Convex Imprecise Previsions for Risk Measurement

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Abstract. In this paper we introduce convex imprecise previsions as a special class of imprecise previsions, showing that they retain or generalise most of the relevant properties of coherent imprecise previsions but are not necessarily positively homogeneous. The broader class of weakly convex imprecise previsions is also studied and its fundamental properties are demonstrated. The notions of weak convexity and convexity are then applied to risk measurement, leading to a more general definition of convex risk measure than the one already known in risk measurement literature.

Keywords: imprecise previsions, risk measures, weakly convex imprecise previsions, convex imprecise previsions.

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1. Introduction

Theories of imprecise probabilities have been increasingly studied and developed in recent years, due to their generality and greater ability to dependably handle uncertainty with respect to more traditional tools. The advantages of an imprecise probability approach are more patent when opinions are based on imprecise or partly conflicting information or beliefs, as often happens in practical problems, or when some measure of our degree of ignorance or uncertainty should be supplied.

The book by P. Walley [9] is a fundamental reference in this area. In [9], imprecise probability theory is developed in terms of *imprecise previsions*, and two major classes of (unconditional) imprecise previsions are considered, relying upon reasonable consistency requirements: *avoiding sure loss* and *coherent* previsions. The avoiding sure loss condition is less restrictive than coherence but many of its properties are often too weak.

Because of their generality, coherent imprecise previsions encompass several other uncertainty measures as special cases [10], including belief functions, possibility or necessity measures, 2-monotone probabilities, precise probabilities and others. Many of these uncertainty formalisms have been by now developed and studied extensively, and were often introduced prior to the theory of imprecise previsions.

Avoiding sure loss previsions received less attention, and it is an interesting question to state whether some special class of avoiding sure loss previsions can be identified, which is such that

(a) its properties are not too far from those of coherent previsions;

(b) it may express beliefs which do not match with coherence but which are useful in formalising and dependably modelling certain kinds of problems.

This is precisely the main purpose of this paper. After recalling some basic notions about imprecise previsions in Section 2, two classes of previsions, weakly convex and convex previsions, are introduced in Section 3 and their fundamental properties are demonstrated. To make comparisons with the theory developed in [9] simpler, we refer to lower previsions throughout Section 3. The theory for upper previsions is specular.

It turns out that weakly convex previsions, whose properties are discussed in Section 3.1, do not necessarily avoid sure loss. The main reason for studying them is that they already retain some interesting properties of coherent previsions. In particular, a notion of convex natural extension may be defined, which is close to that of natural extension, a basic tool of the theory in [9]. As discussed in Section 3.1, the convex natural extension points out in addition a least-committal way to 'correct' certain classes of incoherent or incurring sure loss previsions into weakly convex or also into convex previsions.

Convex previsions are studied in Section 3.2, while generalisations of the important envelope theorem are presented in Section 3.3. Convex previsions always avoid sure loss and share with coherence some additional properties, so they seem appropriate to fulfill the requirement (a). As for (b), we show that, unlike coherent imprecise previsions, convex previsions do not necessarily require the *positive homogeneity* axiom, which requires for a measure μ that

$$\mu(\lambda X) = \lambda\mu(X), \forall\lambda > 0. \tag{1}$$

An upper (lower) imprecise prevision is often interpreted [9] as an infimum selling (supremum buying) price for an agent to exchange a random number X. Positive homogeneity expresses the agent's indifference towards the order of magnitude of the random numbers which are exchanged. However, positive homogeneity cannot always be assumed in real-world exchanges. For instance, if X is a financial asset, it is often the case that the selling price of λX is less than λ times the price for selling X, if λ is large enough, because of what is termed *liquidity risk.*

Consequently, one might expect that also a *risk measure* (that is, roughly speaking, a real function measuring how risky some random numbers are) should not necessarily satisfy (1). In fact, a new class of risk measures, termed convex risk measures, was recently proposed in the risk theory literature [5, 6, 7] to tackle this problem.

Questions in the risk measurement area are closely related to imprecise previsions theory because, as shown in [8], a risk measure may be viewed as a special case of upper prevision. Consequently risk measurement problems can often be treated in the framework of the theory of imprecise previsions. This is done for convex risk measures in Section 4, where (weakly convex and) convex risk measures are defined in terms of (weakly convex and) convex imprecise previsions, and it is shown that their properties can be easily obtained from the theory developed in Section 3. Since weakly convex and convex measures are defined in Section 4 for *arbitrary* sets of random numbers, they generalise the notion of convex risk measure already known in the literature [5, 6], which is defined only on linear spaces of random numbers using a set of axioms. We show that this notion corresponds, on linear spaces, to weak convexity. The results presented in Section 4 also hold referring to arbitrary sets of random numbers, thus improving the generality of convex risk measures.

2. Imprecise Previsions

The purpose of this section is to recall concisely basic facts about imprecise (and precise) previsions, emphasising those aspects which will be needed in later sections.

Imprecise and precise previsions are extensively studied, respectively, in [9] and [4].

Given an arbitrary set $\mathcal{D}$ of bounded random numbers, a *lower prevision* $\underline{P}$ (an *upper prevision* $\overline{P}$, a *prevision* P) is a real-valued function with domain $\mathcal{D}$.

If the set $\mathcal{D}$ contains only indicator functions of events, $\underline{P}$ ($\overline{P}$, P) is termed lower probability (upper probability, probability). Hence, imprecise previsions are a more general tool than imprecise probabilities, even though it is currently customary to use the term 'imprecise probabilities' to indicate a group of theories which includes imprecise previsions theory.

Lower (and upper) previsions should satisfy some consistency requirements, widely discussed in [9], which give rise to the *avoiding sure loss* condition and the stronger *coherence* condition.

DEFINITION 1. *Given a set $\mathcal{D}$ of bounded random numbers, a mapping $\underline{P}$ from $\mathcal{D}$ into $I\!R$ is a lower prevision on $\mathcal{D}$ that* avoids sure loss *iff, for all $n \in I\!N^+$, for each $X_1, \ldots, X_n \in \mathcal{D}$, for each $s_1, \ldots, s_n$ real and non-negative, defining $\underline{G} = \sum_{i=1}^n s_i(X_i - \underline{P}(X_i))$, it is $\sup \underline{G} \geq 0$.*

REMARK 1. *We would state an equivalent definition for the avoiding sure loss condition adding the normalisation constraint $\sum_{i=1}^n s_i = 1$ in Definition 1. This fact will be used later.*

REMARK 2. *Among the properties of avoiding sure loss lower previsions, we shall need the following*

(a) $\underline{P}(X) \leq \sup X$, $\forall X \in \mathcal{D}$

(b) $\underline{P}(X) \leq -\underline{P}(-X)$, $\forall X, -X \in \mathcal{D}$.

The avoiding sure loss condition is too weak under many respects: for instance, it does not require that $\underline{P}(X) \geq \inf X$, nor does it impose monotonicity. On the other hand, it is simpler to assess and to check than coherence.

DEFINITION 2. *Given an arbitrary set $\mathcal{D}$ of bounded random numbers, a mapping $\underline{P}$ from $\mathcal{D}$ into $I\!R$ is a* coherent lower prevision *for the random numbers in $\mathcal{D}$ iff, for all $n \in I\!N^+$, for each $X_0, X_1, \ldots, X_n \in \mathcal{D}$, for each $s_0, s_1, \ldots, s_n$ real and non-negative, defining $\underline{G} = \sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) - s_0(X_0 - \underline{P}(X_0))$, it is $\sup \underline{G} \geq 0$.*

Behaviourally, a lower prevision assessment $\underline{P}(X)$ may be viewed as a supremum buying price for X [9], and $s(X - \underline{P}(X))$ represents an *elementary gain* from a bet on X, with stake s. We shall say that the bet is *in favour* of X if $s \geq 0$, whilst $-s(X-\underline{P}(X))$ $(s \geq 0)$ is an elementary gain from a bet *against* X.[1] Definitions 1 and 2 both require that no admissible linear combination $\underline{G}$ of elementary gains originates a sure loss bounded away from zero. The difference is that the avoiding sure loss concept considers only bets in favour of the X_i, while coherence considers also (at most) one bet against a random number in $\mathcal{D}$.

Coherent precise previsions may be defined by modifying Definition 2 to allow $n \geq 0$ bets in favour of and $m \geq 0$ bets against random numbers in $\mathcal{D}$ $(m, n \in \mathbb{N})$. A coherent precise prevision P is necessarily *linear* and *homogeneous*: $P(aX + bY) = aP(X) + bP(Y)$, $\forall a, b \in \mathbb{R}$, whenever all random numbers involved are in $\mathcal{D}$.

