3)	EUR	0	0	0	0	0	0	0
(4)	USD	-27.5898	1.215	0	56.19	-0.0604	-0.1657	-0.5986
(5)	DEM govt-1m	0.013	-0.003	0	-0.0604	0.001	0.0012	0.0056
(6)	DEM govt-3m	-0.0284	0.0044	0	-0.1657	0.0012	0.0072	0.0222
(7)	DEM govt-12m	-0.1822	0.0468	0	-0.5986	0.0056	0.0222	0.1318

Table 6.2: Variance-covariance matrix rebased with EUR as base currency

6.4 Basket of assets

A basket of assets is a unit composed of a fixed number of units of several component assets. The most common case of basket is the equity index. They are usually made of a given number of a selection of shares.

The other type of basket, and the one on which we will put emphasis, is the currency basket. A currency basket is a unit composed of a fixed number of units of several component currencies. A typical example is the SDR[1] used by several

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[1] SDR: special drawing right, currently composed of USD 0.577, EUR 0.426, GBP 0.0984 and JPY 21.

international institutions. The basket is defined as a fixed number of units and not a fixed weight. With the evolution of the exchange rates the weights of the different currencies in the basket change.

Due to that problem, for the VaR computation the historical time series of the basket exchange rates cannot be used in the way that it is used for a true currency. The computation of the VaR is based on the current value (weight) in the different currencies and the historical time series on a number of units. Those two views cannot be used simultaneously whatever VaR method is used (parametric or historical).

If we look more specifically at the normal parametric method for VaR, one of its founding hypotheses is that the (log) returns of currencies are normal. But it is impossible to have at the same time the returns of series which are normal and the returns of a linear combination of those series which are also normal (the ratio of a linear combination is not a linear combination of the ratios). The use of both the components and the basket as risk factors is incoherent with one of the hypotheses of the model.

One can accept this as part of the method and use historical series to compute the variance-covariance matrix. But then if one owns the exact number of units in each of the underlying currencies, the VaR will not be zero. Or, in other words, the VaR will not be zero even if the risk is null. Numerical examples of the size of the residual VaR are given at the end of Section 6.4.1.

Alternatively, one can try to twist the data slightly to solve the problem and obtain a "clean" risk number. The goal is to obtain no returns if one owns the exact number of units in each of the underlying currencies. A way to do this is explained in Section 6.4.2.

6.4.1 Why it does not work

The basket is composed of n currencies (four for the SDR) with number of units α_i ($i = 1, \ldots, n$). Let $x_{i,t}$ ($T_0 \leq t \leq T$) be the exchange rates of the different currencies with respect to a (arbitrarily selected) base currency. The exchange rate of the basket with respect to that base currency is

$$X_t = \sum_{i=1}^{n} \alpha_i x_{i,t}.$$

A currency neutral position is to hold α_i units of the underlying currencies and to be short of one basket. So the vector

$$(\alpha_1 x_{1,T}, \ldots, \alpha_n x_{n,T}, -X_T)$$

representing this position converted in the base currency, should have no risk.

The relative return of the currencies and the basket with respect to the base currency are

$$c_{i,t} = \frac{x_{i,t} - x_{i,t-1}}{x_{i,t-1}} \qquad C_t = \frac{X_t - X_{t-1}}{X_{t-1}}.$$

The log returns are given by

$$d_{i,t} = \ln(x_{i,t}/x_{i,t-1}) \qquad D_t = \ln(X_t/X_{t-1}).$$

Let R denote the matrix of the returns (weighted or unweighted) used to compute the variance-covariance matrix. In our notations

$$R = \operatorname{diag}(\beta_T, \ldots, \beta_{T_0+1}) \begin{pmatrix} c_{i,T} & \cdots & c_{n,T} & C_T \\ \vdots & \ddots & \vdots & \vdots \\ c_{i,T_0+1} & \cdots & c_{n,T_0+1} & C_{T_0+1} \end{pmatrix}$$

where β is a weighting scheme (see Section 6.1).

The VaR of a position p is given, up to a multiplicative factor depending of the probability level of the VaR, by

$$\mathrm{VaR}(p)^2 = p^T(R^T R)p = (Rp)^T Rp.$$

So to obtain a VaR of zero it is necessary that $Rp = 0$ for the position $p = (\alpha_1 x_{1,T}, \ldots, \alpha_n x_{n,T}, -X_T)$. In the relative return framework[2], this is the case only if for all t

$$\sum_{i=1}^{n} \alpha_i x_{i,T} c_{i,t} - (\sum_{i=1}^{n} \alpha_i x_{i,T}) \frac{\sum \alpha_i c_{i,t} x_{i,t-1}}{\sum \alpha_i x_{i,t-1}} = 0.$$

This would be the case if $x_{i,t-1} = x_{i,T}$, but this is not the case in general.

As the daily returns of the position are not all zero, the implication is that the VaR computed using the historical simulation method would also be non-null.

Note that all the above arguments are valid for an arbitrary base currency. In particular they are valid when the base currency is the currency basket it self. In the following example the SDR is the base currency.

We have computed the VaR of a portfolio made of the component currencies with the exact number of units to have SDR 1 billion (USD 577 million, EUR 426 million, GBP 98.4 million and JPY 21 billion). This was done using five years of data ending on 14 June 2002 (for parametric VaR, the parameters are computed using EWMA and a decay factor of 0.97). The residual VaR for a 97.5% probability level and a 1-day horizon is

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[2] In the log return framework, the formulas are a little bit longer but the results are similar.

Base currency: SDR

Currency	Parametric VaR (in SDR)	Historical VaR (in SDR)
EUR	1,740,279	2,739,039
GBP	403,523	845,471
JPY	1,124,301	1,541,932
USD	1,991,825	2,647,140
Total	96,989	1,061,558

Note that due to the exponential weighting, the parametric VaR is in practice computed with less than one year of data. We have added the historical VaR to emphasized the fact that the problem is not only coming from the normal hypothesis but from the change of weight of the component through time.

The covariance matrix was given by

Currency	Volatilities	GBP	EUR	JPY	USD
GBP	0.1835	1	-0.0103	-0.3674	0.0155
EUR	0.2847	-0.0103	1	0.0361	-0.8604
JPY	0.4393	-0.3674	0.0361	1	-0.5062
USD	0.2282	0.0155	-0.8604	-0.5067	1

6.4.2 How to make it work

Instead of using historical time series of the basket we can rewrite the history using the *current weight* (not the number of units) for the different currencies.

