The Pure Theory of Public Goods: Efficiency, Decentralization, and the Core*

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Abstract

We extend the results of Mas-Colell (1980) and Weber and Wiesmeth (1991) on valuation equilibria and the relationship of cost share equilibria with the core. We allow for any finite number of private goods and a set of public projects without any structure. We show the two welfare theorems for valuation equilibrium, the inclusion of the set of cost share equilibria in the core, and the nonequivalence of these two sets for an economy with a finite number of agents. In the case that the set of public projects is endowed with a topological structure, we provide conditions under which the price system needed to decentralize a Pareto efficient allocation as a valuation equilibrium is continuous.

1 Introduction

The first mathematically general approach to the pure theory of public goods was made by Mas-Colell (1980). He studied a model with one private good and an abstract set of *public projects* without any mathematical structure. In particular, he did not assume that it is part of a linear space or that there is any ordering at all among public projects. In that paper he also proposed the notions of valuation equilibrium and cost share equilibrium. In a valuation equilibrium in the case of a single private good a public project is financed through a system of taxes and subsidies, called a *valuation*. In a *cost share equilibrium* the subsidies are excluded and the costs of the project are distributed among the agents in the economy.

In relation to these two generalizations of the Lindahl equilibrium concept to the case of an economy with an abstract set of public projects, Mas-Colell (1980) introduced the appropriate extensions of the standard normative notions of Pareto efficiency and the core.

Mas-Colell (1980) showed that, in the case of a single private good, valuation equilibria decentralize Pareto efficient allocations and that the set of cost share equilibria is equal to the core. Weber and Wiesmeth (1991) obtained the equivalence of the set of cost share equilibria with the core in a more restricted model with one private and one pure public good, assuming the public good space to be the nonnegative real half-line. They further studied the set of core allocations that are linear cost share equilibria, first defined in Mas-Colell and Silvestre (1989). More recently Vasil'ev, Weber and Wiesmeth (1992) and Gilles and Diamantaras (1994) also discuss core equivalence in a model where the public goods are subject to congestion effects.

Such decentralization and equivalence results are very important for the pure theory of public goods. The main messages of Mas-Colell (1980) and Weber and Wiesmeth (1991) are that (i) valuation equilibria, through the notion of a valuation, provide a "nonlinear price" for the decentralization of the efficient provision of an abstract public project, and (ii) cost share equilibria, in which subsidies are not allowed, cover the whole core. The first result generalizes the Lindahlian approach to public goods provision to a more abstract setting. The second result is interesting given the well-known examples of the failure of core equivalence with Lindahl equilibria in economies with constant returns to scale. However, Mas-Colell (1980) as well as Weber and Wiesmeth (1991) worked in a partial equilibrium model, by virtue of their assumption that there is only one private good.

We obtain in this paper a generalization of the results mentioned of Mas-Colell (1980) in that we allow for any arbitrary finite number of private goods. We develop the appropriate extension of the definition of valuation equilibrium and cost share equilibrium using price systems. We establish the first and second welfare theorems for valuation equilibrium, and show the nonequivalence of the core and the set of cost share equilibria.

Diamantaras, Gilles and Ruys (1994) study a model in which trade infrastructures are treated as public projects. A public infrastructure has a real cost but it reduces transaction costs for agents making more trading possible. The model in that paper has a continuum of agents, and techniques are used which are similar to some of the ones applied in the present paper and in Gilles and Diamantaras (1994).

We emphasize that our results form a bridge between the work on economies with *one* private commodity and an abstract set of public projects (Mas-Colell 1980) on the one hand and the traditional general equilibrium framework for a pure exchange economy on the other hand. Our welfare theorems generalize the traditional results since the latter are obtained after the linearization of the set of public projects. If only one private good is assumed, one arrives at the results developed in Mas-Colell (1980).

2 The Model

We study an economy in which A is a finite set of economic agents, there are $\ell \in \mathbb{N}$ private commodities, and the commodity space is represented by $\mathbb{R}^{\ell}_{+}$. We denote by the function $w\colon A \to \mathbb{R}^{\ell}_{+} \setminus \{0\}$ the *endowment* of private commodities of the agents in A.

