

STRUCTURAL MODELS IN CONSUMER CREDIT

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Fabio Wendling Muniz de Andrade
EAESP-FGV / SERASA – Brazil
fandrade@serasa.com.br

Lyn Thomas
University of Southampton – UK
l.thomas@soton.ac.uk

ABSTRACT

We propose a structural credit risk model for consumer lending using option theory and the concept of the value of the consumer's reputation. Using Brazilian empirical data and a credit bureau score as proxy for creditworthiness we compare a number of alternative models before suggesting one that leads to a simple analytical solution for the probability of default. We apply the proposed model to portfolios of consumer loans introducing a factor to account for the mean influence of systemic economic factors on individuals. This results in a hybrid structural-reduced-form model. And comparisons are made with the Basel II approach. Our conclusions partially support that approach for modelling the credit risk of portfolios of retail credit.

1 – INTRODUCTION

Structural models for credit risk assessment were introduced by Merton (1974). In this approach the stochastic behaviour of the value of a firm's assets is modeled and if the value becomes lower than a threshold, usually a proportion of the firm's debt value, the company is considered to be in default. Merton's model assumes a diffusion process for a firm's asset value and that the firm will default if its asset value is lower than its debt on the maturity date of the debt. Following his work the model was developed in many ways, (Saunders, 1999), including variations in the timing of when default occurs (Black and Cox, 1976) and in the stochastic process that drives the value of a firm assets (Zhou, 1997).

Structural models for corporate credit give a theory of the causes of default based on financial option reasoning at a micro economic level. The shareholders of the firm have a call option on the firm's assets with a strike price equal to the firm's debt. If the value of the firm decreases below the value of the debt it will not be worth exercising the option and the firm will default. As alternatives the reduced-form and intensity-based models (Duffie & Singleton (1999); Jarrow, Lando & Turnbull (1997); Jarrow & Turnbull (1995)) consider default to be a random exogenous event and try to model the timing or intensity of occurrence of default events without worrying about its causes.

Application of structural models in retail and specifically in consumer credit is more of a challenge, since it is difficult to measure a consumer's assets (even for the consumers themselves) nor is it necessarily the case that default occurs when a consumers' debts exceed their assets. So to develop a structural model for this segment it is necessary first to propose a default theory for consumer credit that can use available information on consumers. Perli and Nayda (2004) propose a structural model for revolving retail credit that uses the exactly the same approach of the corporate models, considering that a consumer is in default if his assets are lower than a threshold. Then, following Vasicek (1991), they generate an analytic solution for the cumulative distribution of losses in the portfolio. However, as a great deal of consumer credit is unsecured and it is not the case that a consumer in default will lose the rights over all his assets, just

transposing the corporate default models to consumer default can lead to some aspects of consumer default being missed.

The New Basel Accord uses a formula for capital requirement in retail portfolios that is derived from Merton's model for corporate credit. Academics and practitioners that work with consumer credit had strong doubts concerning the applicability of that framework to consumer credit as pointed out by Thomas (2003).

There are also some works in structural modelling in consumer credit that are not related to individual risk assessment or portfolio modelling. Examples are Longhofer and Peters (2004), who studies lending discrimination and self-selection and Athreya (2004) who analyse the relation between the importance of the stigma of bankruptcy and bankruptcy rates.

Our objective is to develop a structural approach for consumer credit and to compare the Basel II capital requirement formula for consumer credit with such an approach. In section 2 we establish an option-based reasoning for consumer default and propose a theory for default in consumer credit. In section 3 we develop a default prediction model for consumer credit using the structural approach. We also propose the use of a behaviour or credit bureau score as proxy of creditworthiness that will be the consumer's equivalent to corporate asset values in structural models. Based on Brazilian empirical data we test different alternatives for modelling the stochastic behaviour of the creditworthiness proxy and compare the results of these approaches to risk discrimination with the traditional scoring approach for risk assessment.

In section 4 we use the structural approach to model the distribution of default rate in a portfolio. We propose a methodology to insert a systemic risk factor that will account for joint movements of defaults due to economic conditions. This procedure makes our approach for portfolio modelling an hybrid structural-reduced-form model. We compare results obtained by the proposed model with the ones obtained using Vasicek's model (1991) that is the basis of the Basel II formula for capital requirement.

2 – PROPOSITION OF A THEORY FOR CONSUMER DEFAULT

The model of consumer default proposed here is based on the following premises:

1. There is a unobservable stochastic quantity Q_i , the creditworthiness of consumer i that comprises all information about consumer i that is relevant for credit risk assessment.
2. Although Q_i is not directly observable, the market (lending institutions), use information internally or externally available about the consumer, such as past credit experiences, financial conditions of the consumer, credit reports to estimate it or a surrogate for it. Most of the time, the lending institution uses a proxy of Q_i for credit assessment such as a credit or behavioural scoring.
3. The probability of the consumer being accepted as a client by a lending institution, P_{ai} , is a strictly increasing function of Q_i , $P_{ai} = f(Q_i)$.
4. There are credit agencies, credit bureaus and other mechanisms of making default information available to any lending institution. If a consumer is in default all lending institutions will know it and the consumer will lose his reputation and have no more access to credit in the immediate future. We say that when this happens, “the consumer loses his reputation”.
5. Access to credit has a value for the consumer. This value, which is related to the extent of access to credit that the consumer has, is called the value of the consumer’s reputation, R_i . This value is a strictly increasing function of P_{ai} , $R = h(P_{ai})$.

Using these assumptions, it is possible to provide an option-based reasoning for the process of default in consumer credit that is similar to Merton’s approach for corporate credit. The consumer has a call option on his reputation with a strike price equal to the value of the consumer’s debt under consideration. If the value of his reputation is lower than the value of this debt, D_i , the consumer will default.

The lending institution will report the non-payment to the credit agencies or credit bureaus that will make it public to all the market. The consumer will lose the residual value of his reputation and access to credit. On the other hand, if the value of his reputation is greater than the value of the debt, then it is worth the consumer paying off the debt (possibly in installments) or servicing a revolving credit debt and keeping his reputation.