Let w_i be the current weights of the different components:

$$\alpha_i x_{i,T} = w_i X_T.$$

To obtain a clean risk figure, we need a time series that gives the same results with a position in the basket or its decomposition into the component currencies. So for any position $(p_1, \ldots, p_n, p^*)$ with $(p_1, \ldots, p_n)$ the positions in the component currencies[3] and p^* the position in the basket, we should find that in term of risk

$$(p_1, \ldots, p_n, p^*) \equiv (p_1 + \alpha_1 p^*, \ldots, p_n + \alpha_n p^*, 0).$$

For all t the daily returns of the two positions should be equal. For the relative return this implies that for the modified return $\tilde{C}_t$ constructed on the modified time series K_t ($\tilde{C}_t = (K_t - K_{t-1})/K_{t-1}$)

$$\sum_{i=1}^{n} p_i x_{i,T} c_{i,t} + p^* X_T C_t = \sum_{i=1}^{n} (p_i + \alpha_i p^*) x_{i,T} c_{i,t}.$$

---

[3] Some of the units α_i can be zero, so the description we give takes also into consideration the non-component currencies and the non-currency risks

This is equivalent to

$$\frac{K_t - K_{t-1}}{K_{t-1}} = \sum w_i c_{i,t}.$$

To obtain a clean risk figure, we have to take

$$K_T = X_T \quad \text{and} \quad K_{t-1} = \frac{K_t}{1 + \sum w_i c_{i,t}}.$$

For the log return the time series is

$$L_T = X_T \quad \text{and} \quad L_{t-1} = \frac{L_t}{\exp(\sum w_i d_{i,t})}.$$

These series can be used for normal VaR but also for historical VaR. Note that those series have to be recreated every day. They are not cumulative but every day the full history has to be rewritten.

The series to be used depends of the way covariance matrix and historical return are computed.

For the computation of the normal VaR, it depends on the type of return used to compute the covariance matrix.

Parametric VaR

Covariance matrix	Asset return	Series
Relative return	normal	K
Log return	normal	L

For the historical VaR, the series to be used depends on the type of return computed and on the way the historical information is used to compute the profit (see for example [14, Section 3.1] and [9, Section 14.8]).

Historical VaR

Historical return	Profit computation	Series
Log return	$\text{PnL} = P_0(e^r - 1)$	K
Relative return	$\text{PnL} = P_0 r$	K
Log return	$\text{PnL} = P_0 r$	L

A completely coherent choice for the covariance matrix computation, asset return hypotheses in parametric VaR and historical VaR is to choose the relative return, a normal asset return hypothesis and a profit formula of the form $\text{PnL} = P_0 r$ for the historical simulation.

But another set of hypotheses is often chosen. It consists of the log return for the covariance matrix computation, the normal return hypothesis, and a profit formula of the form $\text{PnL} = P_0(e^r - 1)$ for the historical profit series. The incoherence here is that the assets are suppose to be log-normal to compute the parameters and normal to compute the parametric VaR. The historical profit computation is coherent with the returns computation. To compute the VaR with SDR as base currency with those hypotheses, one has to use the "L" series for parametric VaR and the "K" series for historical VaR. Even if the choice seems incoherent it is the one chosen by several implementations.

To the first order the normal and the log-normal hypotheses are equivalent. So if the normal hypothesis is associated to a "delta" approach (first-order approximation of the positions), there is no real incoherence.

We have computed the VaR with the example of the previous section. The log-normal rescaling is used for the parametric VaR and the relative return for the historical VaR as described in the previous paragraph.

Base currency: SDR

Currency	Parametric VaR (in SDR)	Historical VaR (in SDR)
EUR	1,720,794	2,425,955
GBP	403,301	830,502
JPY	1,120,337	1,653,650
USD	2,022,451	2,689,322
Total	683	602

The small residual VaR comes from rounding errors.

Note: Suppose we compute the normal VaR with the log return for the covariance matrix. If a portfolio does not include the basket and its total value is zero, the change of time series has no impact on its VaR even if the basket is used as base currency. To show this, let $p = (p_1, \dots, p_n)$ be a portfolio with $\sum p_i = 0$. The daily historical returns of the position are

$$\begin{aligned}
\sum_{i=1}^{n} p_i d_{i,t} &= \sum_{i=1}^{n} p_i \ln\left(\frac{x_{i,t}}{x_{i,t-1}}\right) \\
&= \ln\left(\prod_{i=1}^{n}\left(\frac{x_{i,t}}{x_{i,t-1}}\right)^{p_i}\right) \\
&= \ln\left(\left(\prod_{i=1}^{n-1}\left(\frac{x_{i,t}}{x_{i,t-1}}\right)^{\sum_{j=1}^{i} p_j}\left(\frac{x_{i+1,t}}{x_{i+1,t-1}}\right)^{-\sum_{j=1}^{i} p_j}\right)\left(\frac{x_{n,t}}{x_{n,t-1}}\right)^{\sum_{j=1}^{n} p_j}\right) \\
&= \ln\left(\prod_{i=1}^{n-1}\left(\frac{x_{i,t}/x_{i+1,t}}{x_{i,t-1}/x_{i+1,t-1}}\right)^{\sum_{j=1}^{i} p_j}\right).
\end{aligned}$$

This value depends only of the ratios $x_{i,t}/x_{i+1,t}$ and then does not depend on the base currency and its time series. So a trading book that starts with a total value of zero and that borrows to invest is not affected at all by the change of the base currency or by the "adjustment" of the basket time series.

If we use the relative return, a change of base currency is not coherent (a series and the series of the inverse of its elements cannot be normal simultaneously). The computations in the different base currencies and series will be slightly different (the same currency loss will be translated in a different way in the base currency).

In any case, whatever the value of the portfolio is, if there is no basket in the portfolio and the the basket is not used as base currency, the VaR of the portfolio is not affected.

Note: Using the log return approach and the "L" series for computing the VaR of a portfolio with SDR as base currency is equivalent to computing the VaR of an "adjusted" portfolio with any other currency as base currency. The adjustment is a short position with a value equal to the total value of the portfolio and a weight in each currency equal to the current weight of the SDR.

This can be seen easily by using the formula of the change of base for the exchange rates and rewriting the historical return matrix R with those rates.

6.5 Prices and rates volatilities

The VaR can be estimated by multiplying the rate volatility by the sensitivities or the price volatility by the cash-flows.

The rates and the prices can not be normal together, so one hypothesis excludes the other. Nevertheless we try to estimate the difference of volatility between the two using first order approximation.

First we estimate the relative change of price of a zero coupon with term t and cash-flow 1 due to a change of rate from s to $s + \Delta s$. The exact value is

$$P(s + \Delta s) = P(s) e^{-\Delta s t}.$$

To the first order, the change is

$$\begin{aligned} \Delta P &= P(s + \Delta s) - P(s) \simeq P'(s) \Delta s \\ &\simeq -s P(s) \Delta s \end{aligned}$$

and thus

$$\frac{\Delta P}{P} \simeq -ts \frac{\Delta s}{s}.$$

The rate volatility is

$$\sigma_s = \sqrt{(1 - \lambda) \sum_{i=1}^{N} \lambda^{i-1} (r_i - \bar{r})^2}$$

where s_i is the rate observed in i, $r_i = (s_i - s_{i-1})/s_{i-1}$ and $\bar{r}$ is the average of the r_i. The price volatility is

$$\sigma_P = \sqrt{(1-\lambda)\sum_{i=1}^{N} \lambda^{i-1}(R_i - \bar{R})^2}$$

where P_i is the price observed in i, $R_i = (P_i - P_{i-1})/P_{i-1}$ and $\bar{R}$ is the average of the R_i (λ is the decay factor, often 0.94).

