Existence of a Coalitionally Strategyproof Social Choice Function: A Constructive Proof*

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Abstract

This paper gives a concrete example of a nondictatorial, coalitionally strategyproof social choice function for countably infinite societies. The function is defined for those profiles such that for each alternative, the coalition that prefers it the most is "describable." The "describable" coalitions are assumed to form a countable Boolean algebra. The paper discusses oligarchical characteristics of the function, employing a specific interpretation of an infinite society. The discussion clarifies within a single framework a connection between the negative result (the Gibbard-Satterthwaite theorem) for finite societies and the positive result for infinite ones.

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1 Introduction

A *social choice function* assigns an alternative to each profile of individual preferences. If the function is not *coalitionally strategyproof*, then at some profile $\mathbf{p}$, a coalition of individuals are better off misrepresenting their preferences (that is, a deviation from $\mathbf{p}$ is profitable for the coalition). I give a concrete example of a nondictatorial, coalitionally strategyproof social choice function for countably infinite societies. The existence of such a function has been shown by Pazner and Wesley [16] in a non-constructive manner for the unrestricted domain of profiles.

The existence of (individually or coalitionally) strategyproof social choice functions has been studied both in the case of a finite set of individuals (voters) and in the case of an infinite set of individuals.

When there are only finitely many individuals, there exists no nondictatorial, individually strategyproof social choice function on the unrestricted domain of profiles (of individual preferences satisfying the usual ordering properties). The Gibbard-Satterthwaite theorem [5, 18] states that.

In contrast, when there are infinitely many individuals, there exists a nondictatorial, *individually* strategyproof social choice function on the unrestricted domain of profiles. Pazner and Wesley [16, p. 254] show the existence by giving a *concrete* example.

In the case of an infinite society, considering *coalitional* strategyproofness, not just *individual* strategyproofness is particularly important, in view of interpretation. (I will comment on this point in Section 4.1.) Pazner and Wesley indeed consider coalitional strategyproofness. They prove [16, Theorem] that when there are infinitely many individuals, there exists a nondictatorial, *coalitionally* strategyproof social choice function on the unrestricted domain. The proof, however, relies on a non-constructive mathematical technique,[1] failing to present any concrete example of a function satisfying the conditions.

In their subsequent work, Pazner and Wesley [17] turn to the problem of explicitly constructing a social choice function that is nondictatorial and coalitionally strategyproof. Their approach is to modify the notion of strategyproofness. They start by defining a nondictatorial social choice function explicitly. Next, they fix an arbitrary *countable* collection of coalitions. Intuitively, the coalitions in the collection are the "describable" ones. Then,

[1] The proof relies on the axiom SPI, "each infinite set carries a free ultrafilter"—an axiom that cannot be derived from the Zermelo-Fraenkel axioms of set theory without the axiom of choice. Brunner and Mihara [3] give a further discussion.

they conclude by showing [17, Theorem 2] that *for almost every profile* the function is strategyproof for all coalitions in the collection: no coalition in the collection can deviate from the profile profitably. A drawback of the function is that it violates *neutrality* (equal treatment of alternatives). In fact, it chooses the same alternative for almost all profiles.

I take a different approach in this paper. Instead of considering social choice functions on the unrestricted domain of profiles, I restrict (Section 2) profiles in a natural way. I admit only those profiles such that for each alternative the set (coalition) of individuals that prefer it the most, is "describable" (or "observable")—and I assume that there are only countably many "describable" coalitions. I then (Section 3) construct a nondictatorial social choice function on the restricted domain of those *admissible profiles* and show the main result (Theorem 2) that it is coalitionally strategyproof in the usual sense (i.e., strategyproof for all admissible profiles and for all coalitions). Though the function satisfies the neutrality condition, it has certain oligarchical characteristics. I observe (Section 4.1) this fact, employing a specific interpretation of an infinite society. The observation clarifies a connection between the negative result (the Gibbard-Satterthwaite theorem) for finite societies and the positive result for infinite ones. I then conclude the paper with a discussion (Section 4.2) that the function may be regarded as a component of a more appealing superrule, suggesting a problem for further investigation.