As f and h are strictly increasing functions of Q_i and P_{ai} respectively, R_i is a strictly increasing function of Q_i , $R_i = g(Q_i)$. It means that values of R_i can be mapped to unique values of Q_i and the strike price, K_{R_i} , can be mapped to a corresponding threshold of creditworthiness, K_{Q_i} . So we can say that the consumer will default if his creditworthiness is lower than K_{Q_i} . As $g(\cdot)$ is an individual specific function unknown function and D_i is a dynamic quantity, $K_{Q_i}(D_i)$ varies over time and from individual to individual.

In this work we use the simplest first passage approach to structural credit models which says that a consumer will default as soon as his creditworthiness hits the barrier K_{Q_i} .

2.1 – Cash flow considerations

It could be argued that the proposed theory does not consider the consumer's cash flow that can play an important role in the consumer's default process. But this effect can be inserted into it through K_{Q_i} . To do that we make an additional assumption:

6. If on the maturity of a debt obligation a consumer does not have enough cash he will raise cash through additional debt.

It means that a consumer with a debt D_i , requiring Z_i to service this debt and with C_i in cash, will raise the debt to $D_i + Z_i - C_i$, or more likely $D_i + Z_i$ to cover the repayment.

This is consistent with reality as far as it is common for a consumer with cash restrictions to raise more credit to pay his actual credit obligations. This behaviour

also leads to a decrease of the consumer's creditworthiness or at least his perceived creditworthiness. So with lower creditworthiness and higher strike price, the consumer with cash flow restrictions will be increasingly more likely to default.

As it is very difficult to track C_i because of its stochastic nature, the effect of cash flow restrictions can be reflected in the model by setting K_{Ri} and consequently K_{Qi} as stochastic quantities.

3 – A MODEL FOR DEFAULT PREDICTION

3.1 – Modelling the stochastic behavior of creditworthiness

Having established a theory for the process of default in consumer credit it is necessary to describe the stochastic behavior of Q_i .

Structural models for corporate credit usually use diffusion processes, like the Merton Model, or jump-diffusion processes (Zhou, 1997) for the log of the asset value. Taking the later more complex model for creditworthiness, we have for a individual's creditworthiness:

$$dQ_t = \mu + \sigma dW_t + a_t dY_t \quad (1)$$

Where:

- dQ_t is the variation of creditworthiness in period t ;
- μ is the drift parameter;
- σ is a volatility parameter;
- W_t is a standard Brownian motion;

- a_t is the jump amplitude in period t , where a_t is a i.i.d. variable with distribution $N(\mu_a, \sigma_a)$;
- dY_t is a Poisson process with intensity λ ;
- dW_t , dY_t and a_t are mutually independent.

The three terms on the right side of equation 1 accounts respectively for the drift, volatility and jump effects of creditworthiness. These effects can be attributed to events that affect the consumer's creditworthiness. The drift effect can be attributed to increase in age or in stability (e.g., a person who has been longer in a job is less likely to lose that job, ageing and experience lead to less irresponsible credit behavior). The volatility effect can be due to events related to the consumer credit behaviour and day-to-day changes in the financial condition of the consumer that can cause a fluctuation in the consumer's creditworthiness. Jump effects are due to sporadic low probability events that can cause a sudden drop in the consumer's creditworthiness such as loss of job, divorce, serious disease or other events that can cause serious financial distress.

One other characteristic of the stochastic behavior of creditworthiness is that, the events that drive it are somewhat infrequent, so there can be periods of constant creditworthiness. This suggests using a zero-inflated process for dQ_t as is shown in equation 2:

$$dQ_t = C_t \times (\mu + \sigma dW_t + a_t dY_t) \quad (2)$$

where C_t is a random term that follows a Bernoulli distribution with probability p_c .

If C_t is one then there is a variation in Q value in period t otherwise Q will remain constant in that period. The parameter p_c is the probability of change in Q value in one period and could vary from consumer to consumer. Thus some consumers are more likely to have movements in their creditworthiness (movers) and others are more likely to have periods with constant creditworthiness (stayers). For an overview and references about zero-inflated models refer to Tu (2002).

Eight different alternatives for the creditworthiness stochastic model are presented in Table 1. These are the models presented in equations 1 and 2 and simplifications of these models got by dropping the drift and/or jump effects. Each one of these eight models was tested with the empirical data which will be described in section 3.2.

Table 1 – Alternative stochastic models for creditworthiness.

Model	Drift effect	Volatility effect	Jump Effect	Zero-Inflated
1	Yes	Yes	Yes	Yes
2	Yes	Yes	Yes	No
3	Yes	Yes	No	Yes
4	Yes	Yes	No	No
5	No	Yes	Yes	Yes
6	No	Yes	Yes	No
7	No	Yes	No	Yes
8	No	Yes	No	No

3.2 – A proxy for creditworthiness

Although Q_i is a non-observable quantity there is available information both within lending institutions and in credit bureaus or credit agencies that can be used to evaluate an individual's creditworthiness. This information is mainly related to the credit behaviour of the consumer within the institution or through credit bureau reports their general credit behaviour .

Actually, lending institutions already make wide use of this information for credit assessment by building and using behavioural scores and credit bureau scores that can be interpreted as proxies of the consumer creditworthiness.

Using a behavioural or credit bureau score as a proxy of Q_i we can implement in practice the proposed model for consumer default. From now on we will

substitute Q_i by S_i where S_i is a behavioural or credit bureau score and which is a proxy for crediworthiness. Similarly, the threshold K_{Q_i} will be replaced by K_{S_i} . To select the most appropriate stochastic model we estimate the parameters of each one of the alternative models using time series data of S_i .