Using the approximation of the first paragraph, we can write

$$\begin{aligned} \sigma_P &\simeq \sqrt{(1-\lambda)\sum_{i=1}^{N} \lambda^{i-1}\left(-ts_i r_i - \frac{1}{N}\sum_{j=1}^{N} -ts_j r_j\right)^2} \\ &\simeq \sqrt{(1-\lambda)\sum_{i=1}^{N} \lambda^{i-1}\left(-ts_i r_i + t\tilde{s}\bar{r}\right)^2} \\ &\simeq t\hat{s}\sigma_s \end{aligned}$$

where $\hat{s}$ is a "special" average of the rates s_i.

The use of the rates volatility instead of the price volatilities in the computation of Value at Risk, is to suppose that the rate change follow better a normal law than those of the prices. The difference at the level of the computation of the VaR is the following. For a position P at t years, in the first case, the distribution of ΔP will be

$$P\sigma_P\mathcal{N},$$

and in the second case, it will be

$$-tP\sigma_s\mathcal{N}$$

where s is the present rate. The VaR is given in the first case by

$$P\sigma_P \simeq t\hat{s}P\sigma_t$$

and in the second by

$$tsP\sigma_t.$$

The difference is the difference between s and $\hat{s}$. It can go up to 5% of the value.

Chapter 7

Related figures

7.1 Marginal VaR

The *marginal VaR* of a portfolio part is simply the extra VaR that results from the incorporation of this part to the portfolio. The marginal VaR is denoted by $\mathrm{VaRM}(p, P)$ where p is a part of portfolio P. It's value is

$$\mathrm{VaRM}(p, P) = \mathrm{VaR}(P) - \mathrm{VaR}(P - p).$$

It is the quantity by which the VaR decreases if the position is removed.

This value can be positive or negative. In particular if a portfolio part hedges some of the risk of the portfolio, its marginal VaR is negative.

Note that the marginal VaR is not an additive decomposition of VaR.

7.2 Incremental VaR

The *incremental VaR* of a position in a portfolio is the first order approximation of the extra VaR that results from adding one unit of position to the portfolio. Let p be the position and P the portfolio. Denote by $f(\epsilon) = \mathrm{VaR}(P + \epsilon p / \mathrm{PV}(p))$ where $\mathrm{PV}(p)$ is the present value of p. The incremental VaR of p in P is

$$\mathrm{VaRI}(p, P) = f'(0).$$

The incremental VaR can be computed analytically. Let $\bar{p}$ denote the position rescale to one unit ($\bar{p} = p / \mathrm{PV}(p)$). The VaR is ($V$ is the variance-covariance matrix rescaled to take the probability level and the time horizon into account ; see Chapter 3).

$$\mathrm{VaR}(P + \epsilon\bar{p}) = \sqrt{(P + \epsilon\bar{p})^T V (P + \epsilon\bar{p})}.$$

So the incremental VaR is

$$\mathrm{VaRI}(p, P) = \partial_\epsilon \mathrm{VaR}(P + \epsilon\bar{p})|_{\epsilon=0} = \frac{1}{\mathrm{PV}(p)} \frac{P^T V p}{\mathrm{VaR}(P)}.$$

The computation of this number requires only the variance-covariance matrix and some matrix multiplication. As this is already available in the implementation of the VaR computation, the implementation of this extra information is almost free.

7.3 Contribution VaR

The *contribution VaR* is an additive decomposition of the VaR of a portfolio. The contribution VaR of a given part (p) is the length (in the VaR metric) of the projection of the vector p on the vector of the portfolio (P). Figure 7.1 gives a representation of this projection.

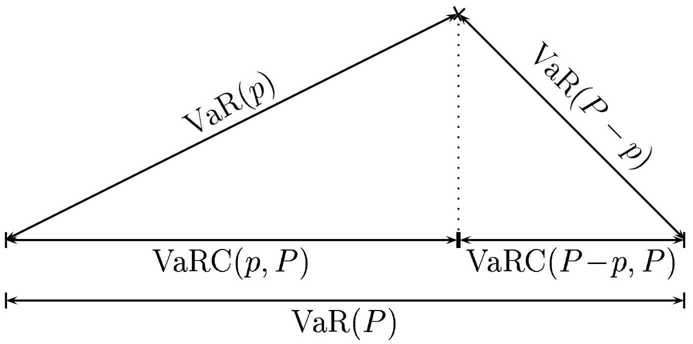

Figure 7.1: Contribution VaR for a position p in the portfolio P.

The contribution VaR can be computed analytically: the contribution VaR of the sub-portfolio p in the portfolio P is

$$\mathrm{VaRC}(p, P) = \frac{P^T V p}{\mathrm{VaR}(P)}.$$

This is equal to the incremental VaR of the part multiplied by its value.

7.4 Relationship between marginal, contribution and incremental VaR

We have

$$\mathrm{VaRI}(p, P)\,\mathrm{PV}(p) = \mathrm{VaRC}(p, P).$$

For p small we have

$$\mathrm{VaRM}(p, P) \sim \mathrm{VaRI}(p, P)\,\mathrm{PV}(p).$$

Even if we are often interested by the marginal VaR, which tell us by how much the VaR would be reduced if the position was removed, we often compute the incremental VaR and scale it linearly.

This number can be used in the following way. Suppose you have a portfolio that is too risky (VaR is too large). You want to decrease the risk by selling some part of the portfolio. You have a number (n) of assets eligible for the selling. You want to sell the smallest amount possible to decrease the VaR by a given number (D). For that you compute $\mathrm{VaRI}(p_i, P)$ $(i = 1, \ldots, n)$ and take the asset with the largest incremental VaR (denoted j). Sell $k = D/\mathrm{VaRI}(p_j, P)$ unit of position p_j and the VaR becomes approximatively

$$\mathrm{VaR}(P - kp_j/\mathrm{PV}(p_j)) \approx \mathrm{VaR}(P) - k\,\mathrm{VaRI}(p_j, P).$$

This approximation is good if the quantity sold k is small enough with respect to the initial portfolio.

7.5 Maximum loss and worst scenario

The information given by the maximum loss and the worst scenario are complementary to the one provided by the VaR. The idea is to restrict the possible scenarios of risk factor changes to a certain domain. This domain is given by the set of risk factor changes with a probability density above a certain level. This level is chosen such that the total probability of the set is equal to a given probability (P), usually 95%.

