Extreme Moves in Daily Foreign Exchange Rates and Risk Limit Setting

Peter Blum* and Michel M. Dacorogna†

November 17, 2002

Abstract

Foreign exchange rates can be subject to considerable daily fluctuations (up to 5 percent within one day). This can, in certain cases, cause serious losses on open overnight positions. Given a maximum tolerable loss for a company, limits have to be set on open overnight positions in foreign currencies. Usually, these limits are determined by using a normal ("Gaussian") model for the daily fluctuations. In our study we illustrate how this common model sometimes quite strongly underestimates the actual extreme risks and, based on methods from Extreme Value Theory (EVT), we propose and justify a more accurate model. We also show how to use these estimations to compute limits that a risk manager can set to open positions to avoid unexpected huge losses.

Keywords: Extreme Value Theory, Foreign Exchange, Time Series Analysis, Risk Management.

*ETH, Department of Mathematics, 8092 Zürich, Switzerland

†Converium Ltd, General Guisan Quai 26, 8022 Zürich, Switzerland, Email: peter.blum@converium.com, michel.dacorogna@converium.com

1 Introduction

Since risk management has been established on a quantitative basis in financial institutions, the prevailing model has been the Gaussian one [J. P. Morgan, 1996]. In particular, it is widely used to determine the Value-at-Risk of assets or more generally of portfolios of assets or in pricing options with the Black and Scholes model [Black and Scholes, 1973]. By using a model, it is possible to determine the probability of a movement of a certain size to occur. The Gaussian model, however, implies that extreme movements are very improbable. Unfortunately, we have learned the hard way that extreme movements in financial markets are more the rule than the exception [Koedijk et al., 1990, Longin, 1996]. This fact pleads for the use of more sophisticated ways for assessing the risk of extreme events. Extreme value theory[1] gives us luckily the possibility to go beyond the normal assumptions and to better quantify the risk of extreme events on financial markets.

In this paper, we first present empirical studies of daily log returns (later called returns) of foreign exchange (FX) rates that demonstrate the existence of extreme events in this market and quantify the failure of the Gaussian model. In a second part, we show how it is possible to estimate the probability distribution of extreme movements using a simple method derived from extreme value theory (EVT). We compare the results obtained with historical data and make predictions for longer periods than those already observed as well as we propose a way to compute the limits that a risk manager would set to open positions in order to avoid the occurrence of large losses.

2 Large Movements Do Occur More Often than Predicted by the Gaussian Model

To examine the presence of extreme movements in the FX market, we use a set of daily observations of four major FX rates, namely CHF/USD, EUR/USD, GBP/USD and JPY/USD[2]. For all rates except EUR/USD,

[1]For the interested reader, there are more and more review books available on the subject. Here are two: one more oriented towards the theory [Leadbetter et al., 1983] and two more practically oriented [Adler et al., 1998, Embrechts et al., 1997].

[2]The data were obtained from Bloomberg L.P. through their Bloomberg Data License; see `www.bloomberg.com`. Free daily observations of certain FX rates can be obtained e.g.

FX rate	$\hat{\mu}$	$\hat{\sigma}$	Max.	One event over number of years	Min.	One event over number of years
EUR/USD	-0.0089%	0.83%	5.05%	10'838'489	-3.86%	1'243
JPY/USD	0.0105%	0.69%	3.98%	1'519'105	-7.20%	impossible
GBP/USD	-0.0077%	0.65%	4.05%	24'533'626	-4.70%	3'269'448'288
CHF/USD	-0.0009%	0.75%	3.76%	21'157	-4.51%	1'543'769

Table 1: Empirical estimates of the averages ($\hat{\mu}$) and the standard deviations ($\hat{\sigma}$) of log returns of major FX rates. We also display the largest (Max.) and the smallest (Min.) returns in the sample together with their probabilities of occurrence according to the Gaussian model parameterized with the estimated empirical averages and standard deviations.

we cover the time interval from January 2nd, 1980 to December 31st, 2001, resulting in some 5600 daily observations per rate. For the EUR/USD rate, we cover the interval from December 29th, 1988 to December 31st, 2001, resulting in 3358 daily observations. For the time before January 1st, 1999, we use a synthetic Euro rate computed from a portfolio of the constituent currencies [3]. Based on these data, we construct the logarithmic returns:

$$r_i = \ln\left(\frac{P_i}{P_{i-1}}\right) \tag{1}$$

where P_i indicates the closing price on day i. This transformation allows us to work with a stationary time series, see [Dacorogna et al., 2001], and is the usual quantity considered in statistical studies of financial data.