REMARK 3. *We recall the following properties of coherent lower previsions:*

(a) $\underline{P}(\lambda X) = \lambda \underline{P}(X)$, $\forall \lambda > 0$ *(positive homogeneity)*

(b) $\inf X \leq \underline{P}(X) \leq \sup X$ *(internality)*

(c) $\underline{P}(X + Y) \geq \underline{P}(X) + \underline{P}(Y)$ *(superlinearity).*

Avoiding sure loss and coherent lower previsions may be characterised using precise previsions as follows.

THEOREM 1. *Let* $\underline{P}$ *be a lower prevision on* $\mathcal{D}$.

(a) $\underline{P}$ *avoids sure loss on* $\mathcal{D}$ *if and only if there exists a coherent precise prevision* P *on* $\mathcal{D}$ *such that* $P(X) \geq \underline{P}(X)$, $\forall X \in \mathcal{D}$

(b) (lower envelope theorem) $\underline{P}$ *is coherent on* $\mathcal{D}$ *if and only if* $\underline{P}$ *is the lower envelope of some set* $\mathcal{M}$ *of coherent precise previsions on* $\mathcal{D}$*, i.e. if and only if*

$$\underline{P}(X) = \inf_{P \in \mathcal{M}} \{P(X)\}, \forall X \in \mathcal{D} \quad \textit{(inf is attained).}$$

Upper previsions are customarily related to lower previsions by the *coniugacy* relation

$$\overline{P}(X) = -\underline{P}(-X). \tag{2}$$

[1] This terminology originates from the fact that when X is an indicator function of some event E, a bettor maximises his gain on the bet in favour (against) X when E is true (is false).

The theory of imprecise previsions can be developed in quite an analogous way referring to upper rather than lower previsions. An upper prevision $\overline{P}(X)$ may be viewed as an infimum selling price for X and an *elementary gain* from a bet concerning X is written as $s(\overline{P}(X)-X)$. The definitions of the avoiding sure loss condition and coherence are modified accordingly. To make an example we report the definition of coherence.

DEFINITION 3. *Given an arbitrary set $\mathcal{D}$ of bounded random numbers, a mapping $\overline{P}$ from $\mathcal{D}$ into $\mathbb{R}$ is a* coherent upper prevision *for the random numbers in $\mathcal{D}$ iff, for all $n \in \mathbb{N}^+$, for each $X_0, X_1, \ldots, X_n \in \mathcal{D}$, for each $s_0, s_1, \ldots, s_n$ real and non-negative, defining $\underline{G} = \sum_{i=1}^n s_i (\overline{P}(X_i) - X_i) - s_0(\overline{P}(X_0) - X_0)$, it is* $\sup \underline{G} \geq 0$.

3. Convex Lower Previsions

3.1. Weakly convex previsions and convex natural extension

DEFINITION 4. *Given a set $\mathcal{D}$ of (bounded) random numbers, a mapping $\underline{P}$ from $\mathcal{D}$ into $\mathbb{R}$ is a* weakly convex lower prevision *on $\mathcal{D}$ iff, for all $n \in \mathbb{N}^+$, for each $X_0, X_1, \ldots, X_n \in \mathcal{D}$, for each $s_1, \ldots, s_n$ real and non-negative such that* $\sum_{i=1}^n s_i = 1$ (convexity condition)*, defining $\underline{G} = \sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) - (X_0 - \underline{P}(X_0))$, it is* $\sup \underline{G} \geq 0$.

Any coherent lower prevision is weakly convex, since Definition 4 is obtained from Definition 2 adding the constraint $\sum_{i=1}^n s_i = s_0 = 1$ (note that we would get a definition equivalent to Definition 4 requiring only $\sum_{i=1}^n s_i = s_0 > 0$).

Conversely, a weakly convex lower prevision is not always coherent, nor does it necessarily avoid sure loss, as shown by the following proposition.

PROPOSITION 1. *Let $\underline{P}$ be a weakly convex lower prevision on $\mathcal{D}$ and let $0 \in \mathcal{D}$. Then $\underline{P}$ avoids sure loss iff* $\underline{P}(0) \leq 0$.

Proof. If $\underline{P}$ avoids sure loss, then necessarily $\underline{P}(0) \leq 0$ by Remark 2, (a). Conversely, if $\underline{P}$ is weakly convex we obtain, putting $X_0 = 0$ in Definition 4, $\sup \sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) + \underline{P}(0) \geq 0, \forall X_1, \ldots, X_n, \forall s_1, \ldots, s_n \geq 0$ such that $\sum_{i=1}^n s_i = 1$. Recalling also Remark 1, this implies that $\underline{P}$ avoids sure loss, since then $\sup \sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) \geq -\underline{P}(0) \geq 0$. □

Weak convexity can be characterised by a set of axioms if $\mathcal{D}$ has a special structure, like in the following theorem (see also Example 2 in Section 4).

THEOREM 2. *Let $\mathcal{L}$ be a linear space of bounded random numbers containing real constants. A mapping $\underline{P}$ from $\mathcal{L}$ into $I\!R$ is a weakly convex lower prevision on $\mathcal{L}$ iff it satisfies the following axioms:*

(T) $\underline{P}(X+c) = \underline{P}(X)+c, \forall X \in \mathcal{L}, \forall c \in I\!R$ *(translation invariance)*

(M) $\forall X, Y \in \mathcal{L}$, *if* $Y \leq X$ *then* $\underline{P}(Y) \leq \underline{P}(X)$ *(monotonicity)*

(C) $\underline{P}(\lambda X + (1-\lambda)Y) \geq \lambda \underline{P}(X) + (1-\lambda)\underline{P}(Y), \forall X, Y \in \mathcal{L}$, $\forall \lambda \in [0,1]$ *(concavity).*

Proof. Let us prove that (T), (M) and (C) imply weak convexity for $\underline{P}$. Let $n \geq 1$, $s_i \geq 0$ $(i = 1, \ldots, n)$ such that $\sum_{i=1}^n s_i = 1$, $X_i \in \mathcal{L}$ $(i = 0, \ldots, n)$. Define $Y = \sum_{i=1}^n s_i X_i$ and $Z = Y - X_0$. From $Y = X_0 + Z \leq X_0 + \sup Z$, by (M) and (T), $\underline{P}(Y) \leq \underline{P}(X_0 + \sup Z) = \underline{P}(X_0) + \sup Z$. Hence $\sup Z \geq \underline{P}(Y) - \underline{P}(X_0) \geq \sum_{i=1}^n s_i \underline{P}(X_i) - \underline{P}(X_0)$ by (C). It follows $\sup\{\sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) - (X_0 - \underline{P}(X_0))\} \geq 0$, so that $\underline{P}$ is weakly convex.

Conversely, suppose $\underline{P}$ is weakly convex. To prove (T), let $n = 1$ (hence $s_1 = 1$), $X_1 = X + c, X_0 = X$ in Definition 4, from which $\underline{P}(X + c) \leq \underline{P}(X) + c$ follows. Interchange $X + c$ with X to get the reverse inequality $\underline{P}(X + c) \geq \underline{P}(X) + c$. To show that (M) holds, suppose $Y \leq X$ and take $n = 1, X_1 = Y, X_0 = X$ in Definition 4. It follows $\sup(Y - X) \geq \underline{P}(Y) - \underline{P}(X)$ and hence $\underline{P}(X) \geq \underline{P}(Y)$, since $\sup(Y - X) \leq 0$. As for (C), take $n = 2, s_1 = \lambda, s_2 = 1 - \lambda, X_1 = X, X_2 = Y, X_0 = \lambda X + (1-\lambda)Y$ $(\lambda \in [0,1])$ in Definition 4. □

REMARK 4. *Axioms (T) and (M) in Theorem 2 are equivalent on $\mathcal{L}$ to the following one:*

$$\textit{(TM)} \quad \forall X, Y \in \mathcal{L}, \underline{P}(X) - \underline{P}(Y) \leq \sup(X - Y).$$

In fact, suppose at first that (M) and (T) hold. For every $X, Y \in \mathcal{L}$ it is $X \leq Y + \sup(X - Y)$. Hence $\underline{P}(X) \leq \underline{P}(Y + \sup(X - Y)) = \underline{P}(Y) + \sup(X - Y)$ by (M) and (T). Conversely, let (TM) be true; if $Y \leq X$, it is $\sup(Y - X) \leq 0$ and therefore $\underline{P}(Y) - \underline{P}(X) \leq 0$, which proves monotonicity. To prove (T), let $c \in I\!R$. By (TM), it is $\underline{P}(X) - \underline{P}(X + c) \leq \sup(X - (X + c)) = -c$ and, symmetrically, $\underline{P}(X + c) - \underline{P}(X) \leq \sup(X + c - X) = c$. Hence $\underline{P}(X + c) = \underline{P}(X) + c$.