2 Framework

Let $N = \{1, 2, 3, \ldots\}$ be a countably infinite set of individuals (voters). Let X be a *finite* set of alternatives, which has at least three elements. Let $\mathcal{S}$ be the set of (strict) preferences, i.e., total, asymmetric, and transitive binary relations on X. (For simplicity, indifference is not allowed.)

A *Boolean algebra* $\mathcal{B}$ consisting of subsets of N satisfies the following: (i) $\emptyset$, $N \in \mathcal{B}$; (ii) $A \cup B$, $A \cap B$, $A^c \in \mathcal{B}$ if A, $B \in \mathcal{B}$ (where A^c denotes the complement of A). Intuitively, an element of a Boolean algebra is a coalition describable (or observable) to the planner (an imaginary person that executes a social choice function). The main theorem assumes a countable Boolean algebra that contains all finite coalitions. The countability condition corresponds to the real-world observation that a language has to be used to describe anything, but there are only countably many sentences in a written language (provided that the alphabet of the language consists of finitely many letters). The condition that the Boolean algebra contains all finite coalitions is equivalent to the condition that it contains all one-individual coalitions. The intuition is that each individual is describable. (I assume a countable set of individuals for this reason.) Each of the following four examples of a Boolean algebra is (i) countable and (ii) contains all finite coalitions.

Example 1. The collection of all finite sets and all cofinite sets (the complements of a finite set) in N is the minimal Boolean algebra that satisfies the two conditions above. Empirical scientists that take an extreme position might reject observability of any infinite object. This is the only Boolean algebra that is acceptable to them.

Example 2. Let REC consist of all *recursive sets* in N. (According to Church's thesis these are the sets whose membership is algorithmically decidable [20, 4].) Then REC is a Boolean algebra. The notion of recursive coalitions (which I used [12, 13] in social choice theory) is a stringent formalization [4, pp. 225 and 197] of the intuitive notion of "describable" coalitions.

Example 3. An *arithmetical set* in N is a set definable in the intended structure for the language of number theory (as described in Enderton [4], especially pp. 235–7, 174–5, and 88). The class of arithmetical sets is a Boolean algebra containing all recursive sets. The notion of arithmetical coalitions is a less stringent formalization of the intuitive notion of "describable" coalitions than Example 2.

Example 4. A *rational interval on* $[0, 1]$ is a set of rational numbers that can be expressed in one of the following forms: $\{x : a < x < b\}$, $\{x : a < x \leq b\}$, $\{x : a \leq x < b\}$, $\{x : a \leq x \leq b\}$, where a and b are some rational numbers such that $0 \leq a \leq b \leq 1$. The collection of all finite unions of rational intervals on $[0, 1]$ is a Boolean algebra. Since there is a one-to-one correspondence between the set of rational numbers and the set N, each element of this Boolean algebra can be regarded as a subset of N. This Boolean algebra (as well as its higher-dimensional extensions) has an obvious interpretation in areas (such as political theory and regional science) where Hotelling locational (spatial) models are used.

The next example is different from the four examples just mentioned. It is a finite Boolean algebra, which does not contain all finite coalitions. The example enables one to treat (as in Section 4.1) the case of finitely many people within the present framework.

Example 5. Consider a finite partition of N. (It does not have to be that all the partition elements are infinite.) The collection of all unions of elements in the partition is a Boolean algebra. In fact, any finite Boolean algebra is of this form, where the partition consists of its *atoms* (nonempty sets containing no proper nonempty subset that belongs to the Boolean algebra).