In the case of the delta-normal approach the change of probability level has only a scaling effect, like for the VaR.

Once the domain is defined, we search for the scenario which gives the largest lost. This loss is called the *maximum loss*; the scenario that causes this loss is called the *worst scenario*.

The maximum loss is a coherent risk measure[1] (see [2]). The worst scenario is an interesting information as it explain to which type of risk factor change the portfolio is the most sensitive.

The domain for the computation of the worst loss is

$$D = \{x : x^T V^{-1} x < \beta\} \quad \text{with} \quad \mathrm{P}(D) = P.$$

---

[1] In the delta-normal approach presented here, the VaR is also a coherent risk measure. But this is not the case for Monte Carlo or historical simulation approach to VaR.

where the number β is given by $F_{\xi^2_n}(\beta) = P$. The function $F_{\xi^2_n}$ is the ξ^2 distribution function with n degrees of freedom. Its value is given by

$$F_{\xi^2_n}(\beta) = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\beta s^{\frac{n}{2}-1} e^{-\frac{s}{2}}\, ds.$$

The worst loss for a position p is the vector x solution of

$$\begin{aligned} &\min\,(p\,|x) \\ &\quad \text{s.t. } x^T V^{-1} x = \beta \end{aligned}$$

Using Lagrange multiplier, we found that x must satisfy for some λ

$$p = 2\lambda V^{-1}x \quad \text{and} \quad x^T V^{-1} x = \beta.$$

By combining those those equation we have $(x|p) = 2\lambda\beta$. As β is positive and we are looking for the minimum, which is a loss, the multiplier λ should be negative. By using those informations we can obtain

$$\lambda = -\frac{1}{2\sqrt{\beta}}\,\mathrm{VaR}(p) \quad \text{and} \quad x = \frac{\sqrt{\beta}}{\mathrm{VaR}(p)} V p.$$

In conclusion, whatever the probability level is chosen, the most dangerous direction for risk factor changes is Vp.

7.6 Shift, twist and cie.

In mathematics, the art of posing a question is more
important than the art of solving one.
Georg Cantor (1867)

7.6.1 Introduction

Very often the movements of a yield curve are divided in *shift*, *twist*, *butterfly* and the rest. This decomposition is coming from the fact that a principal component analysis of yield curves movements very often shows three principal components[2] that explain more than 90% of the curve movements. These three component are called shift, twist and butterfly because of the shape of the eigenvectors is

shift a move of all the rates in the same direction by about the same amount.

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[2] See [12] for an opposite opinion.

twist a movement at one end of the curve and an opposite one at the other end with intermediary rates with interpolated movements.

butterfly equal movements at the two ends of the curve and an opposite movement of equal amplitude in the middle.

Of course those descriptions are not exact and depend of the end point of the curve used and the number of intermediary points. Moreover even if the three effects explain 90% of the curve movements, it does not necessarily explain 90% of the profit and loss of a portfolio. It would be the case if the sensitivities of the portfolio to the different rates were uniformly distributed along the yield curve (or in the 3-dimensional space generated by the 3 vectors). But for trading portfolios or for the position of investment portfolios against their benchmarks, the structure is often more complex. Once the three vectors are chosen one can take a position which has no component in those vectors. The risk evaluated by the three risk factors will be zero but the real risk will be present. Usually a yield curve is constructed using at least 8 points, so there is a least a 5-dimensional space which has no risk with respect to the three vectors chosen. This can be very important if this measure of risk is used to optimize portfolios in some way. In particular, see Gibson and Pritsker [4] and Xu and Pearson [11] for an explanation of why you should never use VaR to optimize portfolios and in particular with low dimension risk decomposition (like shift, twist and butterfly).

7.6.2 The definitions

We can rebase our view of the position, instead of looking at how fixed maturity vertices influence the portfolio value change, we can look how shift, twist, butterfly, and the rest do it. For this we need a precise definition of them. We use the zero-coupon continuously compounded curve to define the movement of the curve. Let $t_1 < t_2 < \ldots < t_{N-1} < t_N = T$ be the different vertices of the yield curve and r_i the corresponding rates. The different movements are

shift	$\bar{r}_i = r_i + 1$
twist	$\bar{r}_i = r_i + (i - \frac{N+1}{2})$
butterfly	$\bar{r}_i = r_i + \left(i - \frac{N}{2}\right)^2 - X$

where $X = 1/N \sum_{i=1}^{N} i^2 - (N-2)N/4$. Section 7.6.4 explains how these numbers were obtained.

We use the following related notations

$$\begin{aligned} R_1 &= S = (1, \ldots, 1) \\ R_2 &= T = (1 - \tfrac{N+1}{2}, \ldots, N - \tfrac{N+1}{2}) \\ R_3 &= B = (T^2/6, \ldots, T^2/6) \end{aligned}$$

The vector S, T and B are orthogonal. Let $R_4, \ldots, R_N$ be a base of the $N-3$ dimensional space orthogonal of R_1, R_2 and R_3. We denote by t the matrix $\mathrm{diag}(t_1, \ldots, t_N)$. The position vectors related to R_i are

$$P_i = tR_i \quad \text{and} \quad \tilde{P}_i = t^{-1}R_i.$$

The vector P_j is the change of market value of cash-flows at the different vertices of the yield curve due to a movement of interest rate. This means that if p is a position then the change of value of this position due to the change of interest rates R_j is given (to the first order) by $(p|P_j)$.

The vector $\tilde{P}_j$ is the position sensible to a P_j movement of price but not to the other movements, i.e. $(\tilde{P}_j|P_i) = \delta_{i,j}$ $(1 \leq i, j \leq N)$.

In general we can chose vectors $\tilde{P}_1, \ldots, \tilde{P}_N$ of positions we want to privilege and work with them. The decomposition proposed in the sequel would work with any set of those vectors, even if there is another financial intuition behind them (or no financial intuition at all).

7.6.3 Decomposition of positions

We now decompose the position in the shift part, the twist part, the butterfly part, and the rest.

Any position can be written in the base $\tilde{P}_1, \ldots, \tilde{P}_N$ as $q = A^{-1}p$ with the matrix A of the change of base which has the coefficient of $\tilde{P}_i$ as columns.

It the new base, we can compute the VaR and VaR contribution of the different components.