There is a set $\mathcal{Y}$ of public projects, on which for the time being we do not impose any structure. Each public project has a cost in terms of each private good, and we capture this by the vector-valued function $c\colon \mathcal{Y} \to \mathbb{R}^{\ell}_{+}$. We remark that on the set $\mathcal{Y}$ one can easily adopt some metric or Euclidean structure. The first case will be employed in Section 4 to discuss continuity properties of price systems that decentralize Pareto efficient allocations. In the case of an Euclidean structure we arrive at the classical public goods model, which might serve as a reference point in the discussion.

Each agent $a \in A$ has preferences defined on $\mathbb{R}^{\ell}_{+} \times \mathcal{Y}$, which are represented by a real-valued function $U_a\colon \mathbb{R}^{\ell}_{+} \times \mathcal{Y} \to \mathbb{R}$.

An **economy** is a collection $\mathbb{E} = \{A, (U_a)_{a\in A}, w, \mathcal{Y}, c\}$ such that $\overline{w} \equiv \sum_{a\in A} w(a) \gg c(y)$ for all $y \in \mathcal{Y}$. Note that this implies that $\overline{w} \gg 0$. (We use the vector inequality convention $\geq, >, \gg$.)

An **allocation** for an economy $\mathbb{E}$ is a pair (f, y) where $f: A \to \mathbb{R}^\ell_+$ and $y \in \mathcal{Y}$. An allocation (f, y) is **feasible** if

$$\sum_{a \in A} f(a) + c(y) = \overline{w}.$$

Note that we do not assume free disposal in production, although our results hold under it. We denote the set of feasible allocations by Φ.

3 Efficiency and Valuation Equilibrium

In this section we define efficient allocations and valuation equilibria and we show the two welfare theorems for valuation equilibria.

Definition 3.1 *A feasible allocation $(f, y) \in \Phi$ is **Pareto efficient** in the economy $\mathbb{E}$ if there is no allocation $(g, z) \in \Phi$ such that*

(i) for every a in A, $U_a(g(a), z) \geq U_a(f(a), y)$, and

(ii) there exists an agent $b \in A$ such that $U_b(g(b), z) > U_b(f(b), y)$.

This definition of efficiency is standard. We will need the usual price space for the private goods:

$$S^{\ell-1} := \left\{ q \in \mathbb{R}^\ell_+ \,\middle|\, \sum_{i=1}^{\ell} q_i = 1 \right\}.$$

We can now introduce our first equilibrium concept.

Definition 3.2 *A feasible allocation $(f, y) \in \Phi$ is a **valuation equilibrium** for $\mathbb{E}$ if there exist a price system $p: \mathcal{Y} \to S^{\ell-1}$ and a valuation function $V: A \times \mathcal{Y} \to \mathbb{R}$, such that*

(i) there is budget neutrality, i.e., $\sum_{a \in A} V(a, y) = p(y) \cdot c(y)$;

(ii) for every agent $a \in A$ the pair $(f(a), y)$ maximizes U_a on the budget set

$$\left\{ (g, z) \in \mathbb{R}^\ell_+ \times \mathcal{Y} \mid p(z) \cdot g + V(a, z) = p(z) \cdot w(a) \right\},$$

and for every $z \in \mathcal{Y}$ we have $V(a, z) \leq p(z) \cdot w(a)$;

(iii) y maximizes the surplus $\sum_{a \in A} V(a, z) - p(z) \cdot c(z)$ on $\mathcal{Y}$.

The feature of this definition that deserves special attention is the price system. For each valuation equilibrium we have a price system that values private goods contingent on *all* public projects, even though, naturally, only one public project occurs at this particular equilibrium.

In a valuation equilibrium an allocation is supported by a price system as well as a valuation. A valuation can be interpreted as a "non-linear" price for the public project that is established, while a price system is a function that assigns to every potential public project some price vector with respect to the private sector of the economy. It is important to note that a valuation equilibrium takes into account the large impact that a change of public project has on the private sector, i.e., each change of the public project leads to a change in the pricing of private commodities. In this sense the valuation equilibrium concept embodies an expression of the changes in the private sector of the economy caused by changes in the public sector.

