# Anonymity and Neutrality in Arrow's Theorem with Restricted Coalition Algebras*

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**Abstract**

In the very general setting of Armstrong (1980) for Arrow's Theorem, I show two results. First, in an infinite society, Anonymity is inconsistent with Unanimity and Independence if and only if a domain for social welfare functions satisfies a modest condition of richness. While Arrow's axioms can be satisfied, unequal treatment of individuals thus persists. Second, Neutrality is consistent with Unanimity (and Independence). However, there are both dictatorial and nondictatorial social welfare functions satisfying Unanimity and Independence but not Neutrality. In Armstrong's setting, one can naturally view Neutrality as a stronger condition of informational simplicity than Independence.

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# 1 Introduction

In social choice theory, along with Arrow's conditions for his impossibility theorem [4], the axioms of anonymity and neutrality have long history. For example, May [9] characterized the simple majority rule, using these axioms. Roughly speaking, Anonymity is a condition of equal treatment of individuals (voters) and Neutrality is a condition of equal treatment of alternatives (candidates). The simple majority rule satisfies both axioms.

In view of Arrow's Theorem, which states that in societies with finitely many individuals any social welfare function satisfying Unanimity and Independence is dictatorial, investigating compatibility of (each or both of) Anonymity and Neutrality with Unanimity and Independence is not very interesting, as far as finite societies are concerned. For the problem is reduced to that of investigating properties of dictatorial social welfare functions. It is trivial that imposing Unanimity and Independence entails violation of Anonymity but not necessarily violation of Neutrality, because dictatorial social welfare functions violate Anonymity but not necessarily Neutrality (though the latter is less obvious).

However, with the appearance of Fishburn's resolution [6] of Arrow's impossibility in which he allowed an infinite set of individuals, the problem has suddenly become more interesting. For, with infinitely many individuals, there are now nondictatorial social welfare functions satisfying Unanimity and Independence. This means that there might be *nondictatorial* social welfare functions that satisfy Anonymity or Neutrality along with Unanimity and Independence.

I explore the axioms of Anonymity and Neutrality in Armstrong's setting [1, 2] for Arrow's Theorem, where there can be infinitely many individuals and where the observable coalitions can be restricted to an arbitrary Boolean algebra on the set of individuals. (Armstrong's papers extend what could be called an "ultrafilter approach" due to Kirman and Sondermann [8], who characterize decisive coalitions associated with a social welfare function satisfying Unanimity and Independence.) There are two reasons for me to adopt Armstrong's setting instead of Fishburn's setting.

An obvious reason is that the results I obtain will be more general since Fishburn's setting is a special case of Armstrong's setting.

But a more important reason is that Armstrong's setting allows me to give [10, 11] a new and practical (as opposed to ethical) significance to the condition of Neutrality. Some social choice theorists (e.g., Kelly [7, ch. 3]) have regarded Independence, a weaker condition than Neutrality, as a condition of computational simplicity. This is because Independence allows a planner to determine the social preference on a pair $\{x,y\}$ of alternatives based only on the individual preferences on the pair. Suppose that Independence is satisfied and suppose that the planner has a procedure ("algorithm") that does this task for a set $\{x,y\}$. Under Independence alone, the procedure for determining the social preference on the set $\{x,y\}$ might not work for determining the social preference on another set $\{x',y'\}$. The planner might thus need many procedures, each working only for a particular pair. I argue [10, 11] however that if Neutrality is satisfied, the planner can use a single procedure for all pairs. In this sense, Neutrality is a stronger condition (than Independence) for efficient information processing. Armstrong's setting allows me to discuss information processing, or computability, more adequately than Fishburn's setting. (The discussion of Neutrality is dropped in some revisions of the discussion paper [10].)

Corollary 3 shows that given an infinite set of individuals and a domain for social welfare functions, there is no social welfare function (on the domain) satisfying Unanimity, Independence, and Anonymity if and only if the domain satisfies a modest condition of richness. The richness condition requires that the planner can observe at least a pair of coalitions of the same cardinality that complement each other. As I argue below, the violation of this condition severely trivializes a social choice problem. While Arrow's axioms can be satisfied for an infinite society, unequal treatment of individuals thus persists. This result is quite negative, strongly qualifying the resolutions (by Fishburn and by Armstrong) of Arrow's impossibility.

The proof of the "only if" direction (Proposition 2) of Corollary 3 goes as follows: If the richness condition is violated, then each coalition $A$ has its complement that is "strictly larger" or "strictly smaller" than $A$. In the latter case, $A$ can be regarded as a "majority" of the society. Then a social welfare function can be defined that follows the preferences of the "majorities." The function satisfies Anonymity. In this construction, it would be clear that the violation of the richness condition makes the social choice problem trivial.