We denote by q_S the vector $(q_1, 0, \ldots, 0)$. The VaR of the shift is

$$\mathrm{VaR}(\mathsf{shift}) = \sqrt{q_S^T A^T V A q_S}.$$

The VaR contribution of the shift is

$$\mathrm{VaRC}(\mathsf{shift}, P) = \sqrt{P^T V A q_S}.$$

Similarly for the twist, we denote $q_T = (0, q_2, 0, \ldots, 0)$ and

$$\mathrm{VaR}(\mathsf{twist}) = \sqrt{q_S^T A^T V A q_T}.$$

As for the VaR, we don't have

$$\mathrm{VaR}(\mathsf{shift}) + \mathrm{VaR}(\mathsf{twist}) + \mathrm{VaR}(\mathsf{butterfly}) + \mathrm{VaR}(\mathsf{rest}) = \mathrm{VaR},$$

but we have

$$\mathrm{VaRC}(\mathsf{shift}) + \mathrm{VaRC}(\mathsf{twist}) + \mathrm{VaRC}(\mathsf{butterfly}) + \mathrm{VaRC}(\mathsf{rest}) = \mathrm{VaR}\,.$$

7.6.4 Supplementary remarks

On the definition of butterfly

For the butterfly we want something like a wing. This can be done with two pieces of lines or, like what we propose here, with a parabola. The curve shape is thus represented by a second order polynomial. Moreover we want a butterfly that does not interfere with the shift and the twist. So we take it with average 0 and even around the middle of the curve.

We have a butterfly like

$$\left(i - \frac{N}{2}\right)^2 - X$$

and take X such that

$$NX = \sum_{i=1}^{N} \left(i - \frac{N}{2}\right)^2 = \sum_{i=1}^{N} i^2 - \frac{N^2}{4}(N-2).$$

On non-orthogonality

The shift, twist and butterfly decomposition is coherent only if the vectors are orthogonal. If not the rest, which is taken here as the orthogonal to S, T and B, is not defined. Without the rest, the decomposition is not possible. Similarly if we want to define directly the position vectors $\tilde{p}_j$ we have to define *all* of them or define the most important one and have a coherent, meaningful way to define the rest.

Chapter 8

Correlation and cie

8.1 Diversification benefits

Suppose we have a portfolio (P) divided in sub-portfolios ($p_1, \dots, p_N$). The diversification of the investment between several types of investments create a *diversification benefit*, i.e. the VaR of the portfolio is smaller than the sum of the individual VaR of the sub-portfolios. We try to give a break-down of this diversification. To simplify the writing, we use the notation $\mathrm{VaR}_i = \mathrm{VaR}(p_i)$. The undiversified VaR is

$$\mathrm{VaRU}(p_1, \dots, p_N) = \sum_{i=1}^{N} \mathrm{VaR}_i \,.$$

The diversification benefit is

$$\mathrm{VaRU}(p_1, \dots, p_N) - \mathrm{VaR}(P) = \sum_{i=1}^{N} \mathrm{VaR}_i - \mathrm{VaR}(P).$$

Let $\rho_{i,j}$ be the correlation between portfolios defined by

$$p_i^T V p_j = \rho_{i,j} \,\mathrm{VaR}_i \,\mathrm{VaR}_j \,.$$

Then

$$\begin{aligned}
&\mathrm{VaRU}(p_1, \dots, p_N) - \mathrm{VaR}(P) \\
&\quad = \frac{\mathrm{VaRU}^2 - \mathrm{VaR}(P)^2}{\mathrm{VaRU} + \mathrm{VaR}(P)} \\
&\quad = \frac{1}{\mathrm{VaRU} + \mathrm{VaR}(P)} \left(\left(\sum_i \mathrm{VaR}_i \right)^2 - \sum_i \mathrm{VaR}_i^2 - 2 \sum_{i<j} \rho_{i,j} \,\mathrm{VaR}_i \,\mathrm{VaR}_j \right) \\
&\quad = \frac{2}{\mathrm{VaRU} + \mathrm{VaR}(P)} \sum_{i<j} (1 - \rho_{i,j}) \,\mathrm{VaR}_i \,\mathrm{VaR}_j \,.
\end{aligned}$$

So the diversification benefit can be divided (in an additive way) between the different pairs of sub-portfolios. The decomposition uses only the correlations between the portfolios and their VaR.

8.2 To hedge or not to hedge

We have a position of which we want to hedge the risk by taking other positions. What type of volatility and correlation relationship should exists between the two positions to decrease the VaR?

In this section we consider only the value-at-risk perspective and not the return. From a return perspective, if you have a liability for which you pay interest you need to invest the proceeds in assets that pay at least the same interest if you don't want to loose money[1]. But as we disregards growth or carry in our estimation of risk, we do not consider this necessity.

We have a position p that we want to hedge. We have at our disposal a instrument r[2]. What is the best use of this hedging instruments? For this we have to minimize the VaR of our position modified by a certain quantity of the instrument. So we have to solve

$$\min_{\lambda\in\mathbb{R}} \mathrm{VaR}(p+\lambda r).$$

This means we have to minimize

$$(p+\lambda r\,|p+\lambda r\,) = \|p\|^2 + 2\lambda\,(p\,|r\,) + \lambda^2\|r\|^2.$$

The minimum is achieved for $\lambda = -\mathrm{VaR}(p)/\mathrm{VaR}(r)\rho$ where ρ is the correlation between p and r. The VaR can be decreased in any case, except when the instrument is independent of the position. In this last case, the extra instrument can never decrease the risk of the initial position.

Suppose you have a position p and you want to invest its proceeds (its value) in another instrument r with same volatility ($\|p\| = \|r\|$). What type of correlation would decrease the risk? Here we have to see when $\|p-r\| < \|p\|$. This is the case when

$$|p|^2 - 2\,(p\,|r\,) + |r|^2 < |p|^2 \Leftrightarrow -2|p||r|\rho + |r|^2 < 0 \Leftrightarrow \rho > -\frac{1}{2}.$$

In this case the investment of the proceeds reduce the risk only when the correlation is larger than $\frac{1}{2}$.

---

[1] If the maturity is the same and the spread is not changing.

[2] Our position and hedging instrument are already decomposed in the risk factor base

Chapter 9

Other instruments

9.1 FRN: second method

As indicated in Section 4.7 we give now a cash-flow distribution of the FRN that differentiate between the projection index and the discounting index (and so incorporate the spread risk).

The FRN pays coupons at times $t_1, \ldots, t_n$ and the principal in t_n. Let $r(t)$ indicate the rate for maturity t for the projection index (Libor curve for example) and $\bar{r}(t)$ the one for the discounting index (issuer curve). The forward rate between t_i and t_{i+1} for the projection index is

$$f(t_i, t_{i+1}) = \frac{t_{i+1} r(t_{i+1}) - t_i r(t_i)}{t_{i+1} - t_i}.$$

Let s_i be the spread above the reference rate for the period from t_i to t_{i+1}. It is usually of the form $s(t_{i+1} - t_i)/360$ or similar. We denote by C_1 the coupon in t_1 that has already fixed.