The empirical data used in this article were monthly observations of Credit Bureau scores supplied by SERASA for the Brazilian market. SERASA is the leading credit bureau company in Brazil. The data comprised 37 observations of the individual's scores from January 2000 to January 2003 of 1,000 consumers randomly selected from the total number of consumers that had credit activity registered at SERASA.

Besides scoring data, information was available on the occurrence of default for each consumer in the twelve months period after the last score observation (February 2003 to January 2004). This information will be used to validate the models developed. The definition of default is any negative report registered in the publicly available files and the private files managed by SERASA. The negative report could relate to any credit operation of the consumer in the market. Usually a consumer is listed in the negative files when they are between 30 and 60 days past due. Due to this wide definition of default and to characteristics of the Brazilian market the default proportion in the sample was high, namely 35.9%.

Note that a consumer in default is a consumer that had any credit problem in any institution within one year. If we had analysed defaults only in the credit operations of a portfolio the default rate would have been considerably lower. Typical default rates in Brazillian consumer credit portfolios are around 10% to 20%. Besides the Brazilian market has quite high interest rates for consumer credit allowing financial institutions to keep profitable portfolios even with relatively high default rates. According to data available at the Brazilian central bank the average annual interest rates for unsecured personal loans and revolving credit in financial institutions were respectively 85.3% and 159.6% in 2003.

SERASA's credit bureau scores have a scale of 0 to 1000 and can be interpreted as $(1 - \text{probability of default}) \times 1000$. This original data was transformed into the natural log of the odds relation:

$$\text{probability of not defaulting} / \text{probability of defaulting}.$$

We used this transformation so that we could work with a quantity that is not restricted to lower and upper bounds that would make the model construction more difficult.

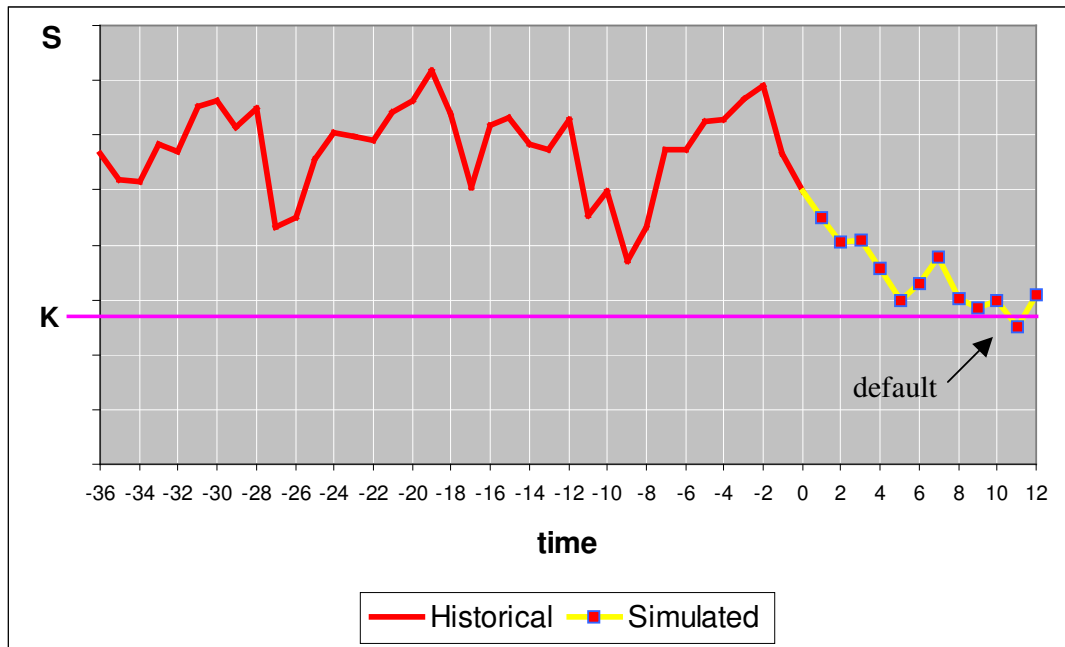
Parameters for each alternative model were estimated using MCMC (Markov Chain Monte Carlo) techniques. The volatility and drift parameters are consumer specific and were estimated in the format of vectors of parameters (one element for each consumer). All other parameters were related to the whole portfolio. MCMC is a simulation technique that uses Bayesian approach and is suitable for parameter estimation of complex non-linear stochastic models. For a description of MCMC technique, algorithms and examples applied to financial econometrics refer to Johannes and Polson (2002). The MCMC method is a suitable choice for estimating parameters of the alternative models since other alternative methods have drawbacks for our models. Standard maximum-likelihood estimation has inconsistencies in models with jumps (Honoré, 1998) and Kalman filtering techniques are not suitable for non-linear non-Gaussian models such as our zero-inflated jump diffusion models.

Comparing the proposed default model with the traditional approach of using a score for risk discrimination, we see that our proposal uses the score and additional information on the stochastic behavior of that score. So we should get better risk discrimination using these option based default model as we are using additional information. To compare the alternative stochastic models for S_i we evaluated how much was the increase in risk discrimination when compared with the traditional scoring approach.

The estimated parameters were used in Monte Carlo simulations of score paths. If the simulated score path of a consumer reaches the barrier K the consumer is considered as in default. Doing many simulations runs, the probability of default of a consumer is just the proportion of the runs when the simulated score path

reached the barrier K . Figure 1 shows how the proposed model can be used for default prediction. In this run the model predicts default because the simulated score path goes lower than the threshold K within the simulate 12 month period.

Figure 1 – Default prediction in a simulated score path.



We assumed previously that K was specific to each consumer, could depend on the debt and might be stochastic. For our initial empirical comparison of the models we simplify these assumptions. We will not estimate K for each consumer, instead we estimate it at a portfolio level.