Theorem 2 indicates that there is a natural class of social welfare functions that satisfy Unanimity and Neutrality (hence Independence). These are the social welfare functions that Armstrong [2, 3] is mainly concerned with in his successive papers. One might thus have an impression that all social welfare functions satisfying Unanimity and Independence also satisfy Neutrality. However, I give an example below that shows there are both dictatorial and nondictatorial social welfare functions violating Neutrality while satisfying Unanimity and Independence.

## 2 Formulation

$I$ is a set of *individuals*, which is either finite or infinite. An example of $I$ is the set $\mathbf{N}$ of nonnegative integers. $X$ is a set of *alternatives*, which has at least three elements. $P$ is the set of (strict) *preferences*, i.e., asymmetric and negatively transitive binary relations on $X$.

A *Boolean algebra* $\mathcal{B}$ consisting of subsets of $I$ satisfies the following: (i) $\emptyset$, $I \in \mathcal{B}$; (ii) $A \cup B$, $A \cap B$, $\overline{A} \in \mathcal{B}$ if $A$, $B \in \mathcal{B}$ (where $\overline{A}$ denotes the complement of $A$). If $I$ denotes the set of individuals, then intuitively, an element of a Boolean algebra is a coalition observable by the planner. For example, the family of all subsets of $I$ forms a Boolean algebra; so does the family of all *recursive* subsets (i.e., effectively decidable sets) of $\mathbf{N}$; so does the family $\mathcal{B}_f$ of finite or cofinite subsets of $I$. A *filter* $\mathcal{F}$ on $\mathcal{B}$ is a family of sets in $\mathcal{B}$ satisfying: (i) $\emptyset \notin \mathcal{F}$; (ii) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$; (iii) if $A$, $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. We may think of a filter as a family of "large" sets. An *ultrafilter* is a filter $\mathcal{U}$ that satisfies: if $A \notin \mathcal{U}$, then $\overline{A} \in \mathcal{U}$. For $\mathcal{B} \supseteq \mathcal{B}_f$, we say an ultrafilter $\mathcal{F}$ is *fixed* if it is of the form $\mathcal{F} = \{ A \in \mathcal{B} : i \in A \}$ for some $i \in I$; otherwise, it is called *free* and does not contain any finite sets.

A *profile* is a list $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i \in I} \in P^I$ of *individual preferences* $\succ_i^{\mathbf{p}}$, $i \in I$. The weak preference $\succeq_i^{\mathbf{p}}$ is the negation of $\prec_i^{\mathbf{p}}$ (defined from $\succ_i^{\mathbf{p}}$ in an obvious manner), and the indifference relation $\sim_i^{\mathbf{p}}$ is the symmetric part of $\succeq_i^{\mathbf{p}}$. A profile $(\succ_i^{\mathbf{p}})_{i \in I}$ is $\mathcal{B}$*-measurable* if $\{ i : x \succ_i^{\mathbf{p}} y \} \in \mathcal{B}$ for all $x, y \in X$. Denote by $P_{\mathcal{B}}^I$ the set of all $\mathcal{B}$-measurable profiles.

A $\mathcal{B}$*-social welfare function* is a function $\succ : P_{\mathcal{B}}^I \to P$ mapping each profile $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i \in I}$ to a social preference $\succ(\mathbf{p}) = \succ^{\mathbf{p}}$. (Using the notation $\succ$ for a function would not cause a confusion since preferences are expressed in the form $\succ_i^{\mathbf{p}}$ or $\succ^{\mathbf{p}}$, with profile $\mathbf{p}$ always present as a superscript.) Social relations $\succeq^{\mathbf{p}}$, $\sim^{\mathbf{p}}$, $\prec^{\mathbf{p}}$, etc., are defined in an obvious manner.

We list Arrow's conditions for $\mathcal{B}$-social welfare functions:

**Unanimity** For any $x, y \in X$ and $\mathbf{p} \in P_{\mathcal{B}}^I$, if $\{ i : x \succ_i^{\mathbf{p}} y \} = I$, then $x \succ^{\mathbf{p}} y$.

**Independence** For any $x$, $y \in X$ and $\mathbf{p}$, $\mathbf{p}' \in P_{\mathcal{B}}^I$, if ($x \neq y$ and) $\succ_i^{\mathbf{p}} \cap \{x,y\}^2 = \succ_i^{\mathbf{p}'} \cap \{x,y\}^2$ for all $i \in I$, then $\succ^{\mathbf{p}} \cap \{x,y\}^2 = \succ^{\mathbf{p}'} \cap \{x,y\}^2$.