Let $\mathcal{B}$ be a Boolean algebra. A *profile* is a list $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i \in N} \in \mathcal{S}^N$ of individual preferences $\succ_i^{\mathbf{p}}$, where $i \in N$. A profile $(\succ_i^{\mathbf{p}})_{i \in N}$ is $\mathcal{B}$-*admissible* if

$$\{\, i \in N : x \succ_i^{\mathbf{p}} y \text{ for all alternatives } y \neq x \,\} \in \mathcal{B}$$

for all $x \in X$. Denote by $\mathcal{S}^N(\mathcal{B})$ the set of all $\mathcal{B}$-admissible profiles. A $(\mathcal{B})$-*social choice function* is a function F from $\mathcal{S}^N(\mathcal{B})$ *onto* X which maps each profile $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i \in N}$ into an alternative. I assume that F is onto (i.e., the image is X) to avoid trivial cases.

Remark 1. The reader may object to the admissibility condition on the grounds that it implies correlation between the preferences of different individuals. To defend the condition, I give two interpretations. The first is the uncertainty interpretation in Section 4.1, where there are only finitely many people, who face infinitely many states. In this case, the admissibility condition simply reflects the reasonable epistemological requirement that each person can describe for each alternative, the set of states in which he prefers it the most. The second interpretation is a society made up of infinitely many people extending into the indefinite future. In this case, it is natural to suppose that we are dealing with preferences reported by a finite number of living voters (who "represent" some future generations), rather than the actual preferences. The admissibility condition reflects the reasonable requirement that what is reported should be describable. ‖

Remark 2. In related papers [11, 12, 13, 14], I used the domain of *measurable* profiles, as in Armstrong [1, 2]. (I say that a profile $(\succ_i^{\mathbf{p}})_{i \in N}$ is $\mathcal{B}$-*measurable* ($\mathbf{p} \in \mathcal{S}_{\mathcal{B}}^N$) if $\{\, i \in N : x \succ_i^{\mathbf{p}} y \,\} \in \mathcal{B}$ for all $x, y \in X$. Note that when X is finite, all $\mathcal{B}$-measurable profiles are $\mathcal{B}$-admissible: $S_B^N \subset S^N(B)$.) I argue below (Remark 5) that the main result (Theorem 2) holds for either domain. ‖

Suppose that a Boolean algebra $\mathcal{B}$ contains all coalitions consisting of only one individual (thus $\mathcal{B}$ contains all finite coalitions). A $(\mathcal{B})$-social choice function F is *dictatorial* if there exists an individual i such that for all $\mathcal{B}$-admissible profiles $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i\in N}$, we have $F(\mathbf{p}) \succ_i^{\mathbf{p}} y$ for all alternatives $y \neq F(\mathbf{p})$. A ($\mathcal{B}$)-social choice function F is said to be *coalitionally manipulable* if for some $\mathcal{B}$-admissible profiles $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i\in N}$, $\mathbf{p}' = (\succ_i^{\mathbf{p}'})_{i\in N}$ and for some nonempty coalition $E \subseteq N$, it is the case that $\succ_i^{\mathbf{p}'} = \succ_i^{\mathbf{p}}$ for all $i \notin E$ and $F(\mathbf{p}') \succ_i^{\mathbf{p}} F(\mathbf{p})$ for all $i \in E$. (Note that E need not be a member of $\mathcal{B}$.) If in the definition of coalitional manipulability, "nonempty coalition" is replaced by "one-individual coalition," we have individual manipulability. F is *coalitionally (individually) strategyproof* if it is not coalitionally (individually) manipulable.

3 Construction of the social choice function

Since the purpose of this paper is to exhibit a nondictatorial, coalitionally strategyproof ($\mathcal{B}$)-social choice function in an explicit fashion, I start by defining a candidate G for such a function. I assume that $\mathcal{B}$ is a countable Boolean algebra that contains all finite coalitions.