The results in the next proposition are quite analogous to corresponding properties of avoiding sure loss and coherent lower previsions ([9], Sections 2.6.4 and 2.6.5) and can be proved in a similar way. We shall follow an alternative technique, used in [3] referring to precise probabilities.

These results also point out ways of obtaining new weakly convex lower previsions from given ones. Another powerful tool for generating weakly convex lower previsions will be discussed in Section 3.3.

PROPOSITION 2.

(a) (Convergence theorem) *Let* $\{\underline{P}_j\}_{j=1}^{+\infty}$ *be a sequence of lower previsions, weakly convex on* $\mathcal{D}$ *and such that* $\forall X \in \mathcal{D}$ *there exists* $\lim_{j\to+\infty} \underline{P}_j(X) = \underline{P}(X)$. *Then* $\underline{P}$ *is weakly convex on* $\mathcal{D}$.

(b) (Convexity theorem) *If* $\underline{P}_1$ *and* $\underline{P}_2$ *are weakly convex lower previsions on* $\mathcal{D}$ *and* $\lambda \in [0, 1]$, *their convex combination* $\underline{P}(X) = \lambda\underline{P}_1(X) + (1-\lambda)\underline{P}_2(X)$ *is weakly convex on* $\mathcal{D}$.

Proof. Prior to proving (a) and (b), consider a random gain $\underline{G}_j = \sum_{i=1}^n s_i(X_i - \underline{P}_j(X_i)) - (X_0 - \underline{P}_j(X_0))$, where $n, s_1, \ldots, s_n, X_0, \ldots, X_n$ are fixed, while $\underline{P}_j$ is not, and let $I\!P$ be any partition whose events describe all possible outcomes of $\underline{G}_j$. Clearly, $\forall\varepsilon > 0$ there exists $\omega_\varepsilon \in I\!P$ such that

$$\sup \underline{G}_j - \underline{G}_j(\omega_\varepsilon) < \varepsilon. \tag{3}$$

Defining $S_j = \underline{P}_j(X_0) - \sum_{i=1}^n s_i\underline{P}_j(X_i)$, $R_j = \sum_{i=1}^n s_iX_i - X_0$, it is $\underline{G}_j = S_j + R_j$. Note that S_j is non-random and a function of $\underline{P}_j$, whilst R_j is random but does not depend on P_j. Hence

$$\sup \underline{G}_j - \underline{G}_j(\omega_\varepsilon) = \sup_{\omega\in I\!P}(S_j + R_j(\omega)) - S_j - R_j(\omega_\varepsilon) = \sup R_j - R_j(\omega_\varepsilon),$$

that is $\sup \underline{G}_j - \underline{G}_j(\omega_\varepsilon)$ does not depend on $\underline{P}_j$. Hence, given a family $\{\underline{G}_j\}$ of random gains which may differ from each other only because of different $\underline{P}_j$, for any $\varepsilon > 0$ there exists $\omega_\varepsilon \in I\!P$ such that

$$\sup \underline{G}_j - \underline{G}_j(\omega_\varepsilon) < \varepsilon, \forall\underline{G}_j. \tag{4}$$

If further $\sup \underline{G}_j \geq 0, \forall\underline{G}_j$, then

$$\underline{G}_j(\omega_\varepsilon) > -\varepsilon, \forall\underline{G}_j. \tag{5}$$

To prove now (a), put $s_0 = -1$. Then, any gain $\underline{G}$ concerning $\underline{P}$ in Definition 4 may be written as

$$\underline{G} = \sum_{i=0}^n s_i(X_i - \underline{P}(X_i)) =$$

$$\sum_{i=0}^{n} s_i(X_i - \lim_{j\to+\infty} \underline{P}_j(X_i)) = \lim_{j\to+\infty} \sum_{i=0}^{n} s_i(X_i - \underline{P}_j(X_i)) = \lim_{j\to+\infty} \underline{G}_j.$$

For any $\varepsilon > 0$, the conditions hold for applying (5) to $\{\underline{G}_j\}_{j=1}^{+\infty}$, so that there exists $\omega_\varepsilon \in I\!P$ such that $\underline{G}(\omega_\varepsilon) = \lim_{j\to+\infty} \underline{G}_j(\omega_\varepsilon) \geq -\varepsilon$. This implies $\sup \underline{G} \geq 0$, and hence $\underline{P}$ is weakly convex by Definition 4.

To prove (b), putting again $s_0 = -1$ any gain $\underline{G}$ concerning $\underline{P}$ in Definition 4 can be written as

$$\underline{G} = \sum_{i=0}^{n} s_i(X_i - \underline{P}(X_i)) =$$

$$\lambda \sum_{i=0}^{n} s_i(X_i - \underline{P}_1(X_i)) + (1-\lambda) \sum_{i=0}^{n} s_i(X_i - \underline{P}_2(X_i)) = \lambda \underline{G}_1 + (1-\lambda)\underline{G}_2.$$

By weak convexity of $\underline{P}_1$ and $\underline{P}_2$, $\sup \underline{G}_j \geq 0$ $(j = 1, 2)$. Applying therefore (5) to $\{\underline{G}_1, \underline{G}_2\}$, for any $\varepsilon > 0$ there exists ω_ε such that $\underline{G}_j(\omega_\varepsilon) > -\varepsilon$. It follows $\underline{G}(\omega_\varepsilon) > \lambda(-\varepsilon) + (1-\lambda)(-\varepsilon) = -\varepsilon$, hence $\sup \underline{G} \geq 0$. □

Definition 4 lets us define a weakly convex prevision on any arbitrary set $\mathcal{D}$ of bounded random numbers. It is clearly important to see whether $\underline{P}$ can be extended to a weakly convex prevision $\underline{P}'$ on any superset of $\mathcal{D}$. The first step is to consider the case of a superset $\mathcal{D}' = \mathcal{D} \cup \{Z\}$ and to examine the range of values $\underline{P}'(Z)$ can assume.

PROPOSITION 3. *Let $\mathcal{D}$ be a non-empty set of bounded random numbers, $\underline{P}$ from $\mathcal{D}$ into $I\!R$ a weakly convex prevision, $Z \notin \mathcal{D}$ an arbitrary bounded random number, $\mathcal{D}' = \mathcal{D} \cup \{Z\}$. Define*

$$U = \left\{ \alpha : \alpha - Z \geq \sum_{i=1}^{n} s_i(X_i - \underline{P}(X_i)) - s_0(X_0 - \underline{P}(X_0)), \right.$$
$$\left. \textit{for some } n \geq 0, X_i \in \mathcal{D}, s_i \geq 0, \sum_{i=1}^{n} s_i + 1 = s_0 \right\},$$

$$L = \left\{ \alpha : Z - \alpha \geq \sum_{i=1}^{n} s_i(X_i - \underline{P}(X_i)), \right.$$
$$\left. \textit{for some } n \geq 1, X_i \in \mathcal{D}, s_i \geq 0, \sum_{i=1}^{n} s_i = 1 \right\}$$

and let $\underline{P}^U(Z) = \inf U$, $\underline{P}^L(Z) = \sup L$. Then $\underline{P}'$ is a weakly convex extension of $\underline{P}$ on $\mathcal{D}'$ iff $\underline{P}' = \underline{P}$ on $\mathcal{D}$ and $\underline{P}'(Z) \in [\underline{P}^L(Z), \underline{P}^U(Z)]$.

Proof. U and L are non-empty. In fact, by putting $n = 0$, $s_0 = 1$, $X_0 = X \in \mathcal{D}$ in the definition of U, it follows $\alpha \in U$ for $\alpha \geq \sup Z - \inf X + \underline{P}(X)$. Analogously, by putting $n = 1$, $s_1 = 1$, $X_1 = X \in \mathcal{D}$ in the definition of L, it is $\alpha \in L$ for $\alpha \leq \inf Z - \sup X + \underline{P}(X)$.