**Nondictatorship** There is no $i \in I$ such that for all $x$, $y \in X$ and all $\mathbf{p} \in P_{\mathcal{B}}^I$, $x \succ_i^{\mathbf{p}} y \Longrightarrow x \succ^{\mathbf{p}} y$.

A $\mathcal{B}$-social welfare function violating Nondictatorship is called *dictatorial*.

*Remark*. Nondictatorship may not be a very meaningful axiom when the Boolean algebra $\mathcal{B}$ is pathological. For example, if $\mathcal{B}$ consists only of $\emptyset$ and $I$, then it can be shown that Nondictatorship is violated if a $\mathcal{B}$-social welfare function $\succ$ satisfies Unanimity. In fact, every individual $i \in I$ is a dictator in this case. One might think having more than one dictators is inconsistent with the sense of the word "dictator." Though I do not require it formally, one natural way to avoid the multiplicity of dictators is to assume that $\mathcal{B}$ contains at least all single-element sets (hence, all finite and cofinite sets). $\diamondsuit$

Finally, we list two more conditions for a $\mathcal{B}$-social welfare function. Recall that a permutation of $I$ is a one-to-one function on $I$ that is onto $I$.

**Anonymity** For all $\mathbf{p}$, $\mathbf{p}' \in P^I_{\mathcal{B}}$ and for any permutation $\pi: I \to I$, if $\succ^{\mathbf{p}'}_i = \succ^{\mathbf{p}}_{\pi(i)}$ for all $i \in I$, then $\succ^{\mathbf{p}'} = \succ^{\mathbf{p}}$.

**Neutrality** For all $x$, $x'$, $y$, $y' \in X$, and all $\mathbf{p}$, $\mathbf{p}' \in P^I_{\mathcal{B}}$, if $[(x \succ^{\mathbf{p}}_i y \leftrightarrow x' \succ^{\mathbf{p}'}_i y')$ & $(y \succ^{\mathbf{p}}_i x \leftrightarrow y' \succ^{\mathbf{p}'}_i x')]$ for all $i \in I$, then $(x \succ^{\mathbf{p}} y \leftrightarrow x' \succ^{\mathbf{p}'} y')$.

Anonymity requires that social preferences be invariant with respect to permutations of individual preferences. Neutrality requires that if a profile $\mathbf{p}$ treats a pair $(x, y)$ in the same way as a profile $\mathbf{p}'$ treats a pair $(x', y')$, then the resulting two social preferences must treat the respective pairs in the same way. Clearly, Anonymity implies Nondictatorship if there are at least two individuals in $I$ and $\mathcal{B} \supseteq \mathcal{B}_f$. Neutrality implies Independence.

*Remark*. Two modifications of the above notion of Anonymity were suggested to me: (i) restricting permutations to those that permute only finitely many individuals and (ii) restricting permutations to those $\pi$ that are measurable in the sense that $\pi^{-1}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$.

As for (i), it is a corollary of Lemma 1 below that *any* $\mathcal{B}$-social welfare function satisfying Independence, Unanimity, and Nondictatorship satisfies the modified notion of anonymity. While trivial, this is certainly an interesting result, which complements the results in this paper.

As for (ii), the problem seems more complicated and I will not enter into details. (Proposition 2 still holds under the modified notion of anonymity.) One nice thing about the modified notion is that we can dispense with the condition that $\mathbf{p}' \in P^I_{\mathcal{B}}$ since the other conditions imply $\mathbf{p}' \in P^I_{\mathcal{B}}$. Since I do not require measurability of $\pi$, I cannot drop the condition in my definition of Anonymity. While measurability of $\pi$ ensures that the inverse images of *all* coalitions in $\mathcal{B}$

are also in $\mathcal{B}$, I do not feel strongly that we should be interested in all coalitions. Given measurable profiles $\mathbf{p}$ and $\mathbf{p}'$, all the coalitions that we should be interested in are those coalitions $\{\, i : x \succ_i^{\mathbf{p}} y \,\}$ and $\{\, i : x \succ_i^{\mathbf{p}'} y \,\}$ that prefer $x$ to $y$, for some $x, y \in X$, and their Boolean combinations. And whether or not $\pi$ is measurable, these "interesting" coalitions *are* in $\mathcal{B}$, since $\mathbf{p}$ and $\mathbf{p}'$ are measurable. As the following example shows, it is not true that the existence of a permutation between two (complementing) measurable coalitions implies the existence of a *measurable* permutation. $\diamondsuit$