To define the function G, I now construct a collection $\mathcal{U}$ of coalitions in the Boolean algebra $\mathcal{B}$. Each coalition in the collection is understood to be a "majority" or "plurality" of the individuals. First, fix an enumeration $C_0, C_1, C_2, \ldots$ of all elements of $\mathcal{B}$. (I allow repetitions.) Then, define the sequence

$$\begin{aligned} \mathcal{U}_0 &= \emptyset \\ \mathcal{U}_{s+1} &= \begin{cases} \mathcal{U}_s \cup \{C_s\} & \text{if this family has an infinite intersection,} \\ \mathcal{U}_s & \text{otherwise.} \end{cases} \end{aligned}$$

The condition for the first case in the definition of $\mathcal{U}_{s+1}$ means that $(\bigcap \mathcal{U}_s) \cap C_s$ is infinite, where $\bigcap \mathcal{U}_s = \bigcap_{C\in\mathcal{U}_s} C$. Finally, let $\mathcal{U} = \bigcup_{s=0}^{\infty} \mathcal{U}_s$. Note that $\mathcal{U}$ does not contain any finite coalitions. (In the trivial case of Example 1, $\mathcal{U}$ consists of all cofinite coalitions. What is significant about the above construction is that it defines a collection $\mathcal{U}$ for nontrivial cases too.)

Having constructed the collection $\mathcal{U}$ of "majorities," I can define the social choice function G from $\mathcal{S}^N(\mathcal{B})$ to X by

$$G(\mathbf{p}) = x \iff \{\, i \in N : x \succ_i^{\mathbf{p}} y \text{ for all alternatives } y \neq x \,\} \in \mathcal{U} \qquad (1)$$

for each profile $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i\in N}$. (I show below that G is well-defined.) Intuitively, G chooses an alternative x when the individuals that prefer x the most, form a "majority." $\mathcal{U}$ is called the set of *decisive* coalitions for G. It is obvious that G is nondictatorial since $\mathcal{U}$ does not contain any coalitions consisting of a single individual.

The next proposition (which is proved in Appendix A) is the key to the main theorem. To state the proposition, I now introduce the notion of an "ultrafilter" (Koppelberg [9, p. 32] gives an exposition). A *filter* $\mathcal{F}$ on a Boolean algebra $\mathcal{B}$ is a family of sets in $\mathcal{B}$ satisfying: (i) $\emptyset \notin \mathcal{F}$; (ii) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$; (iii) if A, $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. Intuitively, a filter is a family of "large" sets. An *ultrafilter* is a filter $\mathcal{F}$ that satisfies: if $A \notin \mathcal{F}$, then $A^c \in \mathcal{F}$. Suppose $\mathcal{B}$ contains all finite sets in N. Then we say that an ultrafilter $\mathcal{F}$ is *fixed* if it is of the form $\mathcal{F} = \{ A \in \mathcal{B} : i \in A \}$ for some $i \in N$; otherwise, it is called *free* and does not contain any finite sets.

Proposition 1 *The set $\mathcal{U}$ of decisive coalitions is a free ultrafilter.*

Remark 3. To see the simplicity of the function G informally, imagine that the planner thinks of coalitions C_0, C_1, C_2, ... one by one, and for each C_s, she determines whether it is decisive. To determine whether C_s is decisive or not, the planner does not have to know much. First, she only needs her "past" decisions about C_0, C_1, ..., C_{s-1}. (In fact, she determines C_s to be decisive as long as doing so does not result in the collection $\mathcal{U}$ failing to be a free ultrafilter: If C_s is put into the collection $\mathcal{U}_{s+1}$ when $\mathcal{U}_s \cup \{C_s\}$ does not have an infinite intersection, then resulting $\mathcal{U}$ has a finite element, namely the intersection of all the decisive coalitions up to C_s. But free ultrafilters cannot have finite elements.) Second, she only needs the "aggregate data" $\bigcap \mathcal{U}_s$, the intersection of all coalitions that she has determined to be decisive up to that point. This simplifies information processing considerably. More formal analysis can be given (as in Mihara [13]) in a framework of *computability analysis of social choice*, which studies algorithmic properties of social choice rules. ‖

The following is the main theorem. It is concerned with the particular G described above. It thereby establishes not only that a social choice function satisfying the properties mentioned exists, but also that such a function can be constructed explicitly.