Let now $\alpha_U \in U$, $\alpha_L \in L$. If it were $\alpha_U < \alpha_L$, there should exist $n \geq 0, m \geq 1, X_0, \ldots, X_n, Y_1, \ldots, Y_m \in \mathcal{D}$, $s_0, \ldots, s_n$, $t_1, \ldots, t_m$ with $s_i \geq 0$, $t_j \geq 0$, $\sum_{i=1}^{n} s_i + 1 = s_0$, $\sum_{j=1}^{m} t_j = 1$ such that, putting $\underline{G}^U = \sum_{i=1}^{n} s_i(X_i - \underline{P}(X_i)) - s_0(X_0 - \underline{P}(X_0)) + Z$ and $\underline{G}^L = Z - \sum_{j=1}^{m} t_j(Y_j - \underline{P}(Y_j))$, it is $\sup \underline{G}^U \leq \alpha_U < \alpha_L \leq \inf \underline{G}^L$; hence $\sup \underline{G}^U - \inf \underline{G}^L < 0$. Defining $\underline{G} = (\underline{G}^U - \underline{G}^L)/s_0$, $\underline{G}$ has the form required in Definition 4, but it is $\sup \underline{G} \leq (\sup \underline{G}^U - \inf \underline{G}^L)/s_0 < 0$, which contradicts the weak convexity of $\underline{P}$ on $\mathcal{D}$. Hence $\alpha_U \geq \alpha_L$.

It ensues from what was proved so far that $[\underline{P}^L(Z), \underline{P}^U(Z)]$ is a bounded and non-empty interval.

We prove now, using Definition 4, that $\underline{P}'(Z) = k \in [\underline{P}^L(Z), \underline{P}^U(Z)]$ implies weak convexity of P' on $\mathcal{D}'$. It is sufficient for this to consider those gains $\underline{G}^*$ concerning $\underline{P}'$ which include exactly one elementary bet on Z, either against or in favour of Z.[2] In the former case, if it were $\sup\{\sum_{i=1}^{n} s_i(X_i - \underline{P}(X_i)) - (Z - k)\} < 0$ for some $n \geq 1$, $X_i \in \mathcal{D}$, $s_i \geq 0$ $(i = 1 \ldots, n)$ such that $\sum_{i=1}^{n} s_i = 1$, there should exist $\varepsilon > 0$ such that $\sum_{i=1}^{n} s_i(X_i - \underline{P}(X_i)) < Z - (k + \varepsilon)$, with $k + \varepsilon > \underline{P}^L(Z) = \sup L$, a contradiction. In the latter, when a bet in favour of Z is considered $k \leq \underline{P}^U(Z)$ implies $\sup\{\sum_{i=1}^{n} s_i(X_i - \underline{P}(X_i)) + t(Z - k) - (X_0 - \underline{P}(X_0))\} \geq 0$ for any choice of $n \geq 0$, $X_i \in \mathcal{D}$, $s_i \geq 0$, $t > 0$ such that $\sum_{i=1}^{n} s_i + t = 1$. Condition $k \in [\underline{P}^L(Z), \underline{P}^U(Z)]$ is therefore sufficient for weak convexity of $\underline{P}'$. In a similar way, it can be shown that this condition is also necessary for weak convexity of $\underline{P}'$, thus concluding the proof. □

Although weak convexity of $\underline{P}$ is sufficient for $\underline{P}^L$ to be well-defined, there are other conditions that guarantee it. For example, if $\underline{P}$ avoids sure loss on $\mathcal{D}$, it is $\sup(Z - \alpha) \geq \sup \sum_{i=1}^{n} s_i(X_i - \underline{P}(X_i)) \geq 0$ for each $\alpha \in L$ and therefore $\underline{P}^L(Z) \leq \sup Z$. Another simple condition is the existence of $k \in \mathbb{R}$ such that $X - \underline{P}(X) \geq k$ $\forall X \in \mathcal{D}$. In this case, which is always true when $\mathcal{D}$ is finite, $\underline{P}^L(Z) \leq \inf Z - k$ follows easily.

$\underline{P}^L$ can therefore be considered also in situations in which the assessment $\underline{P}$ is not weakly convex.

DEFINITION 5. *Let $\underline{P}$ from $\mathcal{D}$ into $\mathbb{R}$ be a lower prevision, $\mathbb{P}$ any partition whose events describe all possible outcomes of the random*

[2] It is easily seen that the supremum of a gain $\underline{G}^{**}$ including one bet in favour and one against Z has the same sign of the supremum of a corresponding gain $\underline{G}^*$ including just one elementary bet against Z, obtained from $\underline{G}^{**}$ by renormalising the stakes. Hence we need not consider the gains of the kind of $\underline{G}^{**}$.

numbers in $\mathcal{D}$, Z a bounded random number defined on $I\!P$. $\underline{P}^L(Z)$, when finite, is termed the convex natural extension *of $\underline{P}$ on Z.*

The following theorem shows that the convex natural extension of a given lower prevision $\underline{P}$, when defined, lets us 'correct' it into a weakly convex lower prevision in a least-committal way. Besides, when $\underline{P}$ is weakly convex $\underline{P}^L$ extends it on any superset $\mathcal{D}' \supset \mathcal{D}$.

THEOREM 3. *Let $\underline{P}$ from $\mathcal{D}$ into $I\!R$ be a lower prevision, $I\!P$ any partition whose events describe all possible outcomes of the random numbers in $\mathcal{D}$, $\mathcal{L}(\supset \mathcal{D})$ the set of all bounded random numbers defined on $I\!P$. Suppose also that the convex natural extension $\underline{P}^L(Z)$ exists for every $Z \in \mathcal{L}$. Then, $\underline{P}^L$ satisfies the following properties:*

- *(a) If $\underline{P}$ avoids sure loss, $\underline{P}^L(Z) \leq \sup Z, \forall Z \in \mathcal{L}$*
- *(b) $\underline{P}^L(Z) \geq \inf Z - \sup X + \underline{P}(X), \forall Z \in \mathcal{L}, \forall X \in \mathcal{D}$*
- *(c) $\underline{P}^L$ is a weakly convex prevision on $\mathcal{L}$*
- *(d) $\underline{P}^L(X) \geq \underline{P}(X), \forall X \in \mathcal{D}$*
- *(e) $\underline{P}$ is weakly convex if and only if $\underline{P}^L = \underline{P}$ on $\mathcal{D}$*
- *(f) If $\underline{P}^*$ is a weakly convex prevision on $\mathcal{L}$ such that $\underline{P}^*(X) \geq \underline{P}(X)$ $\forall X \in \mathcal{D}$, then $\underline{P}^*(Z) \geq \underline{P}^L(Z), \forall Z \in \mathcal{L}$*
- *(g) If $\underline{P}$ is weakly convex then $\underline{P}^L$ is the minimal weakly convex extension of $\underline{P}$ to $\mathcal{L}$*
- *(h) $\underline{P}$ avoids sure loss on $\mathcal{D}$ if and only if $\underline{P}^L$ avoids sure loss on $\mathcal{L}$.*

Proof. Property (a) has been proved previously.

Property (b) follows from $\inf Z - \sup X + \underline{P}(X) \in L, \forall X \in \mathcal{D}, \forall Z \in \mathcal{L}$.

The proofs of $(c) \div (g)$ can be achieved modifying slightly the proofs of the corresponding properties for the natural extension in [9], Section 3.1.2. We stress, among the most relevant modifications, the use of Theorem 2 to prove the weak convexity of $\underline{P}^L$ on $\mathcal{L}$.

As for (h), if $\underline{P}$ avoids sure loss then $\underline{P}^L(0) \leq 0$ from (a) and hence $\underline{P}^L$ avoids sure loss by (c) and Proposition 1. Conversely, if $\underline{P}^L$ avoids sure loss on $\mathcal{L}$, for any choice of $X_1, \ldots, X_n \in \mathcal{D}$ and non-negative $s_1, \ldots, s_n$ such that $\sum_{i=1}^n s_i = 1$, it is $\sup \sum_{i=1}^n s_i(X_i - \underline{P}^L(X_i)) \geq 0$. From the dominance property (d), $\sup \sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) \geq 0$. Hence $\underline{P}$ avoids sure loss. $\square$

Theorem 3 settles the question of the extensibility of a weakly convex prevision $\underline{P}$ defined on an arbitrary set $\mathcal{D}$. Property (g) lets us extend $\underline{P}$ to any $\mathcal{D}' \supset \mathcal{D}$ (maintaining weak convexity) by considering the restriction of $\underline{P}^L$ to $\mathcal{D}'$. Moreover, (h) guarantees that $\underline{P}^L$ inherits the avoiding sure loss condition when $\underline{P}$ satisfies it.

