

# **Stochastics for the worst case : distributions and risk measures for minimal returns**

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**Abstract**

Distributions for returns are used to compute the capital charge for portfolios in investment banks. The mainstream definition of returns is based on closing prices and neglects the important effects of intraday trading activity on the losses . In this paper we introduce "minimal returns", a definition of returns that accounts for intraday trading and gives a worst-case approach on losses. We suggest an appropriate distribution for minimal returns that can be used to compute Value at Risk and coherent risk measures, as suggested by Artzner et al. (1997).

*Key words:* Risk measures, Value at Risk, Stock Returns, Binomial Tree, Brownian Motion, Capital Charge, Risk Management, Worst Case Analysis, Binomial Tree Extremes  
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## 1 Introduction

The interest in adequate stochastic models for returns has steadily increased since the Bank for International Settlements issued her "Amendment To The Capital Accord to Incorporate Market Risks" in 1996 (BCBS (1996)). The amendment allows investment banks to compute their capital charge with an internal stochastic model based on returns and corresponding losses. For internal risk models usually daily closing prices are used to calculate losses and the corresponding continuous compounded returns (see Campbell et al. (1997, pages 9-13)). In order to calculate a risk measure a distribution is fitted to the returns. A widely used risk measure is the so called Value at Risk (VaR) which is the 99%-quantile of the loss/profit distribution. The risk measure will depend on the time horizon and on the prices used in calculating the return and the corresponding loss (see Gouriéroux and Jasiak (2001, page 435)), but not on prices within that time horizon. A very interesting discussion about properties that risk measures should fulfill was initiated by Artzner et al. (1997, 1999), who introduced the concept of coherent risk measures. There is evidence that VaR is not always coherent. A discussion about how to define returns for estimating risk measures is as necessary as a discussion on the coherence of risk measures, since intraday trading is an important fact not accounted for by the usual definition of losses and returns. Intraday trading is not properly reflected in the computed risk measures, even if they are coherent.

## 2 Minimal returns: a solution to account for intraday trading

Suppose that on day  $t$  trading takes place at the times  $(t, 0), (t, 1), \dots, (t, N)$  where  $p_{(t,i)}$  denotes the price at time  $(t, i)$ . A stockholder<sup>1</sup> evaluates the value of its stock in  $(t, 0)$  at  $p_{(t,0)}$  and intends to sell the stock in period  $t$ . If he sells his stock in  $(t, i)$  then his loss/profit will be  $v_t(i) = -(p_{(t,i)} - p_{(t,0)})$ . The continuous compounded return on his investment will be  $r_t(i) = \log(p_{(t,i)}) - \log(p_{(t,0)})$ . In the usual framework the stockholder looks only at the **period return**  $r_t = r_t(N)$  and neglects all other possible returns  $r_t(i)$ . He will also neglect the worst case, that he sells his stock and reaches the maximal loss in period  $t$ ,  $v_t^{\max} = \max\{v_t(i) | 0 \leq i \leq N\}$ . The continuous compounded return he obtains in that case is  $r_t^{\min} = \min\{r_t(i) | 0 \leq i \leq N\}$  and is obviously less than the period return  $r_t(N)$ . We call  $r_t^{\min}$  the **minimal return** of the stock in period  $t$ . By their definition, the minimal return and the maximal loss reflect the information between  $(t, 0)$  and  $(t, N)$ , looking at the worst alternative in period  $t$ . When modeling minimal returns one will therefore look on a systematic basis at the worst case that can be encountered during a holding period.

Computing the VaR for the maximal loss instead of the VaR for the period loss has an important advantage. Since the maximal loss is by definition less than its

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<sup>1</sup> A similar approach may be used for portfolios.

99%-quantile with a probability of 99% and maximal losses are greater than any loss  $v_t(i)$ , the probability that any loss  $v_t(i)$  is covered by the 99%-quantile of the maximal loss is at least 99%. When computing the Value at Risk for the maximal loss one will get an amount covering **any** loss  $v_t(i)$  with a probability of at least 99%. A similar result does not hold for the period loss VaR.