For the rest of this study, we shall assume that the daily logarithmic returns r_i are independent and identically distributed (i.i.d.). In addition, it is fairly popular among practitioners to assume more specifically that each r_i follows a Gaussian distribution, i.e. the probability density function of r_i is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \equiv N(\mu,\sigma) \tag{2}$$

This model is fully determined by two parameters, namely the average μ and the variance σ^2. Empirical estimates $\hat{\mu}$ and $\hat{\sigma}$ to calibrate the model to the characteristics of a specific exchange rate can be easily obtained from

from `www.federalreserve.gov/releases/H10/hist/`.

[3] Actually, the `THEOEURO` index from Bloomberg.

FX rate	Expected Positive	Observed Positive	Expected Negative	Observed Negative
EUR/USD	34	48	34	57
JPY/USD	56	119	56	71
GBP/USD	56	88	56	102
CHF/USD	55	106	55	80

Table 2: Theoretical and observed numbers of exceedances of a VaR at 99% respectively 1% computed according to the Gaussian model.

the historical data. Then, given a specific return r_i, it is possible to compute its probability according to the Gaussian model by using Equation 2.

In Table 1, we report estimates of both μ and σ for the four FX rates as well as the largest negative and positive returns observed in our samples. These values are compared to their probability of occurrence in the Gaussian model. In order to facilitate the understanding, we present the Gaussian probability as the one event in a certain number of years. It is easily seen from the numbers that the Gaussian model gives completely unrealistic probabilities given that our sample, and the extremes observed in it, cover an observation period of only 21 years (resp. 13 for EUR/USD).

To complete the picture, we turn ourselves to the computation of the Value-at-Risk (VaR) as it is given by our historical data. The VaR has become a standard way of measuring the extreme risk of a portfolio [Jorion, 1997]. Given a certain model, the VaR is equivalent to the quantile corresponding to a certain risk threshold. One usual threshold is the 1% event. In the Gaussian model, the VaR is determined by the average μ and the standard deviation σ. Once we have the VaR, we can compute the expected number of exceedances to this threshold according to the Gaussian model. Theoretically, given the quantiles of 1% (left tail) or the 99% (right tail), this number is a function of the number of data. If one looks historically in the data, there should only be 1% of the losses that are beyond the VaR. Our study consisted in counting the number of returns that were beyond the given thresholds. In Table 2, we present the results of the study. It is clear that the number of values that lie beyond the VaR is much larger than expected by the Gaussian model. It is one more clear sign that the Gaussian model is not appropriate to represent the risk of extreme movements in the

FX market.

Both analyses justify the title of this section. Extreme events do indeed occur much more frequently than this is foreseen by the Gaussian model. Hence, in order to get a better understanding of the dangers posed by extreme fluctuations in FX rates, we have to go for alternative models that assign more realistic - i.e. higher - probabilities to these extreme events. This is the subject of the next section.

3 Tail Analysis

As long as we are only interested in the extreme events, we do not need to consider models that cover the full range of possible outcomes as does the Gaussian one. Indeed, we can restrict our attention to dedicated methods for the analysis of extreme events, i.e. the analysis of the tails of the probability distribution. Powerful methods for this tail analysis come from the realm of Extreme Value Theory (EVT) which has become fairly popular in various areas of quantitative risk management during the past few years [Embrechts et al., 1998].

Essentially, EVT aims at estimating tail events, and by definition does not consider fitting the center of the distribution. Depending on the distributional properties of the underlying model, various statistical techniques are available; see for instance [Embrechts et al., 1997] for a survey and pointers to the rich literature on this topic. In our case, we look at so-called heavy-tailed models (also referred to as the Fréchet class) for which

$$1 - F(x) \;=\; x^{-\alpha} L(x). \tag{3}$$

In this formula, the crucial parameter determining the tail properties is $\alpha > 0$. The unknown function $L(x)$ is defined in such a way that it will typically not appear in statistical estimates for models satisfying Formula 3. It does, however, play an important role when it comes to statistical properties of these estimates; see [Embrechts et al., 1997] for all technical details. Hence, the problem becomes to estimate the tail index α from the data. There are many procedures for this estimation, and we concentrate here on the Hill estimator [Hill, 1975], which is a consistent estimator of $\gamma = 1/\alpha$. Given a sequence of n observations, $X_1, X_2, ..., X_n$, drawn from an i.i.d. process whose probability distribution F is unknown, we order the observations in descending order statistics $X_{(1)} \geq X_{(2)} \geq ... \geq X_{(n)}$. We can then define the Hill estimator $\hat{\gamma}^H_{n,m}$ by:

$$\hat{\gamma}^H_{n,m} \;=\; \frac{1}{m-1}\sum_{i=1}^{m-1} \ln X_{(1)} - \ln X_{(m)} \tag{4}$$

where the number of order statistics m is an additional parameter to be determined. In fact the Hill estimator is the maximum likelihood estimator of γ and $\alpha = 1/\gamma$ holds for the tail index. For finite samples, however, the expected value of the Hill estimator is biased. There are many ways of trying to reduce this bias, from graphical techniques [Resnick, 1987] to sophisticated bootstrap procedures [Danielsson et al., 1997, Pictet et al., 1998]. In practice, however, we find that taking a value of $m = \sqrt{n}$ where n is the number of observations in the sample leads to reasonable results.

The purpose of this short communication is not to assess in details the significance or the bias of the estimator[4] but rather to design a simple recipe to quantify the probability of extreme movements from a given data set. The results might vary slightly from one method to the other but the order of magnitude stays the same and, as we shall see, the improvement compared to the Gaussian model is such that a certain imprecision in the estimation is a price worthwhile to pay. However, if one puts the model into practical use, it is nevertheless important to obtain an idea of the accuracy of the estimated parameters. To this end, the Jackknife method is a simple and powerful means. It basically consists of modifying the data sample in 10 different ways, each time removing one tenth of the total sample. The tail index is separately computed for each of the 10 modified samples, and the analysis of the deviations between the 10 results yields an estimate of the standard error. Together with the asymptotic properties of the Hill estimator, this allows to compute e.g. a 95% confidence interval, as given for our estimates in Table 3. It is beyond the scope of this short communication to dwell longer on this issue, but the interested reader can find detailed information in [Pictet et al., 1998] and more examples and background information in [Dacorogna et al., 2001].

Using Equation 4 with $m = \sqrt{n}$ and the Jackknife method, we estimate the tail indices and the 95% confidence bounds for the four FX rates chosen for this analysis. The results are reported in Table 3. The results are relatively stable and remarkably similar across rates, except for CHF/USD. The results of Table 3 differ slightly from the results obtained

[4] The interested reader can consult the different references given in this paper.

FX rate	Number of Observations (n)	Number of Order Statistics (m)	Tail Index $\hat{\alpha} = 1/\hat{\gamma}^H_{n,m}$
EUR/USD	3357	58	3.86 ± 0.80
JPY/USD	5642	75	3.96 ± 0.85
GBP/USD	5634	75	3.76 ± 0.83
CHF/USD	5641	75	4.67 ± 1.07

Table 3: Results of the esimations of the tail index on our sample; the values in parentheses are the 95% Jackknife confidence interval around $\hat{\alpha}$.

in [Dacorogna et al., 2001] but are still clearly within the error bounds given therein. The confidence intervals reported here are not negligible, but much narrower than those given in [Dacorogna et al., 2001]. Differences may be due to the considerably longer data samples and to the simplified estimation procedure in this study. In any case, the differences are not material and do not really affect the rest of our study where we shall use the numbers given in the last column of Table 3.

4 Extreme Risks and Limit Setting

From the practitioner's point of view, one of the most interesting questions that tail studies can answer is what are the extreme movements that can be expected in financial markets. Have we already seen the largest ones or are we going to experience even larger movements? Are there theoretical processes that can model the type of fat tails that come out of the empirical analysis? The answer to such questions are essential for good risk management of financial exposures. It turns out that we can partially answer them here. Once we know the tail index, we can apply extreme value theory *outside* our sample to consider possible extreme movements *that have not yet been observed historically.*

This can be achieved by a computation of the extreme quantiles in the daily returns as proposed in section 5.4.3 of [Dacorogna et al., 2001]. In this reference the derivation of the following quantile estimator is presented for a given probability p :

$$\hat{x}_p \; = \; X_{(m)} \left(\frac{m}{np} \right)^{\frac{1}{\hat{\alpha}}} \tag{5}$$