*Example*. Let $I = \{1, 2, \ldots\}$ and let $\mathcal{B}$ be the $\sigma$-algebra generated by $\{1\}$, $\{2, 3\}$, $\{4, 5, 6, 7\}$, $\ldots$, $\{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\}$, $\ldots$. Measurability of $\pi\colon I \to I$ entails

$$\pi^{-1}(\{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\}) = \{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\} \tag{1}$$

for all $n$. (This algebra was brought to my attention by a referee for a different purpose.) Suppose under $\mathbf{p}, \mathbf{p}' \in P_{\mathcal{B}}^I$, individuals are partitioned into two classes $A$ and $B$: $A$ consists of those belonging to $\{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\}$ for even $n$ and $B$ consists of those belonging to $\{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\}$ for odd $n$. For each of the two profiles, all individuals in the same class have the same preferences. Suppose that no individual is indifferent between any two distinct alternatives, and that for individuals $a \in A$ and for individuals $b \in B$,

$$x \succ_a^{\mathbf{p}} y \iff y \succ_b^{\mathbf{p}} x \iff y \succ_a^{\mathbf{p}'} x \iff x \succ_b^{\mathbf{p}'} y$$

for all $x, y \in X$. It is easy to see that there is a permutation between $A$ and $B$ but there is no *measurable* permutation between them. My position in the above Remark, if applied to these particular profiles on the $\sigma$-algebra, can be expressed as follows: all the coalitions that we should be interested in are $\emptyset$, $A$, $B$, and $I$; hence the measurability conditions (1) need not be imposed. $\diamondsuit$

# 3 Unanimity, Independence, and Anonymity

**Lemma 1 (Armstrong** [1, Proposition 3.2]**)** *Let $\mathcal{B}$ be a Boolean algebra on $I$. Suppose a $\mathcal{B}$-social welfare function $\succ$ satisfies Unanimity and Independence. Then there is a unique ultrafilter $U_\succ$ on $\mathcal{B}$ such that for all $\mathbf{p} = (\succ_i^{\mathbf{p}})_{i \in I} \in P_{\mathcal{B}}^I$ and $x, y \in X$,*

$$\{\, i : x \succ_i^{\mathbf{p}} y \,\} \in U_\succ \Longrightarrow x \succ^{\mathbf{p}} y. \tag{2}$$

*Remark*. The uniqueness follows from Proposition 3.1 of Armstrong [1]. $\diamondsuit$

*Remark*. Armstrong [2] corrects an error in Proposition 3.2 of his earlier work [1]. Lemma 1 is the corrected version. $\diamondsuit$

Before stating Theorem 1, I treat the case corresponding to Fishburn's setting [6], where the domains of a social welfare function are unrestricted.

**Proposition 1** *Let $I$ contain at least two individuals. Let $\mathcal{B}$ be the Boolean algebra consisting of* all subsets *of $I$. Suppose a $\mathcal{B}$-social welfare function $\succ$ satisfies Unanimity and Independence. Then $\succ$ violates Anonymity.*

*Proof*. By Lemma 1, choose the ultrafilter $\mathcal{U}_\succ$ satisfying (2) for all $\mathbf{p}$ and $x$, $y$. If $I$ is finite, then $\succ$ violates Anonymity since it is dictatorial, for the ultrafilter $\mathcal{U}_\succ$ must be fixed.

So, suppose $I$ is infinite, say, of cardinality $\alpha$. We show that $I$ is partitioned into subsets $A$ and $B$ that are equivalent with each other: Since the cardinalities of the disjoint sets $I$ and $I \times \{0\}$ are $\alpha$, the cardinality of the union of these sets is $\alpha + \alpha$; but the last cardinality is equal to the cardinality $\alpha$ of $I$ since $I$ is infinite. It follows that there is a one-to-one mapping $f$ from $I \cup (I \times \{0\})$ onto $I$. Let $A = f(I)$ and $B = f(I \times \{0\})$. Then $A$ and $B$ have the same cardinality and form a partition of $I$.