The properties of $\underline{P}^L$ closely resemble those of the *natural extension* of a lower prevision $\underline{P}$ introduced by Walley in [9], whose definition differs from that of $\underline{P}^L$ only for the lack of the constraint $\sum_{i=1}^{n} s_i = 1$. Among these properties we outline (e): as the natural extension characterises coherence of $\underline{P}$ ($\underline{P}$ is coherent if and only if its natural extension coincides with $\underline{P}$ on $\mathcal{D}$), $\underline{P}^L$ characterises weak convexity of $\underline{P}$. There is, however, an important property which the two extensions do not share: whilst the natural extension of $\underline{P}$ is finite if and only if $\underline{P}$ avoids sure loss, $\underline{P}^L$, as shown above, is finite also in other cases, for example when the set $\mathcal{D}$ is finite. It can therefore be employed to correct $\underline{P}$ also in situations in which $\underline{P}$ incurs sure loss, although property (h) warns us that $\underline{P}^L$ shall continue incurring sure loss. In this case, property (f) in the following Proposition 4 can possibly be applied to $\underline{P}^L$ to achieve the avoiding sure loss condition (cfr. Proposition 1).

PROPOSITION 4. *Let $\underline{P}$ be a weakly convex lower prevision on $\mathcal{D}$. The following properties hold (whenever all random numbers involved are in $\mathcal{D}$):*

- *(a) $\underline{P}$ is translation invariant, monotone and concave (properties (T), (M), (C) of Theorem 2)*
- *(b) If $\underline{P}(0) \geq 0$, $\underline{P}(\lambda X) \geq \lambda \underline{P}(X)$, $\forall \lambda \in [0,1]$*
- *(c) If $\underline{P}(0) \geq 0$, $\underline{P}(\lambda X) \leq \lambda \underline{P}(X)$, $\forall \lambda > 1$*
- *(d) $\underline{P}(X+Y) \geq \lambda \underline{P}(X/\lambda) + (1-\lambda)\underline{P}(Y/(1-\lambda))$, $\forall \lambda \in]0,1[$*
- *(e) $\underline{P}(0) + \inf X \leq \underline{P}(X) \leq \underline{P}(0) + \sup X$*
- *(f) $\forall \mu \in I\!R$, $\underline{P}^*(X) = \underline{P}(X) + \mu$ is weakly convex on $\mathcal{D}$.*

Proof. Any weakly convex $\underline{P}$ satisfies the properties listed in (a): if it were not so, there would exist (by Theorem 3) a weakly convex extension of $\underline{P}$ on a linear space $\mathcal{L} \supset \mathcal{D}$ which does not always satisfy these properties, thus contradicting Theorem 2.

To prove (b), put $Y = 0$ in the concavity axiom of Theorem 2.

For (c), put $Y = \lambda X$ and use (b) to obtain $\underline{P}(X) = \underline{P}(Y/\lambda) \geq \underline{P}(Y)/\lambda = \underline{P}(\lambda X)/\lambda$ and therefore $\underline{P}(\lambda X) \leq \lambda \underline{P}(X)$.

To prove (d), take $n = 2$, $s_1 = \lambda$, $s_2 = 1 - \lambda$, $X_1 = X/\lambda$, $X_2 = Y/(1-\lambda)$, $X_0 = X + Y$ in Definition 4.

For (e), apply monotonicity and translation invariance of $\underline{P}$ (by (a)) to $\inf X \leq X \leq \sup X$.

To prove (f), apply Definition 4 to $\underline{P}^*$ to show that the supremum of each gain $\underline{G}^* = \sum_{i=1}^{n} s_i(X_i - \underline{P}^*(X_i)) - (X_0 - \underline{P}^*(X_0))$ is non-negative. □

Properties (b) and (c) show that weak convexity may match with lack of positive homogeneity, but require the condition $\underline{P}(0) \geq 0$.

Property (d) reduces to superlinearity when the additional property $\underline{P}(\lambda X) = \lambda \underline{P}(X)$, $\forall \lambda \in]0,1[$, holds. We shall comment a version of (d) (for weakly convex risk measures) later in Section 4, comment to Proposition 7.

Property (e) highlights a sore point of weak convexity: $\underline{P}(X)$ need not belong to the closed interval $[\inf X, \sup X]$ (*internality* may fail).[3] This is not surprising, noting that whenever $\mathcal{D} = \{X\}$ internality is the only restriction on $\underline{P}(X)$ required by coherence, whilst weak convexity imposes no restrictions on $\underline{P}(X)$ in this case, since the only $\underline{G}$ to be considered is such that $n = 1, s_1 = 1, X_1 = X_0 = X$, i.e. $\underline{G} = 0$, for every value $\underline{P}(X)$. Property (e) suggests that internality can be restored imposing $\underline{P}(0) = 0$, if $0 \notin \mathcal{D}$; by (f), if $0 \in \mathcal{D}$ and $\underline{P}(0) \neq 0$, then $\underline{P}^*(X) = \underline{P}(X) - \underline{P}(0)$ is weakly convex and $\underline{P}^*(0) = 0$. Requiring $\underline{P}(0) = 0$ is also the only choice to make $\underline{P}$ avoid sure loss (Proposition 1), while assuring that properties (b) and (c) hold.

Thinking of the meaning of a lower prevision, it appears extremely reasonable to add condition $\underline{P}(0) = 0$ to weak convexity: it would be at least weird to give an estimate (even imprecise) of the non-random number 0 which is other than zero.

3.2. Convex previsions

The preceding considerations lead us to a stronger notion of convexity, which we define as follows:

DEFINITION 6. *A lower prevision $\underline{P}$ on domain $\mathcal{D}$ ($0 \in \mathcal{D}$) is* convex *iff it is weakly convex and $\underline{P}(0) = 0$.*

PROPOSITION 5. *Let $\underline{P}$ be a convex lower prevision on $\mathcal{D}$. Then it satisfies the following basic properties (it is understood in (b), (c), (f) that all random numbers involved are in $\mathcal{D}$):*

[3] It may be interesting to observe that if there exists $X \in \mathcal{D}$ such that $\underline{P}(X) > \sup X$, then it is $\underline{P}(Y) \geq \inf Y, \forall Y \in \mathcal{D}$. To see this, suppose $\underline{P}(Y) < \inf Y$ and take $n = 1$, $X_1 = X$, $X_0 = Y$ in Definition 4 to get $\sup \underline{G} < 0$.

(a) $\underline{P}$ *has a convex natural extension (hence at least one convex extension) on any* $\mathcal{D}' \supset \mathcal{D}$

(b) $\underline{P}(\lambda X) \geq \lambda \underline{P}(X)$, $\forall \lambda \in [0,1]$

(c) $\underline{P}(\lambda X) \leq \lambda \underline{P}(X)$, $\forall \lambda \geq 1$

(d) $\inf X \leq \underline{P}(X) \leq \sup X$, $\forall X \in \mathcal{D}$

(e) $\underline{P}$ *avoids sure loss*

(f) $\underline{P}(\lambda X) \leq \lambda \underline{P}(X), \forall \lambda \leq 0.$

Besides, the convergence and convexity theorems hold for convex previsions too (replacing 'weakly convex' with 'convex' in their statements in Proposition 2).

Proof. Properties $(a) \div (e)$ follow easily from previous results.

To prove (f) when $\lambda \in [-1,0]$, use (b) and Remark 2, (b), to get $\underline{P}(\lambda X) \leq -\underline{P}(-\lambda X) \leq \lambda \underline{P}(X)$; when $\lambda < -1$, it is $1/\lambda \in]-1,0[$ and therefore $\underline{P}(X) = \underline{P}((\lambda X)/\lambda) \leq \underline{P}(\lambda X)/\lambda$, from which (f) follows.

The convergence and convexity theorems follow easily from the corresponding theorems for weakly convex previsions. □

Properties (d) and (e) show that convexity is significantly closer to coherence than weak convexity: convex lower previsions are a special class of avoiding sure loss previsions, retaining several properties of coherence, as well as (by (a)) the extension properties of weak convexity, but not requiring positive homogeneity, as shown by (b) and (c). Some comparisons among weak convexity, convexity and coherence are possible in terms of their definition, which we shall do now, and of envelope theorems, which will be discussed later.

With respect to coherence, the definition of weak convexity does not consider those bets on the random numbers of $\mathcal{D}$ where one is betting:

(a) 'in favour' of some random numbers, but 'against' no one (i.e. $\sum_{i=1}^{n} s_i > 0, s_0 = 0$);

(b) in favour of some random numbers and against one (X_0), or just against one, when the sum of the stakes on the first ones is different from the stake on X_0 (i.e. $\sum_{i=1}^{n} s_i \neq s_0 > 0$).