Before we state our generalization of the first welfare theorem to our setting we introduce some terminology. A utility function U_a is *monotone* if for all $f, g \in \mathbb{R}^{\ell}_{+}$ and all $y \in \mathcal{Y}$ with $f \gg g$, $U_a(f, y) > U_a(g, y)$. Furthermore, U_a is *strictly monotone* if for all $f, g \in \mathbb{R}^{\ell}_{+}$ and all $y \in \mathcal{Y}$ with $f > g$, $U_a(f, y) > U_a(g, y)$.

Theorem 3.3 *If for all agents $a \in A$ the utility function U_a is monotone, then every valuation equilibrium in $\mathbb{E}$ is Pareto efficient.*

PROOF

Let $(f, y) \in \Phi$ be a valuation equilibrium with price system p and valuation function V. We must show that (f, y) is Pareto efficient.

Suppose to the contrary that (f, y) is not Pareto efficient. Then there exists an allocation $(g, z) \in \Phi$ with for all a in A

$$U_a(g(a), z) \geq U_a(f(a), y),$$

and there is $b \in A$, such that

$$U_b(g(b), z) > U_b(f(b), y).$$

Since $(g, z) \in \Phi$ it follows that

$$\sum_{a \in A} g(a) + c(z) = \overline{w}. \tag{1}$$

Condition (ii) of the definition of a valuation equilibrium and the monotonicity of the utility functions imply that for all a in A we have that $p(z) \cdot g(a) + V(a, z) \geq p(z) \cdot w(a)$ and for all b we have $p(z) \cdot g(b) + V(b, z) > p(z) \cdot w(b)$. Hence,

$$p(z) \cdot \sum_{a \in A} g(a) + \sum_{a \in A} V(a, z) > p(z) \cdot \overline{w}. \tag{2}$$

Condition (iii) of the definition of valuation equilibrium now implies that

$$\sum_{a\in A} V(a,y) - p(y)\cdot c(y) \geq \sum_{a\in A} V(a,z) - p(z)\cdot c(z) \tag{3}$$

Since equation (1) can be written as

$$\sum_{a\in A}(w(a) - g(a)) - c(z) = 0, \tag{4}$$

we conclude that

$$\begin{aligned} 0 &= \textstyle\sum_{a\in A} V(a,y) - p(y)\cdot c(y) \geq && \text{by (3)}\\ &\geq \textstyle\sum_{a\in A} V(a,z) - p(z)\cdot c(z) > && \text{by (2)}\\ &> p(z)\cdot \textstyle\sum_{a\in A}(w(a) - g(a)) - p(z)\cdot c(z) = && \text{by (4)}\\ &= 0. \end{aligned}$$

This is a contradiction. □

For the second welfare theorem we need the following condition, which is an extension and strengthening of the indispensability condition of Mas-Colell (1980).

Definition 3.4 (Essentiality condition)
The space of public projects $\mathcal{Y}$ satisfies the essentiality condition in $\mathbb{E}$ if it satisfies the following two conditions:

(i) *For every agent $a \in A$, each bundle $f \in \mathbb{R}^{\ell}_{+}$, and all potential public projects $y, z \in \mathcal{Y}$ there exists a bundle $g \in \mathbb{R}^{\ell}_{+}$ such that $U_a(g,z) > U_a(f,y)$.*

(ii) *For every agent $a \in A$, every $x \in \mathbb{R}^{\ell}_{+} \setminus \{0\}$ and all public projects $y, z \in \mathcal{Y}$, $U_a(x,y) > U_a(0,z) = 0$.*

The first condition states that all public projects in principle can be compensated by sufficiently large amounts of private goods. In the second condition, setting $U_a(0,z) = 0$ is just a normalization; the important part of this latter statement is that $U_a(0,y) = U_a(0,z)$ for all projects $y, z \in \mathcal{Y}$. This is similar to the indispensability condition of Mas-Colell (1980).