Since only one of the sets $A$ and $B$ is in $\mathcal{U}_\succ$, assume without loss of generality that $A \in \mathcal{U}_\succ$. Then consider some $\{x, y\}$ and two profiles $\mathbf{p}$ and $\mathbf{p}'$: $x \succ_i^{\mathbf{p}} y$ if $i \in A$, and $y \succ_i^{\mathbf{p}} x$ if $i \in B$; $y \succ_i^{\mathbf{p}'} x$ if $i \in A$, and $x \succ_i^{\mathbf{p}'} y$ if $i \in B$; the two profiles are empty on $X \setminus \{x, y\}$; and $x$ and $y$ are preferred to the other alternatives under both profiles. Obviously, $\mathbf{p}'$ is obtained by permuting the preferences in $\mathbf{p}$ of individuals. However, $x \succ^{\mathbf{p}} y$ since $\{\, i : x \succ_i^{\mathbf{p}} y \,\}$, being the set $A$, is in $\mathcal{U}_\succ$; on the other hand, $y \succ^{\mathbf{p}'} x$. This contradicts Anonymity. $\blacksquare$

The above proof depends on the fact that $\mathcal{B}$ contains complementing sets of the same cardinality. This condition is adopted in the hypothesis of the following theorem. An interesting case is when $I$ is infinite.

**Theorem 1** *Let $I$ contain at least two individuals. Let $\mathcal{B}$ be a Boolean algebra that contains a set that is of the same cardinality as its complement. Suppose that a $\mathcal{B}$-social welfare function $\succ$ satisfies Unanimity and Independence. Then, $\succ$ violates Anonymity.*

*Proof*. We have a set $A \in \mathcal{B}$ that has the same cardinality as its complement $\overline{A}$. Let $B = \overline{A}$ and argue as in the last paragraph of the proof of Proposition 1. Note that both $\mathbf{p}$ and $\mathbf{p}'$ are easily seen to be $\mathcal{B}$-measurable. $\blacksquare$

Some Boolean algebras do not satisfy the hypothesis of Theorem 1. An extreme example is the following:

*Example*. Let $\mathcal{B}$ consist of $\emptyset$ and $I$ only. This is the case in which any $\mathcal{B}$-social welfare function is defined only for unanimous profiles. Then, it is intuitively clear that any $\mathcal{B}$-social welfare function satisfies Anonymity. Indeed, the antecedent $(\forall i \succ_i^{\mathbf{p}'} = \succ_{\pi(i)}^{\mathbf{p}})$ in the statement of Anonymity can be satisfied only when $\mathbf{p} = \mathbf{p}'$, since for any measurable profile $\mathbf{p}$ and for all $x$ and $y$, either (i) $x \succ_i^{\mathbf{p}} y$ for all $i$ or (ii) $\neg x \succ_i^{\mathbf{p}} y$ for all $i$. (If (i), then for all $i$, $x \succ_{\pi(i)}^{\mathbf{p}} y$ for any permutation $\pi$ of $I$; similarly for (ii).) Since the antecedent is satisfied only when $\mathbf{p} = \mathbf{p}'$, in which case the consequent is trivially satisfied, Anonymity is established. $\diamondsuit$

There are less trivial Boolean algebras that do not satisfy the hypothesis of Theorem 1. For example, let

$$\mathcal{B}_f = \{A \subseteq I : A \text{ is finite or cofinite}\}.$$

(A *cofinite* set is a set whose complement is finite.) Then $\mathcal{B}_f$ is a Boolean algebra that does not contain complementing sets of $I$ that have the same cardinality, if $I$ is infinite. The same is true for the Boolean algebra consisting of countable or co-countable elements, if $I$ is uncountable.

However, the following proposition shows that the hypothesis for Boolean algebras is necessary for the conclusion of the theorem if $I$ is infinite.

**Proposition 2** *Let $I$ be infinite. Let $\mathcal{B}$ be a Boolean algebra on $I$ that does not contain complementing sets of the same cardinality. Then, there is a $\mathcal{B}$-social welfare function satisfying Unanimity, Independence, and Anonymity.*

Before giving a proof, we introduce the following:

**Lemma 2 (Armstrong** [1, Proposition 3.1]**)** *Let $\mathcal{B}$ be a Boolean algebra on $I$. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then the map $\succ$ on $P_{\mathcal{B}}^I$ defined for $\mathbf{p} \in P_{\mathcal{B}}^I$ and $x, y \in X$ by*

$$x \succ^{\mathbf{p}} y \iff \{i : x \succ_i^{\mathbf{p}} y\} \in \mathcal{U} \tag{3}$$

*is a $\mathcal{B}$-social welfare function satisfying Unanimity and Independence.*

*Proof of Proposition 2*. (This proof involves some cardinal arithmetic. Enderton [5, ch. 6] is a straightforward exposition.)