Theorem 2 *Let $\mathcal{B}$ be a countable Boolean algebra that contains all finite coalitions. Suppose that C_0, C_1, C_2, ... is an enumeration of all elements of $\mathcal{B}$. Then the nondictatorial $(\mathcal{B})$-social choice function $G\colon \mathcal{S}^N(\mathcal{B}) \to X$ described above is coalitionally strategyproof.*

With the help of Proposition 1, the proof of Theorem 2 goes as in the proof of the Theorem in Pazner and Wesley [16, p.255]. Note that (1) well-defines G since exactly one of the partition elements

$$\{ i \in N : x \succ_i^{\mathbf{p}} y \text{ for all alternatives } y \neq x \},$$

where x ranges over the finite X, belongs to the ultrafilter $\mathcal{U}$ (by a well-known property [9, p. 32] of an ultrafilter).

Remark 4. The proof in Pazner and Wesley carries through for arbitrary deviating coalitions E provided that $\mathcal{U}$ is defined on a Boolean algebra and profiles are admissible. ‖

Remark 5. To show that the main result is unaffected when the domain $\mathcal{S}^N(\mathcal{B})$ of admissible profiles is replaced by the domain $\mathcal{S}_{\mathcal{B}}^N$ of measurable profiles, I now construct a social choice function g on the latter domain. Define g from $\mathcal{S}_{\mathcal{B}}^N$ to X by

$$g(\mathbf{p}) = x \iff \{ i \in N : x \succ_i^{\mathbf{p}} y \text{ for all alternatives } y \neq x \} \in \mathcal{U}$$

for each profile $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i \in N}$ in $\mathcal{S}_{\mathcal{B}}^N$. (The function g is the restriction of G to $\mathcal{S}_{\mathcal{B}}^N$.) By the same argument as that in the proof, one can see that g is well-defined, nondictatorial, and coalitionally strategyproof. (That g is coalitionally strategyproof can be proved directly from the fact that G is coalitionally strategyproof.)

Conversely, I can first construct a nondictatorial, coalitionally strategyproof social choice function on the domain of measurable profiles and then extend [14, Corollary 3] it to a nondictatorial, coalitionally strategyproof social choice function on the domain of admissible profiles. ‖

4 Discussion

The social choice function G constructed in Section 3 has the characteristics of an *oligarchy*, as Pazner and Wesley [16] point out. This means that an infinite but arbitrarily "small" coalition of individuals can decide the social outcome, regardless of the preferences of those outside it. (Kirman and Sondermann [8], Armstrong [1], Sen [19], and Lauwers and Van Liedekerke [10] make a similar observation in the context of Arrow's Theorem for infinite societies.) Section 4.1 discusses this particular lack of anonymity under a specific interpretation of an infinite society. The discussion will clarify a connection between the negative result for finite societies and the positive result for infinite ones. Section 4.2 concludes the paper by suggesting that we view the function as a subrule of some game form.

4.1 Connection with the result for finite societies

According to my interpretation [12], an infinite society of *individuals* consists of finitely many *persons*, whose preferences are conditioned on infinitely many states (expressing uncertainty). If a *person* j prefers an alternative (action) x to an alternative (action) y in a state s, I say that the *individual* (j, s) prefers x to y. (The same person at two different states are thus viewed as two different individuals.) In this way, I can obtain infinitely many *individuals*, who are *persons* at different states. At the time the social choice is made, the real state is not known to the persons.