FX rate	Method of Calculation	Worst daily movement within ... 5 years	10 Years	20 Years	30 Years	40 Years
EUR/USD	Observed	3.79%	3.84%	–	–	–
	Hill	4.30%	5.17%	6.22%	6.93%	7.49%
	Gaussian	2.68%	2.85%	3.01%	3.10%	3.17%
JPY/USD	Observed	3.53%	4.85%	6.09%	–	–
	Hill	3.97%	4.74%	5.67%	6.30%	6.80%
	Gaussian	2.24%	2.38%	2.51%	2.59%	2.64%
GBP/USD	Observed	3.21%	3.88%	4.54%	–	–
	Hill	3.38%	4.08%	4.93%	5.51%	5.96%
	Gaussian	2.09%	2.22%	2.34%	2.41%	2.46%
CHF/USD	Observed	3.22%	3.61%	4.11%	–	–
	Hill	3.54%	4.12%	4.80%	5.24%	5.58%
	Gaussian	2.41%	2.56%	2.71%	2.79%	2.84%

Table 4: Extreme daily returns over periods of 5 to 40 years. The values are computed by three different methods: historically observed in our sample, using the Hill estimator and using the Gaussian model.

where all the quantities on the right-hand side are now known. The reference gives also a way to estimate the error of this quantile computation but here we concentrate on the computation of quantiles using the numbers obtained in Table 3. Since we have a sample covering 21 years, we choose the probabilities of occurrence at which to compute the quantiles so that we obtain at least two numbers that we can compare with our historic data and two that represent an out-of-sample prediction: 1 over 5 years (probability of 0.0008)[5], 1 over 10 years (0.0004), 1 over 20 years (0.0002), 1 over 30 years (0.000133) and 1 over 40 years (0.0001), which are really low probabilities sitting far out in the tails.

In Table 4, we report the results for three different methods of calculation: first the observed quantile (at least for the probabilities still covered by the samples), the quantile computed with the Hill estimator (Equation 5) and then the quantile computed using the Gaussian model as introduced and calibrated in Section 2. The results are fairly striking: as long as we can compare the predicted quantiles to empirically observed ones, we can clearly see that the estimates from the Hill estimator are much closer to the observable reality than the ones from the Normal model. The Gaussian values are clearly underestimating the extremes as we saw in Section 2. One

[5]Meaning the worst daily movement to be expected within 5 (10, 20, 30, 40) years.

FX rate		Worst daily movement within ...				
		5 years	10 Years	20 Years	30 Years	40 Years
EUR/USD	at upper	3.67%	4.25%	4.94%	5.38%	5.73%
	at lower	5.20%	6.53%	8.19%	9.36%	10.28%
JPY/USD	at upper	3.43%	3.96%	4.57%	4.97%	5.28%
	at lower	4.73%	5.91%	7.39%	8.42%	9.24%
GBP/USD	at upper	2.90%	3.38%	3.93%	4.29%	4.57%
	at lower	4.20%	5.22%	6.22%	7.60%	8.39%
CHF/USD	at upper	3.11%	3.51%	3.96%	4.25%	4.47%
	at lower	4.17%	5.05%	6.13%	6.86%	7.43%

Table 5: Quantile estimates according to Equation 5 evaluated at the upper and lower bounds of the 95% confidence interval for $\hat{\alpha}$ as given in Table 3.

can also observe that the increase of the large movement size with respect to the decrease of its probability is more accentuated than in the Gaussian case and here too reproduces better the observed increase in the sample.

Given the non-neglegible width of the confidence intervals for the tail index estimates stated in Table 3, it is worthwhile to explore the variability of the quantile estimates given in Table 4. We can do this by evaluating Equation 5 at the upper and lower bounds of the 95% confidence intervals for $\hat{\alpha}$ given in Table 3. We show the respective values in Table 5. Comparing these bounds with the values given in Table 4, we notice that the observed values - where available - are close to or below the lower bound for the Hill estimates, suggesting that the latter are rather conservative estimates for the potential extreme movements. However, the Hill estimates are still much more closely related to the observable reality than the estimates from the Gaussian model.

These facts give us confidence that the extrapolated values obtained from the Hill estimator will represent a sufficiently conservative estimate of the risk, whereas the results give us the feeling that the Gaussian model dangerously underestimates the actual risk. We are not trying here to quantify very precisely the extreme movements but rather to capture the essentials of them, so that we can set reasonable limits of exposure.