There are two alternatives to finding such a portfolio value of K :

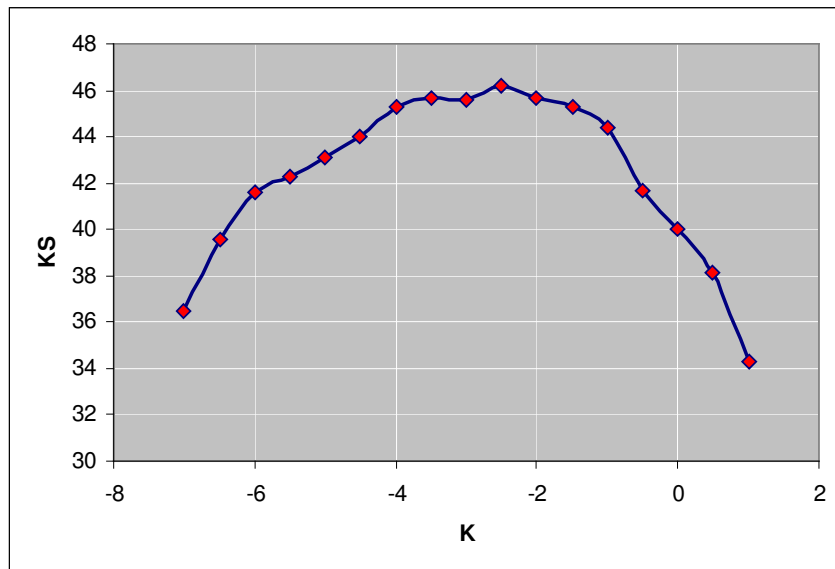
- Set K so that the simulated default rate of the portfolio is equal to the real default rate.
- Set K to maximize the default risk discrimination.

The first alternative is useful when it is necessary to match the predicted probability of default with the empirical probability as in portfolio modelling for capital requirements applications. On the other hand, if the objective is to use the

structural model to improve the risk prediction at an individual level, what really matters is the discrimination between defaulters and non-defaulters. The predicted probability of default works as a ranking measure and can be mapped to real default probabilities by tabulating the proportion of good and bad payers for different bands of predicted default probability in the same way credit scoring modelers do for their scores. In this case the definition of K should be driven by the risk discrimination performance.

Figure 2 shows the relation between K and the Kolmogorov-Smirnov (KS) statistic that is used to assess risk discrimination. The KS statistic measures the maximum difference between the cumulative proportions of defaulters and non-defaulters below a particular score as this score varies. The chosen K should be the one that leads to the highest KS value. The maximum can be reached by a numerical technique like Newton-Raphson or, as we did in this work, simply calculating KS along all values of K with a specified precision in a feasible range.

Figure 2 – Relation between K and the KS statistic.



We estimated K values for each one of the alternative stochastic models. For each consumer in the sample we simulated 10,000 score paths, each of twelve months. Table 2 presents the increase in KS values obtained using the eight

variants of the structural approach compared with the traditional scoring approach for risk discrimination. In each case we chose K as the value that maximized the KS statistic.

Table 2 – KS results for alternative models.

Model	KS	Increase in KS
Behavioural Score (at last observation time)	41.0	0.0
1 – Zero-inflated jump-diffusion	44.9	3.9
2 – Jump-diffusion	44.4	3.4
3 – Zero-inflated diffusion	46.2	5.2
4 – Diffusion	45.9	4.9
5 – Zero-inflated jump-diffusion without drift	45.7	4.7
6 – Jump-diffusion without drift	44.8	3.8
7 – Zero-inflated diffusion without drift	45.7	4.7
8 – Diffusion without drift	46.6	5.6

The results presented above show that best performance in risk discrimination was obtained with the simplest model (diffusion without drift), which considers only a volatility effect. This is a very convenient model since such diffusion models have an analytic solution for the probability of default. The differential equation for this variant of the stochastic model becomes simply:

$$dS_t = \sigma dW_t \quad (3)$$

Using a first passage approach for default occurrence, the first hitting time of a Brownian motion to a barrier has an inverse Gaussian distribution. Using the result presented by Avellaneda e Zhu (2001) and simplifying for a constant barrier (K) we get for a zero drift diffusion model:

$$P(t) = 2\Phi\left(\frac{K - S_0}{\sigma\sqrt{t}}\right) \quad S_0 > K \quad (4)$$

Where:

- $P(t)$ is the probability of default within the time horizon t ;
- K is the default threshold;
- S_0 is the current score of the consumer;
- σ is the standard deviation of the consumer's score;
- $\Phi(\cdot)$ is the cumulative standard normal distribution function.

We can see from equation 4 that for a fixed default horizon t , $(S_0 - K)/\sigma$ gives an equivalent risk ranking to that given by the probability of defaulting within t . The quantity $(S_0 - K)/\sigma$ can be interpreted as the distance to default in standard deviations.

Table 3 shows the actual default percentages for different bands of predicted probability of default. The ordering of the predicted bands does reflect the actual risks, but there is a bias in the predicted values of probability of default for high and low credit quality consumers. The bias can be attributed to the simplification of a unique constant K for the overall portfolio. It also suggests that high credit quality consumers have higher K than the average while low credit quality consumers have lower values of K than the average.

Table 3 – Table of actual defaults by predicted default probability bands.

Predicted Default probability band	Good payers	Bad payers	Percentage of defaults
[0; 0.1[477	108	18.5%
[0.1; 0.3[77	59	43.4%
[0.3; 0.5[25	34	57.6%
[0.5; 0.7[20	29	59.2%
[0.7; 0.9[15	29	65.9%
[0.9; 1.0]	27	100	78.7%
Total	641	359	35.9%

To test the statistical significance of the difference in risk discrimination between the proposed model and the traditional scoring approach we used a bootstrap

procedure to generate the empirical distribution of the difference of KS statistics between both approaches. The Bootstrap method was proposed by Efron (1979) and is a resampling procedure that allows statistical inference of statistics with unknown distribution. We extracted from the original 1,000 consumers sample 50,000 samples with replacement, each one with 1,000 elements. For each sample we calculated the difference of the KS measures, so obtaining 50,000 values that represent a empirical distribution that can be used for inference purposes.