Remark 6. It might appear as natural to model the situation as one with finitely many people having *preferences over an infinite set of (state-specific) alternatives*. In such a model, if a person j prefers an action x to another action y in a state s, then the person j is said to prefer a (state-specific) alternative (x, s) to another (y, s). In this case, the Gibbard-Satterthwaite theorem for finite societies implies the nonexistence of nondictatorial, coalitionally strategyproof social choice function. The difference of the result is due to the fact that in this model, each person (and the society) is required to compare every pair (x, s), (y, t) in the set of alternatives. Where a state is chosen by Nature (chance) rather than by the society, this sort of inter-state comparisons are often meaningless for the decision-making. My interpretation above avoids inter-state comparisons. ||

Note that this interpretation suggests the importance of considering *coalitional* strategyproofness, not just *individual* strategyproofness, in the case of an infinite society. Under the interpretation, a particular person $\bar{j}$ can misrepresent the preferences of many individuals $(\bar{j}, s)$, for various s. She can thus form profitably deviating coalitions by herself, without communicating with other persons.

Now, to see the oligarchical characteristics of the function G under the interpretation above, consider the profiles where each person has the same preference at all states. (Different persons may have different preferences, but all the preferences that belong to a given person is the same across different states. Note that these are the measurable profiles with respect to the Boolean algebra in Example 5, where each partition element corresponds to a person.) This may happen when persons are unable to distinguish any state from other states because their knowledge is severely limited. If G is restricted to these profiles, it can be re-interpreted as a social choice function for a finite set of persons. It turns out (by a property [9, p. 32] of an ultrafilter) in this case that only one person's preferences count: the social outcome will always be the most preferred alternative of that (dictatorial) person. In other words, an "oligarchy" consisting of just one person exists. The restricted function is coalitionally strategyproof (the dictator obviously has no gain misrepresenting the preference; the others have no gain misrepresenting their preferences since doing so simply does not count), but only in the trivial sense. A dictatorial person exists, since the number of persons is finite. Since the number can be arbitrarily large, this means that an arbitrarily small "coalition" (i.e., a person in an arbitrary large society) can dictate the outcome.

A similar argument applies even if each person's preference is more responsive to states. Suppose for example that each person has a finite partition of the set of states such that for each partition element, she has the same preference at all states belonging to the element. Suppose further that this is because she can only distinguish between states belonging to different partition elements. (In the language of the formal model of knowledge (see Osborne and Rubinstein [15, Section 5.1]), the *information function* of a person is partitional here, and the partition consists of finitely many events.) In this case, (by a property [9, p. 32] of an ultrafilter) there are a person and a collection (one of her partition elements) of states that she cannot distinguish between, that dictate the social outcome. This implies (assuming a partitional information function) that in order to avoid this sort of dictatorial person-collection pair, at least one person must have an infinite cognitive power (in the sense that she must be able to distinguish between infinitely many states; formally, her partitional information function must be infinitely valued).

4.2 The oligarchical function as a subrule

The discussion in Section 4.1 suggests that the function G itself is not very appealing as a rule for democratic decision making. (The discussion applies to any nondictatorial, coalitionally strategyproof social choice functions, not just to G. So, G is not exceptional.) This, however, does not mean that the function is not interesting. Consider, for example, a dictatorial social choice function for a finite society. The dictatorial function itself is not very appealing. But when dictatorial functions are used to form a superrule, that superrule may satisfy certain nice properties. Indeed, Hylland [7] shows that the random dictators (where dictatorial functions are combined, given fixed weights) are the only strategyproof rules satisfying a certain condition. There, a dictatorial function can be regarded as a subrule (component of the superrule), which Nature (chance) chooses. Also, Hurwicz and Schmeidler [6] show that a game form in which an individual ("kingmaker") in effect selects a dictatorial function satisfies certain nice properties. A dictatorial function is a subrule there, which the kingmaker chooses when playing the game form.

In a similar fashion, the "oligarchical" function G may be regarded as a subrule of a superrule having nice properties. A plausible candidate for such a superrule is a game form Γ consisting of two groups (Group 1 and Group 2) of individuals, where the individuals in Group 1 first choose a social choice function (such as G) for the society consisting of the individuals in Group 2.