Bets of the kind (a) cannot give rise to a sure loss (bounded away from zero) when $\underline{P}$ is convex, by (e) of Proposition 5 and Definition 1. There remain type-(b) bets to mark the difference between convexity and coherence, but the gap is smaller here than it might appear at first sight. In fact, if $\underline{P}$ is convex, when $s_0 > \sum_{i=1}^n s_i$ it is $\sup\{\sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) - s_0(X_0 - \underline{P}(X_0))\}$ $= \sup\{\sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) + (s_0 - \sum_{i=1}^n s_i)(0 - \underline{P}(0)) - s_0(X_0 - \underline{P}(X_0))\} \geq 0$. This means that the only type of bets which is considered by coherence but not by convexity is the subset of type-(b) bets where $\sum_{i=1}^n s_i > s_0 > 0$.

An interesting implication is that convex lower previsions could be equivalently defined by requiring $\underline{P}(0) = 0$ and replacing the convexity condition $\sum_{i=1}^n s_i = s_0 > 0$ with $\sum_{i=1}^n s_i \leq s_0$.

3.3. Envelope theorems

It was proved in [9] that any lower envelope of coherent lower previsions is coherent. A more general version of this statement, holding for weakly convex lower previsions, is presented in the following proposition.

PROPOSITION 6. *Let $\mathcal{D}$ be a set of bounded random numbers and $\mathcal{P}$ a set of weakly convex lower previsions all defined on $\mathcal{D}$. If $\underline{P}(X) = \inf_{\underline{Q}\in\mathcal{P}} \left\{\underline{Q}(X)\right\}$ is finite $\forall X \in \mathcal{D}$, then $\underline{P}$ is weakly convex on $\mathcal{D}$.*

Proof. Consider a gain $\underline{G} = \sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) - (X_0 - \underline{P}(X_0))$ with $s_i \geq 0\ \forall i$ and $\sum_{i=1}^n s_i = 1$. For every $\varepsilon > 0$ there exists $\underline{Q}_\varepsilon \in \mathcal{P}$ such that $\underline{P}(X) \leq \underline{Q}_\varepsilon(X)\ \forall X \in \mathcal{D}$ and $\underline{Q}_\varepsilon(X_0) < \underline{P}(X_0)+\varepsilon$. Therefore, it is $\sup \underline{G} \geq \sup\{\sum_{i=1}^n s_i(X_i - \underline{Q}_\varepsilon(X_i)) - (X_0 - \underline{Q}_\varepsilon(X_0)) - \varepsilon\} \geq -\varepsilon$ since $\underline{Q}_\varepsilon$ is weakly convex. Because of the arbitrariness of ε, it is $\sup \underline{G} \geq 0$ and $\underline{P}$ is weakly convex. $\square$

Prior to discussing now a generalised version of the lower envelope theorem, we state the next preliminary lemma.

LEMMA 1. *Given $\underline{P}$ on domain $\mathcal{D}$ define, for all $X_0 \in \mathcal{D}$, $\mathcal{D}_{X_0} = \{X - X_0 : X \in \mathcal{D}\}$, $\underline{P}_{X_0}(X - X_0) = \underline{P}(X) - \underline{P}(X_0)$. $\underline{P}$ is weakly convex on $\mathcal{D}$ iff every $\underline{P}_{X_0}$ avoids sure loss on its domain $\mathcal{D}_{X_0}$.*

Proof. Any gain $\underline{G}$ in Definition 4 may be written as (recall that $\sum_{i=1}^n s_i = 1$) $\underline{G} = \sum_{i=1}^n s_i(X_i - \underline{P}(X_i)) - \sum_{i=1}^n s_i(X_0 - \underline{P}(X_0)) = \sum_{i=1}^n s_i[(X_i - X_0) - (\underline{P}(X_i) - \underline{P}(X_0))] = \sum_{i=1}^n s_i[(X_i - X_0) - \underline{P}_{X_0}(X_i - X_0)]$. Therefore $\sup \underline{G} \geq 0$ iff $\sup \sum_{i=1}^n s_i[(X_i - X_0) - \underline{P}_{X_0}(X_i - X_0)] \geq 0$. The thesis follows, recalling Remark 1. $\square$

THEOREM 4. (Generalised envelope theorem) *$\underline{P}$ is convex on its domain $\mathcal{D}$ ($0 \in \mathcal{D}$) if and only if there exist a set $\mathcal{P}$ of coherent precise previsions (all defined on domain $\mathcal{D}$) and a function α from $\mathcal{P}$ into $\mathbb{R}$ such that:*

(a) $\underline{P}(X) = \inf_{P \in \mathcal{P}} \{P(X) + \alpha(P)\}$, $\forall X \in \mathcal{D}$;

(b) $\inf_{P \in \mathcal{P}}\{\alpha(P)\} = 0$.

Moreover inf *is attained in* (a) *and in* (b).

Proof. Suppose $\underline{P}$ is convex.

We prove first that weak convexity of $\underline{P}$ implies (a), where the infimum is actually a minimum.

From Lemma 1 (using the same notation) and Theorem 1, (a), for every $X_0 \in \mathcal{D}$ there exists a coherent precise prevision P_{X_0} defined on $\mathcal{D}_{X_0}$ such that

$$\underline{P}_{X_0}(X - X_0) \leq P_{X_0}(X - X_0), \forall X \in \mathcal{D} \tag{6}$$

or also, by definition of $\underline{P}_{X_0}$ and linearity of P_{X_0}

$$\underline{P}(X) \leq P_{X_0}(X) + \underline{P}(X_0) - P_{X_0}(X_0) = P_{X_0}(X) + \alpha(P_{X_0}), \tag{7}$$

where $\alpha(P_{X_0}) = \underline{P}(X_0) - P_{X_0}(X_0)$.

Considering all possible X_0 in $\mathcal{D}$, we obtain a set $\mathcal{P} = \{P_{X_0} : X_0 \in \mathcal{D}\}$ of precise previsions for which (7) holds, with equality when $X = X_0$. Therefore, omitting here the subscript X_0 for the elements of $\mathcal{P}$ and for function α, we obtain:

$$\underline{P}(X) = \min_{P \in \mathcal{P}} \left(P(X) + \alpha(P)\right), \forall X \in \mathcal{D}. \tag{8}$$

We prove now (b) and precisely that $\inf \alpha(P) = \min \alpha(P) = 0$. In fact, since $\underline{P}(0) = 0$, putting $X = 0$ in (7) gives

$$0 = \underline{P}(0) \leq P_{X_0}(0) + \alpha(P_{X_0}) = \alpha(P_{X_0}), \forall X_0 \in \mathcal{D}. \tag{9}$$

Hence α is non-negative. Further, if $X = X_0 = 0$ equality holds in (7), giving $0 = \underline{P}(0) = P_0(0) + \alpha(P_0) = \alpha(P_0)$, thus showing that $\inf \alpha(\underline{P}_{X_0}) = \alpha(P_0) = 0$.

Suppose now that (a) and (b) hold. Using (a), weak convexity of $\underline{P}$ follows from Proposition 6 and Proposition 4,(f), since every coherent precise prevision is weakly convex. To prove now convexity of $\underline{P}$, use (a) and (b) to write $\underline{P}(0) = \inf_{P \in \mathcal{P}} \{P(0) + \alpha(P)\} = \inf_{P \in \mathcal{P}} \{\alpha(P)\} = 0$.

Finally, having thus proved that $\underline{P}$ is convex on $\mathcal{D}$, it is implied by the first part of the proof that each infimum in (a) and (b) is achieved. □

3.3.1. *Comments on the Generalised Envelope Theorem*

(a) A generalised envelope theorem holds also for weakly convex previsions, as a by-product of the proof of Theorem 4:

THEOREM 5. (Generalised envelope theorem for weakly convex lower previsions) $\underline{P}$ *is* weakly convex *on its domain* $\mathcal{D}$ *if and only if there exist a set* $\mathcal{P}$ *of coherent precise previsions (all defined on domain* $\mathcal{D}$*) and a function* α *from* $\mathcal{P}$ *into* $I\!R$ *such that:* $\underline{P}(X) = \inf\{P(X) + \alpha(P)\}$, $\forall X \in \mathcal{D}$ *(*inf *is attained).*

(b) It may be observed that weak convexity, convexity and coherence can all be characterised by envelope theorems. What makes the difference is function $\alpha(P)$, which is unconstrained in the generalised envelope theorem characterising weak convexity, non-negative and such that $\min \alpha = 0$ with convexity, identically equal to zero with coherence (in this case Theorem 4 reduces to the well known envelope theorem, cfr. Theorem 1, (*b*)).