The 1% percentile of the empirical distribution of the difference ($KS_{\text{structural}} - KS_{\text{score}}$) is 0.15. So the results of Table 2 imply we can reject the null hypothesis that these differences are zero for all models at the 1% significance level.

The proposed model requires one to estimate the volatility parameters using historical time series of scores of the individuals. The estimation of these parameters does not use the default data. The only parameter that is estimated using the default data is the default threshold K, and so this is the only variable that needs to be given when one has a new sample to estimate default predictions for. To test if the value of K is generalizable, we ran the model on a 1,000 consumer validation sample, using the original value of K. The increase in KS for the validation sample was slightly smaller than in the original sample, but the improvement was still very statistically significant. The difference between the KS statistic using the proposed model and the traditional scoring approach was 4.3 in the validation sample instead of the 5.6 obtained for the original sample. The value of KS statistic using the traditional scoring and proposed approaches were respectively 39.0 and 43.3. Choosing the K so that the KS statistic in the validation sample is maximized leads to a KS value equal to 43.5.

4 – PORTFOLIO MODELLING

One of the reasons to develop a new approach to the credit risk in lending to individual consumers, is that it gives the basis to develop portfolio level credit risk models. Portfolio credit risk models seek to estimate the distribution of value or

credit loss for a specific portfolio. Structural models supply a theoretical framework for many corporate credit risk portfolio models, including the popular Creditmetrics (Gupton et al., 1997) and Moody-KMV's model (1993a, 1993b).

In this work we concentrate on the distribution of the number of defaults in a portfolio (the default rate). Corresponding loss distributions could be achieved by adding in a model for recovery rates.

Modelling the credit risk of a portfolio of loans means using the multivariate version (each variate corresponding to one loan or one loan class) of the differential equation that underlies the structural model. In the simplest multivariate extension of our model, we use a multivariate normal distribution with correlation matrix Σ to describe joint movements of creditworthiness proxies. Multivariate normal distributions can be simulated by the use of the Cholesk transformation and we use Romano's (Romano 2001) algorithm for generating such simulations. In one simulation run of the paths for all the behavioural scores in a portfolio of loans, the number of defaults in the portfolio is obtained by counting how many consumers had score paths that reached the default barrier K . Running many iterations of the simulation leads to a distribution of the number of defaults for that portfolio. Since it is very important in portfolio modelling to estimate the unbiased values of the probability of default in the portfolio it is recommended that the portfolio level K is chosen so that the empirical and simulated probability of defaults are matched.

Simulating joint score movements for a consumer credit portfolio is intensive in time and computational power. We used a relatively small portfolio (1,000 consumers) and 100,000 joint simulations of the paths of the next 12 months for all the elements of the portfolio took approximately 10 hours in a Pentium4 2.5 Ghz desktop using SAS IML software. We recognize that such an approach for a large portfolio of 1 million consumers is currently impractical.

4.1 – Correlations

One key element in the portfolio approach is the estimation of the correlation since this is the main driver for increasing the variance of the default rate in a large portfolio. In the limiting case of a portfolio with infinite elements, the only source of variance in the portfolio default rate is the correlation among its elements. As consumer credit portfolios can have millions of elements the correlation assessment is very important. We calculated a correlation matrix for the portfolio using the time series of monthly behavioural scores for the 1,000 consumers. We used 36 observations of score variations for each consumer. Surprisingly, the mean of pairwise correlations was very close to zero, 0.00095, when we had expected a more positive mean correlation. One possible reason for such a low value is that we used monthly time intervals. One month might be too short a period to capture the joint influence of external factors in individuals. But even using 6-monthly basic time periods instead of a monthly time period we obtained a mean pairwise correlation of 0.0047, which is still low.

The low average correlation is due to the fact that the scores used as creditworthiness proxies were not able to capture the influence of economic systemic factors, but only the influence of idiosyncratic characteristics of the individuals. There are two possible alternatives to solve that problem:

- Use as creditworthiness proxy scores that take into account economic factors;
- Include the economic systemic influence in the model by an add-on model for systemic risk.

The first alternative is interesting as it could make the results for risk discrimination better and partly correct the bias in values of predicted default probability. Thomas (2003) and Avery et al. (2004) recognize the importance of incorporating economic factors in the credit score. However to build such a scorecard, one requires data on individual's characteristics and their defaults through various economic cycles, which at present is very scarce in the consumer case. The requirements of Basel II will of course eventually provide

such data. So our current option was the use of a simple add-on model for systemic risk, which does have a compatibility with our original interpretation .

4.2 – A simple model for systemic risk

Define:

- S_{ui} as the unconditional and conditional creditworthiness of individual i .
- $S_{ci}(s)$ as the creditworthiness of individual i conditional on the state of economy being s .

Assume S_{ui} and $S_{ci}(s)$ have the following relationship:

$$S_{ci}(s) = S_{ui} + f_i(s) \quad (5)$$

where $f_i(s)$ is a factor that accounts for the influence of the state of the economy, s , on the creditworthiness of individual i . If we consider the influence of the state of economy as homogenous along the individuals we have:

$$S_{ci}(s) = S_{ui} + f(s) \quad (6)$$

So the influence of an economic scenario can be summarized by a homogeneous additive factor to the individual's creditworthiness.

An alternative explanation is to consider creditworthiness to be an intrinsic characteristic of the individual that is not affected directly by economic factors and consider the influence of systemic risk as movements in the default threshold K . So we would have a conditional threshold K_c that is related to the unconditional threshold K_u by:

$$K_c(s) = K_u - f(s) \quad (7)$$

This suggests that as economic conditions worsen, the value of a consumer's reputation drops, because it is more onerous now to service the debt, since cash flow problems increase and there is more uncertainty about the future cash flow

of the consumer. If it is more costly to keep the reputation then its value to the consumer will drop.

