A Fast Algorithm for Computing Expected Loan Portfolio Tranche Loss in the Gaussian Factor Model.

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Abstract

We propose a fast algorithm for computing the expected tranche loss in the Gaussian factor model. We test it on a 125 name portfolio with a single factor Gaussian model and show that the algorithm gives accurate results. We choose a 125 name portfolio for our tests because this is the size of the standard DJCDX.NA.HY portfolio. The algorithm proposed here is intended as an alternative to the much slower Moody's FT method [1].

1 The Gaussian Factor Model

Let us consider a portfolio of N loans. Let the notional of loan i be equal to the fraction f_i of the notional of the whole portfolio. This means that if loan i defaults and the entire notional of the loan is lost the portfolio loses fraction f_i or $100f_i\%$ of its value. In practice when a loan i defaults a fraction r_i of

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its notional will be recovered by the creditors. Thus the actual loss given default (LGD) of loan i is

$$LGD_i = f_i(1 - r_i) \tag{1}$$

fraction or

$$LGD_i = 100 f_i(1 - r_i)\% \tag{2}$$

of the notional of the entire portfolio.

We now describe the Gaussian m-factor model of portfolio losses from default. The model requires a number of input parameters. For each loan i we are give a probability p_i of its default. Also for each i and each $k = 1, \ldots, m$ we are given a number $w_{i,k}$ such that $\sum_{k=1}^{m} w_{i,k}^2 < 1$. The number $w_{i,k}$ is the loading factor of the loan i with respect to factor k. Let $\phi_1, \ldots, \phi_m$ and $\phi^i, i = 1, \ldots, N$ be independent standard normal random variables. Let $\Phi(x)$ be the cdf of the standard normal distribution. In our model loan i defaults if

$$\sum_{k=1}^{m} w_{i,k}\phi_k + \sqrt{1 - \sum_{k=1}^{m} w_{i,k}^2 \phi^i} < \Phi^{-1}(p_i) \tag{3}$$

This indeed happens with probability p_i. The factors $\phi_1, \ldots, \phi_m$ are usually interpreted as the state of the global economy, the state of the regional economy, the state of a particular industry and so on. Thus they are the factors that affect the default behavior of all or at least a large group of loans in the portfolio. The factors $\phi^1, \ldots, \phi^N$ are interpreted as the idiosyncratic risks of the loans in the portfolio.

Let I_i be defined by

$$I_i = I_{\{loan\ i\ defaulted\}} \tag{4}$$

We define the random loss caused by the default of loan i as

$$L_i = f_i(1 - r_i)I_i, \tag{5}$$

where r_i is the recovery rate of loan i. The total loss of the portfolio is

$$L = \sum_i L_i \tag{6}$$

An important property of the Gaussian factor model is that L_i's are not independent of each other. Their mutual dependence is induced by the dependence of each L_i on the common factors $\phi_1, \ldots, \phi_m$. Historical data supports the conclusion that losses due to defaults on different loans are correlated with each other. Historical data can also be used to calibrate the loadings $w_{i,k}$.

2 Analytic Approximation to the Joint Distribution of $\phi_1, \ldots, \phi_m$ and L

When the values of the factors $\phi_1, \ldots, \phi_m$ are fixed, the probability of the default of loan i becomes

$$p^i = \Phi^{-1}\left(\frac{p_i - \sum_k w_{i,k}\phi_k}{\sqrt{1 - \sum_k w_{i,k}^2}}\right) \tag{7}$$

The random losses L_i become conditionally independent Bernoulli variables with the mean given by

$$E_{cond}(L_i) = f_i(1 - r_i)p^i \tag{8}$$

and the variance given by

$$VAR_{cond}(L_i) = f_i^2(1 - r_i)^2 p^i(1 - p^i) \tag{9}$$

By the Central Limit Theorem the conditional distribution of the portfolio loss L given the values of the factors $\phi_1, \ldots, \phi_m$ can be approximated by the normal distribution with the mean

$$E_{cond}(L) = \sum_i E_{cond}(L_i) \tag{10}$$

and the variance

$$VAR_{cond}(L) = \sum_i VAR_{cond}(L_i) \tag{11}$$

Then the joint distribution of the factors $\phi_1, \ldots, \phi_m$ and the portfolio loss L can be approximated by a distribution with density

$$\rho(\phi_1, \ldots, \phi_m, L) = \rho_{G,E_{cond}(L),VAR_{cond}(L)}(L) \prod_{k=1}^{m} \rho_{G,0,1}(\phi_k), \tag{12}$$

where $\rho_{G,m,v}(x)$ stands for the Gaussian density with mean m and variance v.