With those notations the value[1] of the note is

$$C_1 e^{-t_1 \bar{r}(t_1)} + \sum_{i=1}^{n-2} \left(e^{(t_{i+1}-t_i) f(t_i, t_{i+1})} - 1 + s_{i+1} \right) e^{-t_{i+1} \bar{r}(t_{i+1})} + \left(e^{(t_n - t_{n-1}) f(t_{n-1}, t_n)} + s_n \right) e^{-t_n \bar{r}(t_n)}.$$

We write the cash-flow equivalent of this value. For the discounting curve we have

$$\begin{array}{ll} \text{in } t_1 & C_1 \\ \text{in } t_i & e^{(t_i - t_{i-1}) f(t_{i-1}, t_i)} - 1 + s_i \quad i = 2, \ldots, n-1 \\ \text{in } t_n & e^{(t_n - t_{n-1}) f(t_{n-1}, t_n)} + s_n \end{array} .$$

---

[1] The valuation is done by *discounting the forward rate*. This gives a correct result when the two curves are the same. To have a perfect valuation here we should use a *spread model*.

For the projection index we have

$$\begin{array}{lll} \text{in } t_1 & e^{-t_2(\bar{r}(t_2)-r(t_2))} & \\ \text{in } t_i & -\left(e^{t_i r(t_i)-t_{i-1}r(t_{i-1})}-1\right)e^{-t_i(\bar{r}(t_i)-r(t_i))} & i=2,\ldots,n-1\ . \\ \text{in } t_n & -e^{t_n r(t_n)-t_{n-1}r(t_{n-1})}e^{-t_n(\bar{r}(t_n)-r(t_n))} & \end{array}$$

The interpretation is that with respect to the discounting curve we have the equivalent of the forward coupon and the principal at maturity. On the projection curve we have the principal in t_1 adjusted by the difference between the projection and discounting rate. The next payments are outflows of the forward coupon also adjusted by the difference of rate. The last one is the outflow of the principal plus the forward coupon also adjusted.

Note that if the spread s_i are 0 and the projection and the discounting index are the same, we obtain the simplified method of Section 4.7.

9.2 Equity: relation to index

When only an equity index (and not each equity) is available, the equities can be replaced by their beta equivalent. The beta is the slope of the best-fit line obtained when equity return is regressed against the index return.

An equity of value p and beta β with respect to his index will be replaced by a position of $p\beta$ in the index.

If the portfolio of equity is diversified enough the beta measure tends to be stable and this representation is good enough if equities represent only a small part of the risk.

9.3 Index future

We treat futures as forwards. The value of a forward in t at a price F when the current price is S is

$$Se^{-qt} - Fe^{-r(t)t}$$

where $r(t)$ is the interest rate and q is the dividend rate. This is equivalent to a position Se^{-qt} in the index and a cash-flow $-F$ in t.

9.4 Forex option

When the current price of the underlying is S, the strike is X, and the maturity is T, the price of a call in the Black-Scholes model is

$$C(t,S) = e^{-r_1 T}SN(d_1) - e^{-r_0 T}XN(d_2),$$

where

$$d_2 = \frac{\ln(S/X) + (r_0 - r_1 - \frac{1}{2}\sigma^2)T}{\sigma\sqrt(T)}, \quad d_1 = d_2 + \sigma\sqrt(T).$$

The price of the put is

$$P(t,S) = e^{-r_0T}XN(-d_2) - e^{-r_1T}SN(-d_1).$$

For a A currency option against B currency, we denote it $\Delta^A_{C/P}$ the A currency delta equivalent to the option and $\Delta^B_{C/P}$ the B delta equivalent.

Call

$$\Delta^A_C(t,S) = e^{-r_1T}N(d_1) \quad \text{and} \quad \Delta^B_C(t,S) = \Delta^A_C(t,S)S + C(t,S)$$

Put

$$\Delta^A_P(t,S) = e^{-r_1T}\left(N(d_1) - 1\right) \quad \text{and} \quad \Delta^C_P(t,S) = \Delta^A_P(t,S)S + P(t,S)$$

9.5 Bond options and swaption

We treat the bond options and swaptions in the same way. they both are, for valuation perspectives, equivalent to an option to enter into a set of cash-flows. The cash-flows are the fixed coupon, the notional at the end and the initial amount (strike for bond options and notional for swaptions) with opposite sign at delivery.

The option has now to be replaced by its *delta equivalent.* There are various ways to compute it. Different techniques will lead to different results. Even with standard ways, the difference can easily reach 10% of the notional. Detailed explanation and examples can be found in [8].

Even the term *delta equivalent* can be ambiguous as it can mean the theoretical delta of the option multiplied by the underlying cash-flow or the standard grid-point delta cash-flow equivalent.

It is beyond the goal of this document to choose between or explain those choices. We give some references where those concepts are explained: Hull's book [9] for Black model on swaption; Henrard article [7] for Hull-White model, Hagan et all [5] for stochastic volatility model.

Chapter 10

Reports

10.1 Standard

This section presents some possible reports based on VaR concepts. The list is certainly not exhaustive and the order is not a value judgment.

The reports are based on the portfolios and matrices presented in Appendix A.

Portfolio	VaR com.	VaR FX	VaR IR	VaR total
P_1	0	4,245	2,106	5,163
P_2	0	525,475	99,414	560,238
P_3	66,044	66,572	0	70,467
Total	66,044	465,723	101,154	533,778

Table 10.1: Standard VaR report for the portfolios P_1, P_2 and P_3

10.2 Correlation between portfolios

Portfolio	Total	P_1	P_2	P_3
Total	1	0.95	0.99	-0.39
P_1	0.95	1	0.96	-0.51
P_2	0.99	0.96	1	-0.50
P_3	-0.39	-0.51	-0.50	1

Table 10.2: Correlation report between portfolios

Note the negative correlation between P_3 and the other portfolios. One is short EUR and the other are long.

10.3 Diversification benefits

Portfolio	VaR	Diversification benefit	
		P_2	P_3
P_1	5,163	200	939
P_2	560,238	–	100,951
P_3	70,467	Sum	102,090
Sum of VaR	635,870		
Total VaR	533,778		

Table 10.3: Diversification benefit

This correspond to a split of the diversification benefit between the portfolio pairs as explained in Section 8.1.

10.4 Evolution

Portfolio	last week	position	volatilities	correlation	this week
P_1	10,587	-3,786	-1,512	-126	5,163
P_2	550,289	10,412	-2,445	1,982	560,238
P_3	100,820	-23,057	-7,371	76	70,467
Total	590,720	-49,343	-5,100	-2,498	533,778

Table 10.4: Evolution of the VaR over one week

The evolution of the portfolio VaR from one reporting moment to the next can be divided in several explaining factors. The first one is the change of position. The VaR of the current position is computed with the previous variance-covariance matrix. The difference between that VaR and the previous VaR is reported in the position column. In a similar way the volatilities and the correlations are successively updated to obtain the current VaR figures. The order of the changes (position, volatility, correlation) is arbitrary. Another order is possible and will give different results.

A variant is to change the positions, volatilities and correlations independently and report the resulting changes in VaR. The residual number needed to obtain the current VaR is the "cross" effect. The change of volatilities on the change of positions and so on.