### 3 The distribution of minimal returns when intraday returns follow a binomial tree

In order to compute risk measures for maximal losses, one needs appropriate distributions to model minimal returns. First we will give the distribution for minimal returns under the simple assumption that the returns  $r_t(i)$  follow a symmetric N-Step binomial tree starting at 0. We assume that  $r_t(0) = 0$  and:

$$r_t(i) = \sum_{k=1}^i x_{(t,k)} \quad (1)$$

with  $x_{(t,k)}$  independent random variables on a probability space  $(\Omega, F, P)$  taking values  $u$  and  $-u$  with probabilities  $p$  and  $1-p$ . We look for the distribution function for minimal returns in this model, noting that minimal returns take only the values  $-ju$  with  $j \in \{0, 1, \dots, N\}$ .

**Proposition 3.1.** *Let  $x$  be a real number, if  $x \in [-Nu, 0]$  then let  $-j = \left\lfloor \frac{x}{u} \right\rfloor$ . The distribution function for the minimal return  $r_t^{\min}$  is:*

$$F_{r^{\min}}(x) = P(r_t^{\min} \leq x) = \begin{cases} 0 & \text{if } x < -Nu \\ \sum_{k=0}^{\lfloor \frac{N-j}{2} \rfloor} \binom{N}{k} p^k (1-p)^{N-k} + \sum_{k=\lfloor \frac{N-j}{2} \rfloor + 1}^{N-j} \binom{N}{j+k} p^k (1-p)^{N-k} & \text{if } -Nu \leq x \leq 0 \\ 1 & \text{if } x \geq 0 \end{cases} \quad (2)$$

For the proof note that a given path with exactly  $k \in \{0, \dots, N\}$  upward movements occurs with probability  $p^k \cdot (1-p)^{N-k}$ . Let  $A(k, j, N)$  be the number of paths with exactly  $k$  upward movements having a minimal return  $r_t^{\min} \leq -ju$ . The probability that any path with  $k$  upward movements has a minimal return  $r_t^{\min} \leq -ju$  is  $A(k, j, N)p^k(1-p)^{N-k}$ . Summing after all  $k \in \{0, \dots, N\}$  gives the corresponding distribution function.

To compute  $A(k, j, N)$  one can regard three different cases. Note that for a fixed  $k$  all paths have the same period return  $r_t(N) = (2k - N)u$ . In the first case  $k$  is such that  $r_t(N) \leq x$ . Since  $r_t^{\min} \leq r_t(N)$  all paths with  $k$  upward movements have a minimal return  $r_t^{\min} \leq x$  and  $A(k, j, N) = \binom{N}{k}$ . In the second case, the number

of downward movements  $N - k$  is less than  $j$ , so one cannot reach the level  $-ju$  when starting from zero and  $A(k, j, N) = 0$ . In the third case, we can compute  $A(k, j, N)$  by the reflection principle, see Pliska (1999, pages 104-106). Reflect all paths with  $k$  upward movements that touch or cross  $-ju$  in the level  $-ju$  until they first reach  $-ju$ . We obtain paths in an  $N$ -Step binomial tree starting at  $-2ju$  and ending in  $(2k - N)u$  and we can compute  $A(k, j, N)$  in the new binomial tree as  $A(k, j, N) = \binom{N}{j+k}$ .

#### 4 The distribution of minimal returns in a continuous case

A known stochastic model for stock returns is the brownian motion, a continuous analog of the  $N$ -Step binomial tree (see Campbell et al. (1997); Korn and Korn (2000)). One can obtain the brownian motion with drift  $\mu \in \mathbb{R}$  and volatility  $\sigma > 0$  as a limiting case for the binomial tree in section 3 assuming, as in Cox et al. (1979), that  $p$  and  $u$  are functions of  $N$ ,  $\sigma$  and  $\mu$  given by:

$$p := p(\mu, \sigma, N) := \frac{1}{2} + \frac{\mu}{2\sigma\sqrt{N}} \quad (3)$$

$$u := u(\sigma, N) := \frac{\sigma}{\sqrt{N}} \quad (4)$$

and letting  $N \rightarrow \infty$ . The distribution of the period return will then be a normal distribution with expectation  $\mu$  and variance  $\sigma^2$ . For the distribution of the minimal return one can show the following result.