3 Expected Loss of a Tranche of Loan Portfolio

Let $0 \leq a < b \leq 1$. We define a tranche loss profile $Tl_{a,b}(x)$ by

$$Tl_{a,b}(x) = \frac{min(b-a, max(x-a,0))}{b-a} \quad (13)$$

Number a is called the attachment point of a tranche, while b is called the detachment point of a tranche. The expected loss of a tranche is then

$$TLoss(a,b) = \int Tl_{a,b}(L)\rho(\phi_1,\ldots,\phi_m,L)d\phi_1\ldots\phi_m L \quad (14)$$

This can be rewritten as a double integral

$$TLoss(a,b) = \int\int Tl_{a,b}(L)\rho_{G,E_{cond}(L),VAR_{cond}(L)}(L)dL\prod_{k=1}^{m}\rho_{G,0,1}(\phi_k)d\phi_1\ldots\phi_m \quad (15)$$

The inside integral with respect to L is easily done analytically because L has a simple normal distribution for fixed values of the factors $\phi_1,\ldots,\phi_m$. The outside integral has to be computed numerically. However, since it is an integral of a bounded smooth function with respect to m-dimensional Gaussian density, it is one of the simpler integrals to compute numerically.

4 Numerical Example

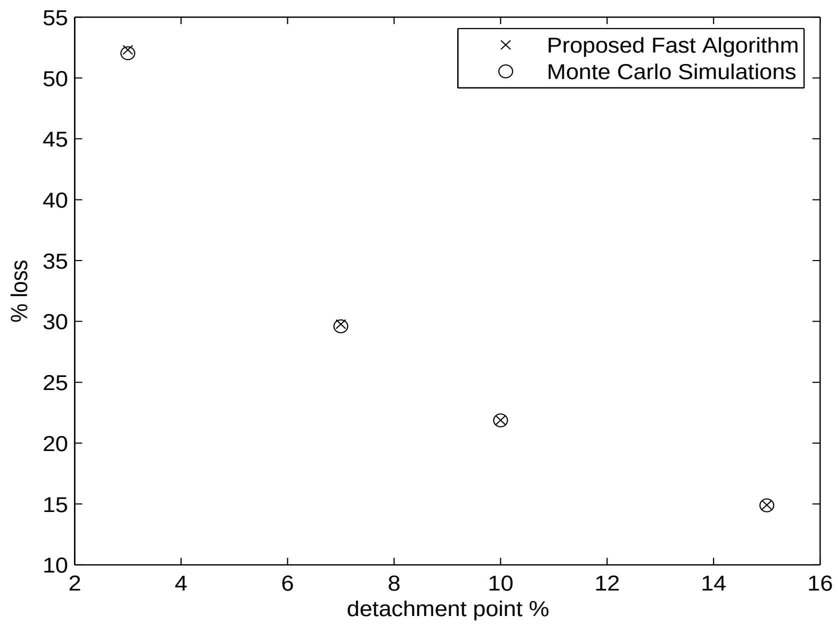
In this section we apply the proposed algorithm to the single factor Gaussian model of a portfolio with 125 names. We choose a 125 name portfolio because it is the size of the standard DJCDX.NA.HY portfolio. We choose a single factor model because it is the one most frequently used in practice. We evaluate the expected loss for four different tranches. All tranches have attachment point $a = 0$ or $a = 0\%$. The detachment points are 3%, 7%, 10% and 15%. We take these detachment points because they are the ones most frequently used in practice in order to evaluate the base correlation. The parameters of the porfolio are

$$f_i = \frac{1}{125}$$

$$
\begin{aligned}
p_i &= 0.015 + \frac{0.05(i-1)}{124} \\
r_i &= 0.5 - \frac{0.1(i-1)}{124} \\
w_{i1} &= 0.5 - \frac{0.1(i-1)}{124} \qquad (16)
\end{aligned}
$$

In Figure 1 we compare the expected loss computed using 10^6 Monte Carlo samples with the expected loss computed using formula (15). The agreement between the two is good.

Figure 1: Loan Portfolio Tranche Loss in the Gaussian Single Factor Model



5 Conclusions.

To obtain the results in Figure 1 we only needed to perform a single one dimensional numerical integration for each tranche. This is an improvement over the Moody's FT method [1] which requires computing a large number of Fourier transforms for each tranche. Each individual Fourier transform is as computationally expensive as (15).

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