10.5 Spread positions

Portfolio	VaR
P_4 Interest rate risk	21,594
P_4 spread risk	35,916
P_4 Total	49,398

Table 10.5: VaR report for the interest rate and spread risk of Portfolio P_4

This report is obtain by mapping all the risk factors on the swap curve for each currency to create a new portfolio. The Interest rate risk reported is the risk of this new portfolio. The spread risk is the risk between the original portfolio and the new one. The total risk is lower than the sum of the interest rate risk and the spread risk, this illustrate the diversification benefit between different type of positions.

10.6 Shift, twist and butterfly

Portfolio	P_5	P_6
Shift VaR	0	57,485
Twist VaR	0	19,777
Butterfly VaR	0	522
Rest VaR	4,143	1,671
Total VaR	4,143	62,248

Table 10.6: VaR report shift, twist and butterfly VaR

Portfolio P_5 was obviously created to obtain this strange effect. The position of the portfolio can be explained easily in term of curve movement: a steepening of the short term curve and a flattening on the medium term (4y-5y). This illustrate that a portfolio can enter in the fourth dimension without being too exotic.

Portfolio P_6 is a pure investment portfolio with a large part on short term (liquidity part) and the rest nicely distributed on the curve. For such a portfolio, the shift, twist and butterfly work quite well. But we insist that this is the VaR of the portfolio and not the VaR of the portfolio against a benchmark (tracking error). In this last case in it easy to have effects similar to the one of Portfolio P_5.

Appendix A

Examples

A.1 Matrix

All the examples of this book uses the same covariance matrix. The actual algorithm for the production of the matrix doesn't affect the way we use it in the examples. We have rounded the numbers to 2 decimals in the presentation and we have used the rounded numbers for the computations. It should be possible to reproduce all the figures obtained in the different examples.

The base currency is USD. We have reduced the number of risk factors to 26.

Currencies: EUR, DKK

Commodity Gold (1 oz)

Interest rate: EUR swap: 1m, 3m, 6m, 12m, 2y, 3y, 4y, 5y

DEM govt: 1m, 3m, 12m, 2y, 3y, 4y, 5y

USD swap: 1m, 3m, 6m, 12m, 2y, 3y, 4y, 5y

The values for April 2, 2001 are given in the table A.1. The interest rates are zero-coupon continuously compounded with 365 days as unit of time.

The correlations between the different risk factors are presented in Table A.2 and the volatilities in Table A.3.

EUR	DKK	Gold					
0.882	0.1182	255.35					
			EUR swap (in %)				
1m	3m	6m	12m	2y	3y	4y	5y
4.72	4.57	4.43	4.31	4.41	4.54	4.66	4.80
			DEM govt (in %)				
1m	3m	12m	2y	3y	4y	5y	
4.60	4.49	4.25	4.17	4.24	4.29	4.37	
			USD swap (in %)				
1m	3m	6m	12m	2y	3y	4y	5y
5.13	4.80	4.60	4.60	4.87	5.17	5.37	5.54

Table A.1: Values of the risk factors on April 2, 2001

A.2 Portfolios

We present the portfolios by giving the different cash-flow equivalent on the different risk factors. The instruments that have produced those cash-flows are irrelevant for the use we do of the portfolios. Nevertheless some of them have been created by specific instruments. Those examples are given in Chapter 4.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)
(1)	1	0.43	0.44	0.05	-0.04	-0.06	0.06	0.01	0.03	-0.04	-0.02	0.03	0	-0.03	-0.08	-0.1	-0.13	-0.13	0.1	0.09	0.11	0.08	0	-0.03	-0.06	-0.07
(2)	0.43	1	0.99	0.25	0.27	0.24	0.34	0.33	0.3	0.23	-0.05	0.12	0.13	0.17	0.18	0.15	0.12	0.1	0.35	0.36	0.34	0.29	0.11	0.1	0.1	0.08
(3)	0.44	0.99	1	0.26	0.26	0.22	0.3	0.3	0.26	0.2	-0.05	0.12	0.13	0.18	0.15	0.12	0.09	0.07	0.35	0.34	0.32	0.27	0.08	0.07	0.07	0.05
(4)	0.05	0.25	0.26	1	0.46	0.5	0.25	0.27	0.29	0.24	0.3	0.42	0.39	0.44	0.28	0.23	0.29	0.25	0.09	0.09	0.08	0.06	0.06	0.01	0.03	0.02
(5)	-0.04	0.27	0.26	0.46	1	0.72	0.47	0.51	0.54	0.5	0.42	0.54	0.47	0.41	0.5	0.47	0.44	0.4	0.11	0.3	0.29	0.29	0.16	0.15	0.17	0.17
(6)	-0.06	0.24	0.22	0.5	0.72	1	0.78	0.82	0.84	0.83	0.23	0.42	0.42	0.46	0.83	0.81	0.81	0.8	0.33	0.6	0.61	0.59	0.51	0.51	0.49	0.48
(7)	0.06	0.34	0.3	0.25	0.47	0.78	1	0.93	0.92	0.91	0.1	0.27	0.25	0.26	0.85	0.87	0.88	0.87	0.43	0.67	0.66	0.62	0.6	0.61	0.61	0.6
(8)	0.01	0.33	0.3	0.27	0.51	0.82	0.93	1	0.98	0.98	0.09	0.28	0.23	0.24	0.89	0.9	0.9	0.91	0.48	0.69	0.72	0.69	0.59	0.6	0.59	0.58
(9)	0.03	0.3	0.26	0.29	0.54	0.84	0.92	0.98	1	0.99	0.11	0.29	0.26	0.27	0.91	0.91	0.92	0.92	0.43	0.69	0.73	0.71	0.63	0.63	0.61	0.6
(10)	-0.04	0.23	0.2	0.24	0.5	0.83	0.91	0.98	0.99	1	0.07	0.26	0.25	0.26	0.92	0.93	0.94	0.94	0.44	0.67	0.7	0.67	0.6	0.62	0.6	0.59
(11)	-0.02	-0.05	-0.05	0.3	0.42	0.23	0.1	0.09	0.11	0.07	1	0.84	0.73	0.51	0.18	0.17	0.22	0.12	-0.05	-0.15	-0.16	-0.14	-0.08	-0.1	-0.11	-0.14
(12)	0.03	0.12	0.12	0.42	0.54	0.42	0.27	0.28	0.29	0.26	0.84	1	0.93	0.78	0.37	0.35	0.38	0.31	0.06	0.1	0.06	0.04	0.02	-0.01	-0.01	-0.04
(13)	0	0.13	0.13	0.39	0.47	0.42	0.25	0.23	0.26	0.25	0.73	0.93	1	0.93	0.42	0.37	0.4	0.32	0.1	0.11	0.08	0.06	0.05	0.03	0.02	0
(14)	-0.03	0.17	0.18	0.44	0.41	0.46	0.26	0.24	0.27	0.26	0.51	0.78	0.93	1	0.46	0.4	0.41	0.35	0.16	0.19	0.17	0.14	0.09	0.1	0.09	0.08
(15)	-0.08	0.18	0.15	0.28	0.5	0.83	0.85	0.89	0.91	0.92	0.18	0.37	0.42	0.46	1	0.99	0.95	0.95	0.39	0.63	0.67	0.65	0.56	0.57	0.55	0.54
(16)	-0.1	0.15	0.12	0.23	0.47	0.81	0.87	0.9	0.91	0.93	0.17	0.35	0.37	0.4	0.99	1	0.95	0.96	0.36	0.61	0.66	0.64	0.58	0.6	0.58	0.57
(17)	-0.13	0.12	0.09	0.29	0.44	0.81	0.88	0.9	0.92	0.94	0.22	0.38	0.4	0.41	0.95	0.95	1	0.98	0.38	0.57	0.6	0.58	0.59	0.6	0.57	0.56
(18)	-0.13	0.1	0.07	0.25	0.4	0.8	0.87	0.91	0.92	0.94	0.12	0.31	0.32	0.35	0.95	0.96	0.98	1	0.37	0.6	0.64	0.63	0.62	0.63	0.61	0.6
(19)	0.1	0.35	0.35	0.09	0.11	0.33	0.43	0.48	0.43	0.44	-0.05	0.06	0.1	0.16	0.39	0.36	0.38	0.37	1	0.68	0.54	0.39	0.11	0.11	0.08	0.06
(20)	0.09	0.36	0.34	0.09	0.3	0.6	0.67	0.69	0.69	0.67	-0.15	0.1	0.11	0.19	0.63	0.61	0.57	0.6	0.68	1	0.94	0.84	0.61	0.6	0.58	0.55
(21)	0.11	0.34	0.32	0.08	0.29	0.61	0.66	0.72	0.73	0.7	-0.16	0.06	0.08	0.17	0.67	0.66	0.6	0.64	0.54	0.94	1	0.97	0.77	0.75	0.73	0.7
(22)	0.08	0.29	0.27	0.06	0.29	0.59	0.62	0.69	0.71	0.67	-0.14	0.04	0.06	0.14	0.65	0.64	0.58	0.63	0.39	0.84	0.97	1	0.85	0.84	0.83	0.8
(23)	0	0.11	0.08	0.06	0.16	0.51	0.6	0.59	0.63	0.6	-0.08	0.02	0.05	0.09	0.56	0.58	0.59	0.62	0.11	0.61	0.77	0.85	1	0.98	0.96	0.94
(24)	-0.03	0.1	0.07	0.01	0.15	0.51	0.61	0.6	0.63	0.62	-0.1	-0.01	0.03	0.1	0.57	0.6	0.6	0.63	0.11	0.6	0.75	0.84	0.98	1	0.98	0.97
(25)	-0.06	0.1	0.07	0.03	0.17	0.49	0.61	0.59	0.61	0.6	-0.11	-0.01	0.02	0.09	0.55	0.58	0.57	0.61	0.08	0.58	0.73	0.83	0.96	0.98	1	0.99
(26)	-0.07	0.08	0.05	0.02	0.17	0.48	0.6	0.58	0.6	0.59	-0.14	-0.04	0	0.08	0.54	0.57	0.56	0.6	0.06	0.55	0.7	0.8	0.94	0.97	0.99	1