Let

$$\mathcal{U} = \{A \in \mathcal{B} : |A| > |\overline{A}|\}, \tag{4}$$

where $|A|$ denotes the cardinal number of $A$. We show that $\mathcal{U}$ is an ultrafilter:

(i) $\emptyset \notin \mathcal{U}$ since $|\emptyset| < |\overline{\emptyset}| = |I|$.

(ii) Suppose $A \in \mathcal{U}$ and $A \subseteq B \in \mathcal{B}$. Then $|A| > |\overline{A}|$ and $|A| \leq |B|$. Also, $|\overline{A}| \geq |\overline{B}|$ since $\overline{A} \supseteq \overline{B}$. From these inequalities, we get

$$|B| \geq |A| > |\overline{A}| \geq |\overline{B}|.$$

Hence, $|B| > |\overline{B}|$, implying $B \in \mathcal{U}$.

(iii) Suppose $A$, $B \in \mathcal{U}$. Then $|\overline{A}| < |A|$ and $|\overline{B}| < |B|$. Without loss of generality, assume $|\overline{B}| \leq |\overline{A}|$. We have to show that $|A \cap B| > |\overline{A \cap B}|$. Suppose otherwise. Then $|A \cap B| \leq |\overline{A \cap B}|$. So,

$$\begin{aligned}
|I| &= |(A \cap B) \cup (\overline{A \cap B})| \\
&= |A \cap B| + |\overline{A \cap B}| \\
&\leq |\overline{A \cap B}| + |\overline{A \cap B}| \\
&= |\overline{A} \cup \overline{B}| + |\overline{A} \cup \overline{B}| \\
&\leq |\overline{A}| + |\overline{B}| + |\overline{A}| + |\overline{B}| \\
&\leq |\overline{A}| + |\overline{A}| + |\overline{A}| + |\overline{A}|.
\end{aligned}$$

Let $\alpha$ be the last expression. If $\overline{A}$ is finite, then $\alpha < |I|$ since $I$ is infinite. If $\overline{A}$ is infinite, then by the absorption law, $\alpha = |\overline{A}| < |A| \leq |I|$. In either case, we have established that $|I| < |I|$, which is contradiction.

(iv) Suppose that $\overline{A} \in \mathcal{B}$ but $\overline{A} \notin \mathcal{U}$. Then, $|\overline{A}| \leq |A|$. But $|\overline{A}| \neq |A|$ by the hypothesis on $\mathcal{B}$. Hence, $|A| > |\overline{A}|$. So, $A \in \mathcal{U}$.

Now, define a $\mathcal{B}$-social welfare function $\succ$ by (3). $\succ$ satisfies Unanimity and Independence.

To show Anonymity, let $\mathbf{p}$, $\mathbf{p}' \in P_{\mathcal{B}}^{I}$, let $\pi$ be a permutation of $I$, and suppose that $\succ_i^{\mathbf{p}'} = \succ_{\pi(i)}^{\mathbf{p}}$ for all $i$. Fix $x$ and $y$. (We have to show $x \succ^{\mathbf{p}'} y \leftrightarrow x \succ^{\mathbf{p}} y$.) Then, the cardinal numbers of $\{\, i : x \succ_i^{\mathbf{p}'} y \,\}$ and of $\{\, i : x \succ_i^{\mathbf{p}} y \,\}$ are the same. Also, the complements of these sets are of the same cardinality. It follows that *either* both $\{\, i : x \succ_i^{\mathbf{p}'} y \,\}$ and $\{\, i : x \succ_i^{\mathbf{p}} y \,\}$ are in $\mathcal{U}$ *or* neither is in $\mathcal{U}$. Hence, Lemma 2 implies that $x \succ^{\mathbf{p}'} y$ iff $x \succ^{\mathbf{p}} y$. ∎

*Remark*. It can be shown that $\mathcal{U}$ in (4) is in fact $\{\, A \in \mathcal{B} : |A| = |I| \,\}$. One can give a shorter proof using this fact. ◇

**Corollary 3** *Let $I$ be infinite and let $B$ be a Boolean algebra on $I$. Then there is no $B$-social welfare function satisfying Unanimity, Independence, and Anonymity if and only if $B$ contains a set that has the same cardinality as its complement.*