To investigate the properties of such game forms formally is beyond the scope of the present paper. To do that, one has to specify the game form Γ, preferably in extensive form. Also, one has to be careful in choosing a suitable concept of equilibrium: While the individuals in Group 2 may reasonably be expected to have dominant strategies, those in Group 1 cannot be expected to have dominant strategies (they are not likely to play best responses either).

A Proof of Proposition 1

This appendix gives a proof of Proposition 1.

First, the following two lemmas are easily obtained by mathematical induction.

Lemma 1 *For all s and s',* (i) *$\mathcal{U}_s$ is a finite family consisting of infinite coalitions in $\mathcal{B}$, and* (ii) *if $s \leq s'$, then $\mathcal{U}_s \subseteq \mathcal{U}_{s'}$.*

Lemma 2 *For all s and s',* (i) *$\bigcap \mathcal{U}_s$ is infinite, and* (ii) *if $s \leq s'$, then $\bigcap \mathcal{U}_s \supseteq \bigcap \mathcal{U}_{s'}$.*

Lemma 3 *For all s, if $C_s \in \mathcal{U}$, then $(\bigcap \mathcal{U}_s) \cap C_s$ is infinite and $C_s \in \mathcal{U}_{s+1}$.*

Proof. Suppose $C_s \in \mathcal{U}$. If $(\bigcap \mathcal{U}_s) \cap C_s$ is infinite, then by the definition of $\mathcal{U}_{s+1}$, $C_s \in \mathcal{U}_{s+1}$, and we are done.

So, suppose $(\bigcap \mathcal{U}_s) \cap C_s$ is finite. Since $C_s \in \mathcal{U}$, it must be that $C_s \in \mathcal{U}_t$ for some t. Without loss of generality, assume $t > s$. (If $t \leq s$, then $C_s \in \mathcal{U}_s$. So, $(\bigcap \mathcal{U}_s) \cap C_s = \bigcap \mathcal{U}_s$, which is infinite by Lemma 2. This contradicts the assumption that $(\bigcap \mathcal{U}_s) \cap C_s$ is finite.) Then $\bigcap \mathcal{U}_t \subseteq \bigcap \mathcal{U}_s$ by Lemma 2. Then

$$(\bigcap \mathcal{U}_t) \cap C_s \subseteq (\bigcap \mathcal{U}_s) \cap C_s.$$

But the expression on the right is finite, by assumption. Hence $(\bigcap \mathcal{U}_t) \cap C_s$ is finite. Since $C_s \in \mathcal{U}_t$, we have $\bigcap \mathcal{U}_t = (\bigcap \mathcal{U}_t) \cap C_s$, which is finite. This contradicts Lemma 2. ∎

Lemma 4 *$\mathcal{U}$ is a filter.*

Proof. (i) Since only infinite coalitions belong to $\mathcal{U}$ by Lemma 1, $\emptyset \notin \mathcal{U}$.

(ii) Suppose $C_{s'} \in \mathcal{U}$ and $C_s \supseteq C_{s'}$. We show that $C_s \in \mathcal{U}$.

(*Case:* $s' < s$). Since $s' + 1 \leq s$ and $C_{s'} \in \mathcal{U}_{s'+1}$, we have $C_{s'} \in \mathcal{U}_s$. Hence $(\bigcap \mathcal{U}_s) \cap C_{s'}$ is infinite, being equal to $\bigcap \mathcal{U}_s$. But $C_{s'} \subseteq C_s$ implies that

$$(\bigcap \mathcal{U}_s) \cap C_{s'} \subseteq (\bigcap \mathcal{U}_s) \cap C_s.$$

It follows that the right hand side expression is infinite. So, $C_s \in \mathcal{U}_{s+1} \subseteq \mathcal{U}$.