Obviously as far as the model of the default process is concerned, the reasoning that underlies equations 6 and 7 lead to the same result. Since $f(s)$ is now part of the portfolio model, it is no longer a pure structural approach, but a kind of hybrid structural-reduced-form approach.

$f(s)$ can be estimated empirically by a Market Default Index (MDI). To do this, we used a time series of balances of consumer credit operations that are available at the central bank of Brazil. The data included monthly balances classified by risk categories and includes all private financial institutions in Brazil. The balances did not include residential mortgage credit operations, which correspond only to 9.4% of the total balance of consumer credit operations in Brazilian private financial institutions. We used balances on loans which were the equivalent of 60 days past due or more as a measure of the balance in default and constructed a time series of default rates that represent our MDI. The period used was from July 1994 to April 2004.

The values of the MDI were sorted by their values and classified in four states of the economy in the following way:

Table 4 – Classification of states of the economy.

State (s)	Interpretation	Observations
1	Very favorable	1 st quartile (25% of observations with lowest default rates)
2	Favorable	2 nd quartile
3	Unfavorable	3 rd quartile
4	Very unfavorable	4 th quartile (25% of observations with highest default rates)

The factor $f(s)$ for each of the four states was calculated by:

$$f(s) = \frac{1}{n_s} \sum_{t \in s} \ln \left(\frac{1 - MDI_t}{MDI_t} \right) - \frac{1}{n} \sum_t \ln \left(\frac{1 - MDI_t}{MDI_t} \right) \quad (8)$$

where n_s is the number of observations of the MDI in state of the economy s and n is the total number of observations of the MDI. Thus $f(s)$ is the difference between the averages of the natural log of the “not default/default” odds in state s compared with the odds averaged over all the states. This transformation is necessary to make $f(s)$ compatible with the scale of S or K . It is strongly related to Shannon’s definition of entropy (Shannon, 1948). The factors obtained for the four states of the economy were:

Table 5 – Additive systemic factors for each state of economy.

State of Economy	$f(s)$
1	0.2795
2	0.1044
3	-0.0023
4	-0.3722

The evolution of the economy over these four states were modeled as a first-order Markov process using monthly time intervals. Table 6 presents the transition matrix estimated from the 118 months of empirical data.

Table 6 – Transition matrix among states of economy.

$s(t)$	$s(t+1)$			
	1	2	3	4
1	89.7%	10.3%	0.0%	0.0%
2	10.3%	69.0%	17.2%	3.5%
3	0.0%	16.7%	73.3%	10.0%
4	0.0%	6.7%	10.0%	83.3%

To test if a first-order Markov-chain is suitable for the evolution of states of the economy we used the test proposed by Anderson and Goldman (1957). We tested the null hypothesis that the transition probability to state s_t conditional on

s_{t-1} ($p(s_t|s_{t-1})$) is equal to the transition probability to s_t conditional on states s_{t-1} and s_{t-2} ($p(s_t|(s_{t-1},s_{t-2}))$) for all states of economy. The statistic:

$$X^2 = \sum_{s_t, s_{t-1}, s_{t-2}} n_{s_{t-1}, s_{t-2}} \frac{[p_{s_t | s_{t-1}} - p_{s_t | (s_{t-1}, s_{t-2})}]^2}{p_{s_t | s_{t-1}}} \quad (9)$$

where $n_{s_{t-1}, s_{t-2}}$ is the number of times that the path $s_{t-2} \rightarrow s_{t-1}$ occur in the series of states desconsidering the last period.

The statistic X^2 has a Chi-square distribution with $m(m-1)$ degrees of freedom, where m is the number of states. The critical value for rejecting the null hypothesis that the process is a first order Markov chain is 51.0 with 5% significance level. The markovity test applied to our data of states of economy evolution resulted in a value of X^2 equal to 10.6, meaning that we cannot reject the null hypothesis and so we are not far from reality in assuming a first order Markov chain model of the economy.

The systemic factor can be easily inserted in the simulation process to produce the distribution of defaults in the portfolio. For each period a migration from the current state of economy (s_t) to the state of economy in the following period (s_{t+1}) is simulated. The variation in scores simulated in that period have a factor $(f(s_{t+1}) - f(s_t))$ added to them, or, equivalently, the default threshold is lowered by $(f(s_{t+1}) - f(s_t))$. In each run of the simulation (one run is the joint simulation of the score paths for all elements of the portfolio) a new migrations of the economy are simulated.

4.3 – Capital requirement under Basel II

The Basel II formula for capital requirement to cover credit risk in retail exposures (BIS, 2004) is based on the work of Vasicek (1991), where he derives an analytic solution for the distribution of default rate of a portfolio of corporate credit. Basel II

applies this formula to retail credit. Vasicek uses Merton's diffusion model and, assuming an infinite number of exposures of equal amounts in the portfolio and equi-correlation among the asset value of the borrowing companies, shows that the cumulative default rate distribution at default rate x is given by:

$$F(x) = \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(\text{PD})\right)\right) \quad (10)$$

Where:

- PD is the mean probability of default of the portfolio;
- ρ is the correlation among firm's assets value;
- Φ and Φ^{-1} are the cumulative standard normal distribution function and its inverse function respectively.