Table A.2: Correlations April 2, 2001

	Risk factor	Vol. (in %)	P_1	P_2	P_3
1)	Gold	0.8365	0	0	4,800,000
2)	DKK	0.7408	0	0	10,000,000
3)	EUR	0.7496	1,226,330	42,618,233	-15,000,000
	USD	–	0	300,016	300,000
4)	DEM govt-1m	0.0031	0	362,906	0
5)	DEM govt-3m	0.0085	2,759,636	9,721,424	0
6)	DEM g.-12m	0.0363	3.055,896	2,746,995	0
7)	DEM govt-2y	0.1041	0	1,912,455	0
8)	DEM govt-3y	0.1432	0	1,900,010	0
9)	DEM govt-4y	0.1760	0	25,203,329	0
10)	DEM govt-5y	0.2059	0	5,436,820	0
11)	EUR swap-1m	0.0020	2,002,924	4,407,454	0
12)	EUR swap-3m	0.0066	-5,039,914	-2,411,907	0
13)	EUR swap-6m	0.0136	2,070,455	-10,065,094	0
14)	EUR s.-12m	0.0307	24,177	12,332,747	0
15)	EUR swap-2y	0.0738	0	-241,617	0
16)	EUR swap-3y	0.1131	0	-239,963	0
17)	EUR swap-4y	0.1296	0	-4,322,606	0
18)	EUR swap-5y	0.1618	0	-361,227	0
19)	USD swap-1m	0.0047	0	0	0
20)	USD swap-3m	0.0114	0	0	0
21)	USD swap-6m	0.0253	0	199,924	0
22)	USD s.-12m	0.0611	0	374,600	0
23)	USD swap-2y	0.1284	0	5,258,512	0
24)	USD swap-3y	0.1888	0	0	0
25)	USD swap-4y	0.2550	0	0	0
26)	USD swap-5y	0.3201	0	0	0

Table A.3: Exposure of the portfolios to the different risk factors

	Risk factor	P_4	P_5	P_6
1)	Gold	0	0	0
2)	DKK	0	0	0
3)	EUR	0	0	0
	USD	0	0	0
4)	DEM govt-1m	-2,500,000	66,666,667	0
5)	DEM govt-3m	-1,500,000	-33,333,333	0
6)	DEM g.-12m	-5,000,000	0	0
7)	DEM govt-2y	-12,000,000	0	0
8)	DEM govt-3y	-14,000,000	0	0
9)	DEM govt-4y	-5,000,000	2,053,388	0
10)	DEM govt-5y	-5,000,000	-1,095,290	0
11)	EUR swap-1m	2,500,000	0	0
12)	EUR swap-3m	2,500,000	0	0
13)	EUR swap-6m	2,500,000	0	0
14)	EUR s.-12m	5,000,000	0	0
15)	EUR swap-2y	10,000,000	0	0
16)	EUR swap-3y	10,000,000	0	0
17)	EUR swap-4y	3,000,000	0	0
18)	EUR swap-5y	2,000,000	0	0
19)	USD swap-1m	0	0	25,000,000
20)	USD swap-3m	0	0	2,000,000
21)	USD swap-6m	0	0	2,000,000
22)	USD s.-12m	0	0	10,000,000
23)	USD swap-2y	0	0	5,000,000
24)	USD swap-3y	0	0	5,000,000
25)	USD swap-4y	0	0	4,000,000
26)	USD swap-5y	0	0	3,000,000

Table A.4: Exposure of the portfolios to the different risk factors

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