(*Case:* $s' \geq s$). We have $\mathcal{U}_{s'} \supseteq \mathcal{U}_s$ and $C_s \supseteq C_{s'}$. To show $C_s \in \mathcal{U}$, suppose $C_s \notin \mathcal{U}$. Then $(\bigcap \mathcal{U}_s) \cap C_s$ is finite. (Otherwise, $C_s \in \mathcal{U}_{s+1} \subseteq \mathcal{U}$; contradiction.) Then, since

$$(\bigcap \mathcal{U}_{s'}) \cap C_s \subseteq (\bigcap \mathcal{U}_s) \cap C_s,$$

$(\bigcap \mathcal{U}_{s'}) \cap C_s$ is finite. But $C_{s'} \subseteq C_s$ implies that $(\bigcap \mathcal{U}_{s'}) \cap C_{s'}$ is finite, being a subset of $(\bigcap \mathcal{U}_{s'}) \cap C_s$. Hence $C_{s'} \notin \mathcal{U}_{s'+1}$. By Lemma 3, $C_{s'} \notin \mathcal{U}$, which is a contradiction.

(iii) Suppose that C_s, $C_{s'} \in \mathcal{U}$. We show that $C_s \cap C_{s'} \in \mathcal{U}$. Without loss of generality, we may assume that $s \geq s'$. Then C_s, $C_{s'} \in \mathcal{U}_{s+1}$ by Lemma 3 and Lemma 1. Choose t such that $C_t = C_s \cap C_{s'}$.

(*Case:* $t > s$). In this case, C_s, $C_{s'} \in \mathcal{U}_t$. Since $\bigcap \mathcal{U}_t$ is infinite and $C_t = C_s \cap C_{s'} \supset \bigcap \mathcal{U}_t$, it follows that $(\bigcap \mathcal{U}_t) \cap C_t = \bigcap \mathcal{U}_t$ is infinite. So, $C_t \in \mathcal{U}_{t+1} \subseteq \mathcal{U}$.

(*Case:* $t \leq s$). In this case, $\bigcap \mathcal{U}_t \supseteq \bigcap \mathcal{U}_{s+1}$. It follows that

$$(\bigcap \mathcal{U}_t) \cap C_t \supseteq (\bigcap \mathcal{U}_{s+1}) \cap (C_s \cap C_{s'}).$$

Since the set on the right is equal to $\mathcal{U}_{s+1}$, which is infinite, the set on the left is also infinite. Therefore, $C_t \in \mathcal{U}_{t+1} \subseteq \mathcal{U}$. ∎

Lemma 5 *$\mathcal{U}$ is an ultrafilter.*

Proof. Suppose that C_s, $C_t \in \mathcal{B}$, $C_s^c = C_t$, where $s < t$ without loss of generality. To show that $C_s \in \mathcal{U}$ or $C_t \in \mathcal{U}$, suppose $C_s \notin \mathcal{U}$ and $C_t \notin \mathcal{U}$.

Then, both $(\bigcap \mathcal{U}_s) \cap C_s$ and $(\bigcap \mathcal{U}_t) \cap C_t$ are finite. (Otherwise, $C_s \in \mathcal{U}_{s+1}$ or $C_t \in \mathcal{U}_{t+1}$.) Since $\bigcap \mathcal{U}_s \supseteq \bigcap \mathcal{U}_t$, it follows that

$$(\bigcap \mathcal{U}_s) \cap C_s \supseteq (\bigcap \mathcal{U}_t) \cap C_s.$$

Since the set on the left is finite, the set on the right is also finite. Now,

$$\begin{aligned}
\bigcap \mathcal{U}_t &= (\bigcap \mathcal{U}_t) \cap N \\
&= (\bigcap \mathcal{U}_t) \cap (C_s \cup C_t) \\
&= [(\bigcap \mathcal{U}_t) \cap C_s] \cup [(\bigcap \mathcal{U}_t) \cap C_t].
\end{aligned}$$

Hence, $\bigcap \mathcal{U}_t$, being the union of finite sets, is finite. This contradicts Lemma 2. ∎

We conclude that $\mathcal{U}$ is a free ultrafilter. This is immediate from Lemma 1 since any element of $\mathcal{U}$ has to be infinite.

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