# 4 Unanimity, Independence, and Neutrality

Let $B$ be a Boolean algebra on $I$. Let $U$ be an ultrafilter on $B$. Lemma 2 above shows that the map $F_U$ from $\mathbf{p} \in P_B^I$ to a binary relation $F_U^{\mathbf{p}}$ on $X$ defined by

$$xF_U^{\mathbf{p}}y \iff \{\, i : x \succ_i^{\mathbf{p}} y \,\} \in U. \tag{5}$$

is a $B$-social welfare function satisfying Unanimity and Independence. We say that a $B$-social welfare function $\succ$ is *precisely diktatorial* if $\succ = F_U$ for some ultrafilter $U$. (Armstrong [2] calls such social welfare functions "precisely dictatorial." I reserve the word *dictatorial* for those social welfare functions violating Nondictatorship.) Precisely diktatorial social welfare functions are the social welfare functions $\succ$ whose ultrafilter $U_\succ$ (in Lemma 1) of decisive coalitions precisely determines the social preference in the sense that "$\Longrightarrow$" in (2) can be replaced by "$\iff$."

There are some $B$-social welfare functions satisfying Unanimity and Independence that are not precisely diktatorial. An example is a social welfare function with a primary dictator (dictator in the true sense) and a secondary dictator such that the latter's preference determines the social preference between two alternatives in case the primary dictator is indifferent between the alternatives. It is easy to see that a social welfare function can be constructed similarly for any finite "hierarchy" of dictators (or of ultrafilters). In fact, the following discussion generalizes this construction to any well-ordered sequence of ultrafilters.

Let $B$ be a Boolean algebra on $I$. Let $U = \{\, U^\alpha : \alpha \in \Gamma \,\}$ be a well-ordered family of ultrafilters on $B$. Define a map $H_U$ from $\mathbf{p} \in P_B^I$ to a binary relation $H_U^{\mathbf{p}}$ on $X$ as follows: for $\mathbf{p} \in P_B^I$, and for $x, y \in X$,

$$xH_U^{\mathbf{p}}y \iff (\exists \alpha \in \Gamma)[(\forall \beta < \alpha)(\{\, i : x \sim_i^{\mathbf{p}} y \,\} \in U^\beta) \;\&\; \{\, i : x \succ_i^{\mathbf{p}} y \,\} \in U^\alpha]. \tag{6}$$

Then, by [3, Proposition 17], $H_U$ is a $B$-social welfare function satisfying Unanimity and Independence. We call $H_U : P_B^I \to P$ the *hierarchical* social welfare function *associated with* the sequence $U$.

**Theorem 2** *Let $H_U : P_B^I \to P$ be the hierarchical social welfare function associated with a well-ordered sequence $U$ of ultrafilters on $B$. Then, $H_U$ satisfies Unanimity and Neutrality.*

*Proof.* It is discussed above that $H_U$ is a $B$-social welfare function satisfying Unanimity.

To show Neutrality, suppose that $(x \succ_i^{\mathbf{p}} y \leftrightarrow x' \succ_i^{\mathbf{p}'} y')$ and $(y \succ_i^{\mathbf{p}} x \leftrightarrow y' \succ_i^{\mathbf{p}'} x')$ for all $i \in I$. Then

$$\{\, i : x \sim_i^{\mathbf{p}} y \,\} = \{\, i : x' \sim_i^{\mathbf{p}'} y' \,\}$$

and

$$\{\, i : x \succ_i^{\mathbf{p}} y \,\} = \{\, i : x' \succ_i^{\mathbf{p}'} y' \,\}.$$

So, (6) implies that

$$x H_U^{\mathbf{p}} y \iff x' H_U^{\mathbf{p}'} y'.$$

Therefore, Neutrality is satisfied. ∎

**Corollary 4** *Precisely diktatorial $B$-social welfare functions satisfy Neutrality.*

Theorem 2 implies that if $I$ is infinite and $B \supseteq B_f$ (i.e., $B$ contains all finite and cofinite subsets of $I$), then there are both (i) dictatorial and (ii) nondictatorial $B$-social welfare functions satisfying Unanimity and Neutrality (hence Independence). To see that (i) there is a dictatorial one, just let an ultrafilter $U$ in (5) be fixed. To see that (ii) there is a nondictatorial function, let $U$ in (5) be a free ultrafilter on $B$. (A free ultrafilter can be obtained by extending the filter of cofinite elements of $B$ using Zorn's lemma.)

Because of Theorem 2, in order to find a social welfare function violating Neutrality but satisfying Unanimity and Independence, we have to consider a social welfare function that is not the hierarchical $H_U$ associated with a well-ordered sequence $U$ of ultrafilters. The following example shows that there are such social welfare functions both among dictatorial ones (set $U$ be a fixed ultrafilter) and among nondictatorial ones (set $U$ be free), provided that $I$ is infinite and $B \supseteq B_f$. (The example is a special instance of attaching a *Deux ex Machina* to a hierarchical social welfare function, discussed in Armstrong [3, p. 39]. It is easy to see that addition of a *Deux ex Machina* to a hierarchy destroys Neutrality.)