**Proposition 4.1.** *The distribution for the minimal returns in the considered continuous case is:*

$$F_{r_{\min}}(x) = \begin{cases} \Phi\left(\frac{x - \mu}{\sigma}\right) + \exp\left(\frac{2\mu x}{\sigma^2}\right) \Phi\left(\frac{x + \mu}{\sigma}\right) & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \quad (5)$$

where  $\Phi(\cdot)$  denotes the  $N(0, 1)$  distribution function.

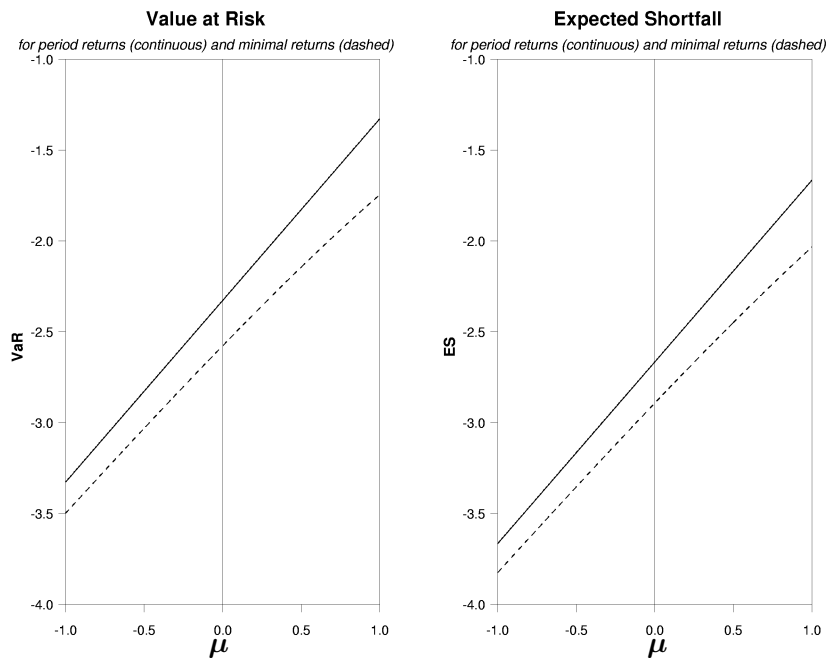
One can either prove Proposition 4.1 by starting from the discrete distribution in section 3, or in a continuous framework, as in Korn and Korn (2000, Lemma 4.5, page 168).

The distribution in Proposition 4.1 appeared previously in the context of lookback option pricing in Goldman et al. (1979). It was also used for down-and-out option pricing as in Korn and Korn (2000, pages 167-170) and for estimating default probabilities as in Duffie and Singleton (2003). Because it depends only on the expectation  $\mu$  and volatility  $\sigma$  of the period return, it is as simple to handle for minimal returns as the normal distribution for period returns.

## 5 Conclusions

Assuming that returns follow a brownian motion with drift  $\mu$  and volatility  $\sigma$  and using Proposition 4.1 one can compute risk measures for period stock returns and minimal stock returns. In Figure 1 we compare the VaR for the period return (VaR is here the 1%-quantile of the period return) with the VaR for the minimal return ( $\text{VaR}^{\min}$  is the 1%-quantile of the minimal return) and the expected shortfall for the period return  $\text{ES} = \text{E}(r_t | r_t < \text{VaR})$  with the expected shortfall for the minimal return  $\text{ES}^{\min} = \text{E}(r_t^{\min} | r_t^{\min} < \text{VaR}^{\min})$  for different values of  $\mu$  keeping  $\sigma = 1$ . VaR's and ES's for minimal returns and period returns differ significantly. Minimal returns account for intraday trading and their distribution can be modeled in a simple way using the result of Proposition 4.1. Thus we have a simple tool for implementing risk measures that account for intraday trading.

Fig. 1. Value at Risk and Expected Shortfall for period and minimal returns



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