It can be shown (Smithson, 2003) that using the inverse of $F(x)$ the j^{th} -percentile ($j = 100 \times \alpha$) of the distribution of default rate is:

$$x(\alpha) = \Phi\left(\frac{1}{\sqrt{1-\rho}}\Phi^{-1}(\text{PD}) + \sqrt{\frac{\rho}{1-\rho}}\Phi^{-1}(\alpha)\right) \quad (11)$$

Basel's formula for capital requirement in retail credit uses this $x(0.999)$, the default rate in the percentile 99.9, multiplies it by the loss given default (LGD) and subtracts the expected loss to get the required capital:

$$\text{CR} = \text{LGD} \times \Phi\left(\frac{1}{\sqrt{1-\rho}}\Phi^{-1}(\text{PD}) + \sqrt{\frac{\rho}{1-\rho}}\Phi^{-1}(0.999)\right) - \text{PD} \times \text{LGD} \quad (12)$$

According to Basel II the correlation parameter is set to 15% for residential mortgages exposures and 4% for revolving exposures. For other retail credit

exposures the correlation has to be calculated as a weighted average of two extreme values by the following equation:

$$\rho = \rho_{\min} \times \left(\frac{1 - e^{-b \times PD}}{1 - e^{-b}} \right) + \rho_{\max} \times \left[1 - \left(\frac{1 - e^{-b \times PD}}{1 - e^{-b}} \right) \right] \quad (13)$$

where:

$$b = 35;$$

$$\rho_{\min} = 0.03;$$

$$\rho_{\max} = 0.16.$$

To test the Basel formula we try first to estimate the correlation that is implicit in the Brazilian consumer credit market to check if it is compatible with the value of correlation obtained through equation 13. Again we use macroeconomic data from the Brazilian central bank, namely the time series of market default index (MDI) described in section 4.3. The 118 monthly observations of default rate (this time between July 1994 to April 2004) in the MDI represent an empirical distribution of default in the market portfolio of consumer credit in Brazil. This market portfolio follows closely the Vasicek's assumption of an infinite portfolio. This data supplies values of x and $F(x)$ that can be used in equation 10. The probability of default (PD) is the average default rate of the MDI, which turns out to be 14.8% for the period considered. So the only remaining unknown quantity in equation 9 is ρ , and that can be estimated by non-linear regression. Using the same data set again, Table 7 displays the least square analysis for the regression. The parameter ρ was significant with an estimated value of 2.28%.

Using the average default rate of the MDI, 14.8%, in the Basel correlaton formula for other retail exposures we got a correlation of 3.07%. Although we did not have available data on the Brazilian market to test the relationship between PD and correlation in equation 13, at other values of the PD, the Basel estimate for correlations in revolving and other retail

exposures seem to be slightly higher than the correlation implicit in the theoretical model of Vasicek. If the Basel committee has stayed with the correlation formula that was proposed in Consultative Paper 3 (BIS, 2003) the value of the correlations would have been 2.01% for revolving credit and 2.08% for other retail exposures. These values are considerably closer to the implicit correlation in the Vasicek model for the Brazilian consumer credit market.

Table 7 – Least square analysis for implicit correlation estimation.

Source	DF	Sum of Squares	Mean Square	F Value	Approx Pr > F
Regression	1	38.9318	38.9318	3716.10	<.0001
Residual	118	1.2362	0.0105		
Uncorrected Total	119	40.1681			
Corrected Total	118	9.9160			

Parameter	Estimate	Std Error	Approximate 95% Confidence Limits
RHO	0.0228	0.00233	0.0182 0.0275

To compare the results of the Vasicek models with our proposed model we made point estimates of the 99% and 99.9% percentiles of the distribution of default rate for the portfolio of 1,000 consumers used in our empirical work. The results are shown in Table 8. We tested the Vasicek model using the correlation that is proposed by the Basel's committee for other retail credits (since this is the predominant type of credit product in the empirical data) and by using the estimated implicit correlation from the Brazilian consumer credit market.

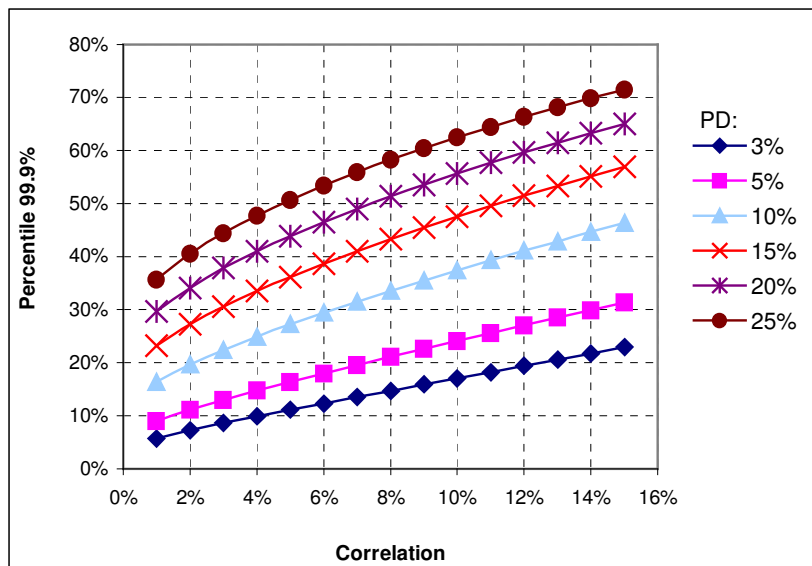
Table 8 – Extreme percentiles derived from Vasicek and proposed models.

<i>Model</i>	<i>Percentile 99% of default rate distribution</i>	<i>Percentile 99.9% of default rate distribution</i>
Proposed model	46.3%	47.9%
Vasicek model with Basel correlation formula	51.7%	57.0%
Vasicek model with estimated implicit correlation	49.6%	54.2%

Comparing the results of our proposed model and the Vasicek model we see that both the latter are more conservative for high percentiles of the default rate distribution. This difference is smaller when we use Vasicek model with the estimated implicit correlation instead of the Basel correlation.

Figure 3 shows the 99.9% percentile point estimate of the default rate distribution using the Vasicek model as a function of different values of the probability of default and of different correlation values. It immediately shows that the correlation value is the key parameter in the Vasicek model and that even small changes in the correlation can cause significant differences in the required capital. Thus using an arbitrary value of correlation for calculating the required capital can be very misleading as it ignores the specific characteristics of different markets, products and portfolios.

Figure 3 – Effect of the correlation and PD on the value of 99.9% percentile predicted by Vasicek model.