A convex prevision may be obtained as a lower envelope of 'translated' previsions ($P(X)$ is replaced by $P(X) + \alpha(P) = P(X + \alpha(P))$), provided that the translation represented by the $\alpha(P)$ is always in the same sense (i.e. it is $\alpha \geq 0$) and that there is at least one non-translated prevision P_0, which (as appears from the proof of Theorem 4) determines the value of $\underline{P}$ at 0 (and hence at all constants in $\mathcal{D}$).

In this interpretation, the envelope theorem for coherent previsions assures further that the value of $\underline{P}$ at X is equal to that of some non-translated prevision $P(X)$, for *all* $X \in \mathcal{D}$.

(c) With respect to other proofs of a similar theorem in risk measurement theory [5], the proof of Theorem 4 is simpler, since it follow directly from Theorem 1, (*a*) (which relies on a version of the separating hyperplane theorem [9]). Further, Theorem 4 does not require any structure on the set $\mathcal{D}$.

4. Convex Risk Measures

A *risk measure* ρ is a real mapping, defined on a set $\mathcal{D}$ of random numbers, which associates a real number $\rho(X)$ to every $X \in \mathcal{D}$. The basic idea is that $\rho(X)$ should reliably measure how 'risky' X is, and whether it is acceptable to buy or hold X. Intuitively, X should be acceptable (not acceptable) if $\rho(X) \leq 0$ (if $\rho(X) > 0$), and $\rho(X)$ should measure the amount of money which could be subtracted from X, keeping it acceptable (the amount of money to be added to X to make it acceptable).

The use of risk measures is widespread in financial and insurance practice, where they help in taking decisions in many problems, like evaluating portfolios or settling capital requirements to face various banking and insurance risks.

In recent years, the dependability of traditional measures of risk, the most known of which is probably *Value-at-Risk* or *VaR*, has been questioned under many respects by both practitioners and academicians. The introduction of a new family of risk measures, *coherent risk measures*, by Artzner, Delbaen, Eber and Heath in a series of papers (including [1] and [2]) gave therefore rise to a considerable interest. We report here the definition:

DEFINITION 7. *Let $\mathcal{L}$ be a linear space of random numbers which contains real constants. A mapping ρ from $\mathcal{L}$ into $I\!R$ is a* coherent risk measure *iff it satisfies the following axioms:*

(T1) $\forall\, X \in \mathcal{L}, \forall\, \alpha \in I\!R, \rho(X+\alpha) = \rho(X) - \alpha$ *(translation invariance)*

(PH) $\forall\, X \in \mathcal{L}, \forall\, \lambda \geq 0, \rho(\lambda X) = \lambda \rho(X)$ *(positive homogeneity)*

(M1) $\forall\, X, Y \in \mathcal{L}$, *if* $X \leq Y$ *then* $\rho(Y) \leq \rho(X)$ *(monotonicity)*

(S) $\forall\, X, Y \in \mathcal{L}, \rho(X+Y) \leq \rho(X) + \rho(Y)$ *(subadditivity).*

Unlike most older risk measures, operating with coherent risk measures does not require assessing just one probability distribution for each random number X as a necessary preliminary step. Therefore coherent risk measures are very useful when the probability evaluations for some or all X are not quite reliable.

In the approach of [1, 2] risk measures are not explicitly related to imprecise previsions. This has been done in [8], where it is shown that a risk measure can be behaviourally viewed as a special case of upper prevision. In short, to evaluate $\rho(X)$ an agent can identify it with the infimum of the amounts he/she would ask to shoulder the random number X. Since getting a specific amount for receiving X is the same as selling $-X$ for the same amount, $\rho(X)$ can be equivalently viewed as an infimum selling price for $-X$.[4] This is the behavioural interpretation given in [9] for the upper prevision $\overline{P}$ of $-X$. Hence, a risk measure may be considered a special case of upper prevision:

$$\rho(X) = \overline{P}(-X). \tag{10}$$

[4] In principle we should also take account of the gap between time t_0, when the agent evaluates X, and time t, when X becomes non-random. This was done in [8], considering that the agent evaluates an infimum selling price for the discounted value of X at t_0 (rather than for the value of X in t) and assuming (like in [2]) t fixed. For the sake of simplicity, we shall neglect this aspect here.

Starting from this, a general notion of coherent risk measure was introduced in [8]. A mapping ρ from $\mathcal{D}$ into $\mathbb{R}$ is a *coherent risk measure* iff there exists a coherent upper prevision $\overline{P}$ (cfr. Definition 3) defined on $\mathcal{D}^- = \{-X : X \in \mathcal{D}\}$ such that $\rho(X) = \overline{P}(-X)$, for each $X \in \mathcal{D}$. In this way, standard results from imprecise probability theory can be applied and coherent risk measures are defined on *arbitrary* sets of random numbers [8]. In particular, if $\mathcal{D}$ is a linear space containing real constants this notion reduces to the concept of coherent risk measure already known in the literature [1, 2].

A generalisation of the notion of coherent risk measure (according to [1, 2]) was suggested by Föllmer and Schied in [5, 6]. They defined convex risk measures on linear spaces of random numbers by substituting axioms (S) and (PH) in Definition 7 with the *convexity axiom*

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y) \ \forall X, Y \in \mathcal{L}, \lambda \in [0,1]. \quad (11)$$

The idea here is to weaken the positive homogeneity axiom PH), to cope with situations where $\rho(\lambda X) \geq \lambda\rho(X)$, for $\lambda > 1$.

In fact, $\rho(\lambda X)$ might be greater than $\lambda\rho(X)$ because of *liquidity risks*: if we were to sell immediately a large amount λX, with $\lambda > 1$, of a financial investment, we might be forced to accept a smaller reward than λ times the current selling price for X.

In the sequel of this section we show how the theory developed for convex previsions can be applied to define convex risk measures on arbitrary sets of random numbers and to investigate their properties. Proofs of the results can be easily obtained from the corresponding ones in Section 3, recalling (2) and (10).

DEFINITION 8. *Given an arbitrary set $\mathcal{D}$ of random numbers, a mapping ρ from $\mathcal{D}$ into $\mathbb{R}$ is a* weakly convex risk measure *on $\mathcal{D}$ iff for all $n \in \mathbb{N}^+$, for each $X_0, X_1, \ldots, X_n \in \mathcal{D}$, for each $s_1, \ldots, s_n$ real and non-negative such that $\sum_{i=1}^n s_i = 1$, defining $\overline{G} = \sum_{i=1}^n s_i(X_i + \rho(X_i)) - (X_0 + \rho(X_0))$, it is* $\sup \overline{G} \geq 0$.

REMARK 5. *Because of (10), defining weak convexity on $\mathcal{D}$ for a risk measure corresponds to defining it for an upper prevision on $\mathcal{D}^- = \{-X : X \in \mathcal{D}\}$. This is done quite analogously to Definition 4, using the elementary gains for upper previsions (cfr. Section 2).*

If $\mathcal{D}$ is a linear space containing real constants, the notion of weakly convex risk measure reduces to that of convex risk measure according to [5, 6], by the following theorem (analogous to Theorem 2):

THEOREM 6. *Let $\mathcal{L}$ be a linear space of bounded random numbers containing real constants. A mapping ρ from $\mathcal{L}$ into $\mathbb{R}$ is a weakly* *convex risk measure (according to Definition 8) iff it satisfies axioms (T1), (M1) of Definition 7 and the convexity axiom (11).*

Hence, weakly convex risk measures make it possible to drop the sometimes overly restrictive assumption that convex risk measures be defined on a linear space.

The convergence and convexity theorems (cfr. Proposition 2) hold for weakly convex risk measures; also, weakly convex risk measures can be extended on any $\mathcal{D}' \supset \mathcal{D}$, preserving weak convexity. On the other hand, they avoid sure loss iff $\rho(0) \geq 0$ (ρ avoids sure loss on $\mathcal{D}$ iff $\overline{P}(-X) = \rho(X)$ avoids sure loss on $\mathcal{D}^-$).

Like the general case in Section 3, it appears quite appropriate to put $\rho(0) = 0$, and hence to use *convex* risk measures: 0 is precisely what we would ask to shoulder the random number $X = 0$.