To conclude this paper, we turn now to the problem of setting limits on open FX positions in order to restrict the risk of large losses. We first have to decide on the maximum size of the loss we are prepared to incur in a

FX rate	Method of Calculation	Position limits for one daily loss of USD 1 mn. within ... 5 years	10 Years	20 Years	30 Years	40 Years
EUR/USD	Observed	26	26	–	–	–
	Hill	23	19	16	14	13
	Gaussian	37	35	33	32	32
JPY/USD	Observed	28	21	16	–	–
	Hill	25	21	18	16	15
	Gaussian	47	42	40	39	38
GBP/USD	Observed	31	26	22	–	–
	Hill	29	24	20	18	17
	Gaussian	47	45	43	41	41
CHF/USD	Observed	31	28	24	–	–
	Hill	28	24	21	19	18
	Gaussian	41	39	37	36	35

Table 6: Proposed limits for open overnight FX positions of according to three different methods. The numbers are given in million USD.

certain period of time. Let us assume that the maximum loss we are ready to risk is 1'000'000 USD any one day and that we do not want to risk losing it more than once every 5, 10, 20, 30 and 40 years. Using the numbers given in Table 4, it is now easy to compute the limits to be set for different FX rates.

We provide in Table 6 the limits for the major FX rates considered in this study and one sees that if we are using the Gaussian model open positions would be permitted that are about twice as large as it would be prudent. We can see from this that risk limit setting according to the Gaussian model may easily result in unaffordably high losses.

5 Conclusion

Considering the results of our study, we can conclude that daily fluctuations in FX rates are higher than usually assumed, and that the Gaussian model is not able to reflect these extreme fluctuations properly. Luckily, we can quantify this risk by concentrating our attention on the tails of the distribution. Thus, it is possible to set realistic limits for trading on the FX market and, as we use to say among insurers, quantifying the risks transforms them from a nuisance into an opportunity.

References

[Adler et al., 1998] Adler R., Feldman R., and Taqqu M., editors, 1998, *A Practical Guide to Heavy Tails*, Birkhäuser, Boston.

[Black and Scholes, 1973] **Black F. and Scholes M.**, 1973, *The pricing of option and corporate liabilities*, Journal of Political Economy, **81**, 637–659.

[Dacorogna et al., 2001] **Dacorogna M. M., Gençay R., Müller U. A., Olsen R. B., and Pictet O. V.**, 2001, *An Introduction to High Frequency Finance*, Academic Press, San Diego,CA.

[Danielsson et al., 1997] **Danielsson J., de Haan L., Peng L., and de Vries C. G.**, 1997, *Using a bootstrap method to choose the sample fraction in tail index estimation*, Tinbergen Institute discussion paper TI97-016/4, 1–30.

[Embrechts et al., 1997] **Embrechts P., Klüppelberg C., and Mikosch T.**, 1997, *Modelling Extremal Events*, volume 33 of *Applications of Mathematics Stochastic Modelling and Applied Probability*, Springer, Berlin.

[Embrechts et al., 1998] **Embrechts P., Resnick S., and Samorodnitzky G.**, 1998, *Living on the edge*, Risk, 96–100.

[Hill, 1975] **Hill B. M.**, 1975, *A simple general approach to inference about the tail of a distribution*, Annals of Statistics, **3**(5), 1163–1173.

[J. P. Morgan, 1996] **J. P. Morgan**, 1996, *RiskMetrics – technical document*, Technical report, J. P. Morgan and International marketing – Reuters Ltd.

[Jorion, 1997] **Jorion P.**, 1997, *Value at Risk : The New Benchmark for Controlling Market Risk*, Irwin Professional, Chicago.

[Koedijk et al., 1990] **Koedijk K. G., Schafgans M. M. A., and De Vries C. G.**, 1990, *The tail index of exchange rate returns*, Journal of International Economics, **29**, 93–108.

[Leadbetter et al., 1983] **Leadbetter M., Lindgren G., and Rootzén H.**, 1983, *Extremes and related properties of random sequences and processes*, Springer Series in Statistics. Springer-Verlag, New York Berlin.

[Longin, 1996] **Longin F. M.**, 1996, *The asymptotic distribution of extreme stock market returns*, Journal of Business, **69**(3), 383–407.

[Pictet et al., 1998] **Pictet O. V., Dacorogna M. M., and Müller U. A.**, 1998, *Hill, bootstrap and jackknife estimators for heavy tails*, published in "A practical guide to heavy tails:"Statistical Techniques for Analysing Heavy Tailed Distributions", edited by Murad Taqqu and published by Birkhäuser, 283–310.

[Resnick, 1987] **Resnick S.**, 1987, *Extreme Values, Regular Variation, and Point Processes*, Springer, New York.