5 – CONCLUSION

In this work we have suggested a structural approach to modelling the credit risk of retail lending both at the individual and portfolio level. Our model indicates a way of generalizing the structural corporate credit models to retail credit by substituting for the value of a firm's assets a behavioural score that is a proxy of the individual's creditworthiness.

We have shown that this approach could add significant predictive power to the traditional approach based on behavioural scoring models. Our results for Brazilian credit bureau data revealed that a simple diffusion model with no drift term and no zero-inflation has the best performance in modelling the stochastic behavior of the scores. This surprising result means that there is a simple analytic solution for an individual's probability of default and there is no need for computer intensive simulations.

Much research still needs to be done on structural models for retail credit risk. One promising field is to study more closely extensions of the simple default barrier considered here, by finding ways to differentiate the barrier among segments or elements of a portfolio and by studying the stochastic behaviour of the barrier. Another area worth more study would be to develop more sophisticated models of how to incorporate economic factors into the score so as to improve its use as a proxy for creditworthiness, or alternatively to develop more sophisticated model of how to include them in the dynamics of the default barrier.

Clearly there is also the need to further test and validate structural models in retail credit as the results obtained for one specific portfolio and market may not be generalizable. Tests on portfolios in other markets (especially ones where the level of default is historically much lower) and in other economic periods in order to allow empirical validation of the distribution of default still have to be done before one can consolidate the structural modelling approach for retail credit.

Concerning portfolio modelling and the Basel II Accord, our results support the Basel approach in the sense that, by finding that the simple diffusion models are

adequate for consumer credit modelling, it supports the use of a diffusion based structural model like Vasicek model for retail portfolio modelling. Moreover some of the assumptions of the Vasicek model – large numbers of relatively small and equal loans – seem to be more plausible for retail credit than for corporate credit.

On the other hand we indicated how the capital required under the Vasicek model is very dependent on the correlation coefficients chosen and that the use of a fixed value or fixed formula for these may not be appropriate for all the product types and economic conditions that occur in the world. Comparing the results of the Basel approach with the results of the proposed structural model built on Brazilian empirical data, showed that the Basel approach leads to more conservative results increasing the amount of required capital. One half way house between these two extremes would be if the financial institutions could estimate the correlation parameter of their loan portfolios in the same way that they do for PD, LGD and EAD in the Internal Ratings Based approach. This would lead to capital requirement more closely adjusted to the institution's real distribution of default.

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REFERENCES

- Anderson T. W., & Goodman, L. A. (1957). Statistical inference about Markov chains. *Annals of Mathematical Statistics*, 28, 89-110.
- Athreya, K. (2004). Shame as it ever was: Stigma and personal bankruptcy. *Federal Reserve Bank of Richmond Economic Quarterly*, v.90(2).
- Avery, R. B., Calem, P. S., & Canner, G. B. (2004). *Consumer credit scoring: do situational circumstances matter?*. BIS Working Papers, n. 146.
- Avellaneda, M., & Zhu, J. (2001). Distance to default. *Risk*, 14(12).
- Bank of International Settlements (2003). *Consultative Document*. Basel Committee on Banking Supervision.
- Bank of International Settlements (2004). *International convergence of Capital measurement and Capital Standards*, June 2004. Basel Committee on Banking Supervision.
- Black, F., & Cox, J. C. (1976). Valuing corporate securities: some effects of bond indenture provisions, *Journal of Finance*, 31, 351-367.
- Duffie, D., & Singleton, K. J. (1999). Modeling term structures of defaultable bonds. *Review of Financial Studies*, 12, 687-720.
- Efron, B. (1979). Bootstrap methods: another look at the jackknife. *Annals of Statistics*, 7, 1-26.
- Honoré, P. (1998). Pitfalls in estimating jump-diffusion models. Working Paper. Centre for Analytical Finance. University of Aarhus.
- Jarrow, R. A., & Turnbull, S. (1995). Pricing derivatives on financial securities subject to credit risk. *Journal of Finance*, 50, 53-86.
- Jarrow, R. A., Lando, A., & Turnbull, S. (1997). A Markov model of the term structure of credit risk spreads. *Review of Financial Studies*, 10(2), 481-523.
- Johannes, M., & Polson, N. (2002). *MCMC methods for financial econometrics*. Working Paper. Columbia University.
- Gupton, G. M., Finger, C. C., & Bhatia, M. (1997). *Credit Metrics*. Technical Report. New York: J. P. Morgan & Company.
- KMV Corporation (1993a). *Modeling default risk*. São Francisco: KMV Corporation.
- KMV Corporation (1993b). *Portfolio management of default risk*. São Francisco: KMV Corporation.

- Longhofer, S., & Peters, S. (2004). Self-selection and discrimination in credit markets. Working paper.
- Merton, Robert C. (1974). On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance*, 29, 449-470.
- Perli, R., & Nayda, W. I. (2004). Economic and Regulatory Capital Allocation for revolving retail exposures. *Journal of Banking and Finance*, 28, 789-809.
- Romano C. (2001). *Applying copula function to risk management*. Working paper.
- Saunders A. (1999). Credit risk measurement : new approaches to value-at-risk and other paradigms, New York: Wiley.
- Shannon, C. E. (1948). A Mathematical Theory of Communication. *The Bell System Technical Journal*, 27, 379-423 and 623-656.
- Smithson, C. (2003). *Credit portfolio management*. Hoboken: John Wiley.
- Thomas, L. C. (2003). Consumer credit modelling: context and current issues. Working Paper presented on the Banff Credit Risk Conference 2003, Banff International Research Station.
- Tu, Wanzhu (2002). Zero inflated data. In: *Encyclopedia of environmetrics*. Ed. El-Shaarawi, A. H. & Piegorisch, W. W., Chichester: John Wiley & Sons, v.4, 2387-2391.
- Vasicek O. (1991). *Limiting loan loss probability distribution*. KMV Corporation.
- Zhou, C. (1997). *A jump-diffusion approach to modeling credit risk and valuing defaultable securities*. Working paper 1997-15. Federal Reserve Board.