DEFINITION 9. *Given an arbitrary set $\mathcal{D}$ of random numbers* ($0 \in \mathcal{D}$)*, a mapping ρ from $\mathcal{D}$ into $\mathbb{R}$ is a* convex risk measure *on $\mathcal{D}$ iff ρ is weakly convex and $\rho(0) = 0$.*

Convex risk measures have further nice additional properties, corresponding to those of convex lower previsions: they always avoid sure loss, and

$$-\sup X \leq \rho(X) \leq -\inf X, \forall X \in \mathcal{D}. \tag{12}$$

Condition (12) corresponds to internality ((d) of Proposition 5), and is a rationality requirement for risk measures: for instance, $\rho(X) > -\inf X$ would mean that to make X acceptable we would require adding to it a sure number ($\rho(X)$) higher than the maximum loss X may cause.

The following proposition considers two further properties for convex or weakly convex risk measures.

PROPOSITION 7. *Let ρ be a risk measure from $\mathcal{D}$ into $\mathbb{R}$.*

(a) If ρ is weakly convex, then, $\forall \lambda \in]0, 1[$

$$\rho(X + Y) \leq \lambda\rho(X/\lambda) + (1 - \lambda)\rho(Y/(1 - \lambda)). \tag{13}$$

(b) If ρ is convex, then

$$\rho(\lambda X) \geq \lambda\rho(X), \forall \lambda \geq 1. \tag{14}$$

In general, weakly convex and convex risk measures are not subadditive. If ρ were subadditive and positively homogeneous, we would get

$$\rho(X + Y) \leq \lambda\rho(X/\lambda) + \mu\rho(Y/\mu), \forall \lambda, \mu > 0. \tag{15}$$

Weak convexity guarantees that (15) holds whenever $\mu = 1 - \lambda$; for instance, if λ is close to 1 (so the range of the values of X/λ is moderately larger than that of X), Y has to be multiplied by a very large scaling factor $1/(1-\lambda)$ to ensure (13).

If ρ is convex, using (14) inequality (13) may be strengthened to $\max(\rho(X+Y), \rho(X)+\rho(Y)) \leq \lambda\rho(X/\lambda)+(1-\lambda)\rho(Y/(1-\lambda))$. Anyway (14) itself is more significant, since it clearly shows that convex risk measures may express evaluations which take account of liquidity risks (property (b) holds for weakly convex risk measures if ($0 \in \mathcal{D}$ and) $\rho(0) \geq 0$).

A notion of convex natural extension may also be given for convex (or weakly convex) risk measures and its properties correspond to those listed in Theorem 3. When well-defined, it gives in particular a standard way of 'correcting' other kinds of risk measures into convex risk measures.

Of course the generalised envelope theorem holds too. We report the version for convex risk measures.

THEOREM 7. *ρ is convex on its domain $\mathcal{D}$ ($0 \in \mathcal{D}$) if and only if there exist a set $\mathcal{P}$ of coherent precise previsions (all defined on domain $\mathcal{D}^-$) and a function β from $\mathcal{P}$ into $I\!R$ such that:*

(a) $\rho(X) = \sup_{P\in\mathcal{P}} \{P(-X) + \beta(P)\}$, $\forall X \in \mathcal{D}$

(b) $\sup_{P\in\mathcal{P}} \beta(P) = 0$.

Moreover sup *is attained in* (a) *and in* (b). *To obtain the version for weakly convex risk measures, omit* (b)*; for coherent risk measures, put* $\beta = 0$.

EXAMPLE 1. *Given X, let $\mathcal{D} = \{\lambda X : \lambda \geq 0\}$. The random numbers in $\mathcal{D}$ might for instance represent the possible investment choices of an investor who only acts as a buyer of a financial activity X and considers whether to buy it or not and to what extent. This choice for $\mathcal{D}$ is rather peculiar, because it is easy to see that any coherent upper prevision $\overline{P}$ on $\mathcal{D}$ is confined to a coherent* precise *prevision. Consequently any coherent risk measure on $\mathcal{D}$ is a linear function of λ: $\rho(\lambda X) = \lambda\rho(X)$, with $\rho(0) = 0$. In this example, considering ρ as a function of λ, convex real functions are natural candidates to accomodate liquidity risk into the risk measure, but only some of them correspond to a convex risk measure. Consider for instance the piecewise linear function*

$$\rho(\lambda X) = \begin{cases} \lambda\rho(X) & \text{if } \lambda \in [0,1] \\ k(\lambda-1)+\rho(X) & \text{if } \lambda > 1 \qquad \text{with } k \in [\rho(X), -\inf X] \end{cases}$$

where $\rho(X) \in [-\sup X, -\inf X]$ *is given. We may use the generalised envelope theorem (Theorem 7) to show that* ρ *is convex. Define for this* $P_1(-\lambda X) = \lambda\rho(X) = \rho(\lambda X)$, $P_2(-\lambda X) = k\lambda$, $\mathcal{P} = \{P_1, P_2\}$, $\beta(P_1) = 0$, $\beta(P_2) = \rho(X) - k$. *Then Theorem 7 holds for such a choice of* $\mathcal{P}$ *and* β, *and it is precisely* $\rho(\lambda X) = P_1(X) + \beta(P_1)$ *if* $\lambda \leq 1$, $\rho(\lambda X) = P_2(X) + \beta(P_2)$ *if* $\lambda \geq 1$. *Note also that at* $\lambda = 1$ *it is* $P_2(-\lambda X) = k$, *and that the necessary condition for coherence of* P_2, $P_2(-X) \in [\inf(-X), \sup(-X)] = [-\sup X, -\inf X]$, *is guaranteed to hold by the constraints on* ρ *and* k.

Other intuitively appealing convex functions are not necessarily convex risk measures. For instance,

$$\rho^*(\lambda X) = \begin{cases} \lambda\rho(X) & \text{if } \lambda \in [0,1] \\ \lambda^r\rho(X) & \text{if } \lambda > 1 \end{cases} \quad \text{with } r > 1$$

is not a convex risk measure. In fact, by (12), for every $\lambda \geq 0$

$$-\sup\{\lambda X\} \leq \rho(\lambda X) \leq -\inf\{\lambda X\} \tag{16}$$

should hold for any convex risk measure ρ *defined on* $\mathcal{D}$. *It is not difficult to see that (16) is not satisfied by* ρ^* *when* λ *is large enough.*

EXAMPLE 2. *Let now* $\mathcal{D}$ *be a convex cone. Weak convexity of any risk measure defined on* $\mathcal{D}$ *is characterised by two simple axioms.*

PROPOSITION 8. *If* $\mathcal{D}$ *is a convex cone, a mapping* ρ *from* $\mathcal{D}$ *into* $\mathbb{R}$ *is a weakly convex risk measure if and only if it satisfies the following axioms,* $\forall X, Y \in \mathcal{D}$:

(C1) $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \forall\lambda \in [0,1]$ *(convexity)*

(M2) $\forall\mu \in \mathbb{R}$, *if* $X \geq Y + \mu$ *then* $\rho(Y) \geq \rho(X) + \mu$ *(monotonicity).*

The proof resembles with simple modifications that of Theorem 2 and is omitted.

If in particular $\mathcal{D} = \{\lambda X : \lambda \geq 0\}$ *as in Example 1, it can be shown that axiom (M2) is equivalent for* $\lambda_1 \neq \lambda_2$ *(if* $\lambda_1 = \lambda_2$ $(M2)$ *is trivially true) to the following condition:*

$$-\sup X \leq \frac{\rho(\lambda_2 X) - \rho(\lambda_1 X)}{\lambda_2 - \lambda_1} \leq -\inf X, \forall\lambda_1, \lambda_2 \geq 0, \lambda_1 \neq \lambda_2. \tag{17}$$

To prove the equivalence, let $\lambda_2 > \lambda_1 \geq 0$ *(note that this is not restrictive). Assuming that* $(M2)$ *holds, (17) follows by applying* $(M2)$ *to* $(\lambda_2 - \lambda_1) supX \geq \lambda_2 X - \lambda_1 X \geq (\lambda_2 - \lambda_1)\inf X$. *Conversely, suppose that* (17) *holds. For* $X, Y \in \mathcal{D}$, *condition* $X \geq Y + \mu$ *reduces* *to either* $\lambda_2 X \geq \lambda_1 X + \mu$ *or* $\lambda_1 X \geq \lambda_2 X + \mu$. *If* $\lambda_2 X \geq \lambda_1 X + \mu$, $-(\lambda_2 - \lambda_1) \inf X \leq -\mu$ *and* $(M2)$ *follows using the second inequality in (17). If* $\lambda_1 X \geq \lambda_2 X + \mu$, *this is equivalent to* $-(\lambda_2 - \lambda_1) \sup X \geq \mu$ *and* $(M2)$ *follows from the first inequality in (17).*

Condition (17) and (C1) can be alternatively employed for checking convexity of the risk measures in Example 1.

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