*Example*. Let $P$ be an arbitrary nonempty preference in $\mathcal{P}$. Let $B$ be a Boolean algebra on $I$. Let $U$ be an arbitrary ultrafilter on $B$. Define a map $\succ$ from $\mathbf{p} \in \mathcal{P}_B^I$ into a binary relation $\succ^{\mathbf{p}}$ on $X$ as follows: for $\mathbf{p} \in \mathcal{P}_B^I$, and for $x, y \in X$,

$$x \succ^{\mathbf{p}} y \quad \text{iff} \quad \begin{array}{l} \text{(a) } \{\, i : x \succ_i^{\mathbf{p}} y \,\} \in U \text{ or} \\ \text{(b) } \{\, i : x \sim_i^{\mathbf{p}} y \,\} \in U \;\&\; xPy. \end{array} \tag{7}$$

First, note that (7) can be rewritten as:

$$x \succ^{\mathbf{p}} y \quad \text{iff} \quad \begin{array}{l} \text{(a) } xF_U^{\mathbf{p}}y \text{ or} \\ \text{(b) } \neg xF_U^{\mathbf{p}}y \ \& \ \neg yF_U^{\mathbf{p}}x \ \& \ xPy. \end{array} \tag{8}$$

where $F_U$ is given by (5). Armstrong [2, Lemma 1] proves the following:

**Lemma 3** *Let $P$, $P' \in \mathcal{P}$. Define a binary relation $P''$ on $X$ by $xP''y$ if* (a) *$xP'y$ or* (b) *$\neg xP'y$ & $\neg yP'x$ & $xPy$. Then, $P'' \in \mathcal{P}$.*

For the $\succ$ defined by (7), we show the following 1–4:

1. $\succ$ *is a $\mathcal{B}$-social welfare function*. We must show that for each $\mathbf{p} \in \mathcal{P}_{\mathcal{B}}^I$, $\succ^{\mathbf{p}}$ is in $\mathcal{P}$. Fix $\mathbf{p} \in \mathcal{P}_{\mathcal{B}}^I$. Then since $\succ^{\mathbf{p}}$ is defined from $F_U^{\mathbf{p}}$ and $P \in \mathcal{P}$ as in (8), above Lemma applies immediately and we get $\succ^{\mathbf{p}} \in \mathcal{P}$.

2. $\succ$ *satisfies Unanimity*. This is obvious from (7).

3. $\succ$ *satisfies Independence*. Suppose that $\succ_i^{\mathbf{p}} \cap \{x,y\}^2 = \succ_i^{\mathbf{p}'} \cap \{x,y\}^2$ for all $i \in I$. To show contradiction, assume $x \succ^{\mathbf{p}} y$ but $\neg x \succ^{\mathbf{p}'} y$ without loss of generality. That $x \succ^{\mathbf{p}} y$ implies that either (a) or (b) in (7). In case (a), $\{i : x \succ_i^{\mathbf{p}'} y\} \in \mathcal{U}$ since $\{i : x \succ_i^{\mathbf{p}} y\} \in \mathcal{U}$ and these two coalitions are the same. By (7), $x \succ^{\mathbf{p}'} y$, contradicting the assumption. In case (b), $\{i : x \sim_i^{\mathbf{p}'} y\} \in \mathcal{U}$ (since $\{i : x \sim_i^{\mathbf{p}} y\} \in \mathcal{U}$ and these two coalitions are the same) as well as $xPy$. By (7), $x \succ^{\mathbf{p}'} y$, contradiction.

4. $\succ$ *violates Neutrality*. Let $\mathbf{p} \in \mathcal{P}_{\mathcal{B}}^I$ and $x, y \in X$ be such that $\{i : x \sim_i^{\mathbf{p}} y\} = I$ and $xPy$. (Since $P$ is not empty, there are such $x$ and $y$.) Then, $x \succ^{\mathbf{p}} y$. Let $\mathbf{p}' = \mathbf{p}$, and $x' = y$, $y' = x$. Clearly, $y' \succ^{\mathbf{p}'} x'$. Hence, $\neg x' \succ^{\mathbf{p}'} y'$, violating the consequent in the statement of the Neutrality condition. On the other hand, the antecedent of the Neutrality condition is satisfied since all individuals are indifferent between the two alternatives. Hence, Neutrality is violated. $\diamondsuit$

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