We note that in the standard case of pure public goods ($\mathcal{Y} = \mathbb{R}^{m}_{+}$), continuity of preferences, indispensability of the private goods and strict monotonicity in one private good together imply essentiality.

Theorem 3.5 *Let for every agent $a \in A$ the utility function U_a be continuous, quasi-concave and strictly monotone on $\mathbb{R}^{\ell}_{+}$. Assume that $\mathcal{Y}$ satisfies the essentiality condition. Then every Pareto efficient allocation in $\mathbb{E}$ can be supported as a valuation equilibrium.*

PROOF

Let (f, y) be a Pareto efficient allocation in $\mathbb{E}$, and let $a \in A$ and $z \in \mathcal{Y}$ be arbitrary. We define

$$F(a,z) := \left\{ g \in \mathbb{R}^{\ell}_{+} \,|\, U_a(g,z) > U_a(f(a),y) \right\}.$$

Note that $F(a,z) \neq \emptyset$ for all $a \in A$ and $z \in \mathcal{Y}$ by part (i) of the essentiality condition. Moreover, from the assumptions on U_a it follows that $F(a,z)$ is open, convex, and bounded from below. Let, for every $z \in \mathcal{Y}$,

$$F(z) := \sum_{a \in A} F(a,z) + c(z) - \overline{w}.$$

$F(z)$ is nonempty. Moreover, from the above it follows that $F(z)$ is also open, convex, and bounded from below. Finally, because (f, y) is efficient, we have that

$$\mathbb{R}^{\ell}_{-} \cap F(z) = \emptyset.$$

By Minkowski's separation theorem, there exists a hyperplane separating $\mathbb{R}^{\ell}_{-}$ from $F(z)$. Hence, there exists a normal vector $p(z) \in S^{\ell-1}$ such that $p(z) \cdot F(z) > 0$. Obviously, this defines a function $p\colon \mathcal{Y} \to S^{\ell-1}$. We now show that p satisfies the conditions as required in Definition 3.2.

Let for every agent $a \in A$ and every $z \in \mathcal{Y}$ the bundle $x(a,z) \in \mathbb{R}^{\ell}_{+}$ be chosen such that, in case $z \neq y$,

(i) $p(z) \cdot x(a,z) = \inf\ p(z) \cdot F(a,z)$;

(ii) $U_a(x(a,z),z) \geq U_a(f(a),y)$,

and in case $z = y$, $x(a,z) = x(a,y) = f(a)$. Clearly, such bundles exist. Finally, we define a valuation function $V\colon A \times \mathcal{Y} \to \mathbb{R}$ by

$$V(a,z) := p(z) \cdot w(a) - p(z) \cdot x(a,z).$$

Note that $V(a,z)$ is finite and $V(a,z) \leq p(z) \cdot w(a)$ for all $a \in A$ and all $z \in \mathcal{Y}$ by definition. We now check the three requirements of Definition 3.2.

CONDITION (I)

By the feasibility of (f, y) and the definition of V,

$$\sum_{a \in A} V(a,y) = p(y) \cdot \overline{w} - p(y) \cdot \sum_{a \in A} f(a) = p(y) \cdot c(y).$$

CONDITION (III)

By construction, $p(z)\cdot\inf F(z)\geq 0$, and, hence, for all $z\neq y$:

$$p(z)\cdot\sum_{a\in A}x(a,z)+p(z)\cdot c(z)\geq p(z)\cdot\overline{w}.$$

(The inequality becomes weak because $x(a,z)$ is at the infimum.) From this we obtain

$$\sum_{a\in A}V(a,z)=p(z)\cdot\overline{w}-p(z)\cdot\sum_{a\in A}x(a,z)\leq p(z)\cdot c(z),$$

while

$$\sum_{a\in A}V(a,y)=p(y)\cdot\overline{w}-p(y)\cdot\sum_{a\in A}x(a,y)=p(y)\cdot c(y),$$

by feasibility. Thus, (iii) is shown.

CONDITION (II)

Note that, by part (ii) of the essentiality condition and the continuity and strict monotonicity of preferences, it follows for every $a\in A$ that $U_a(x(a,z),z)=U_a(f(a),y)$ for all $z\in\mathcal{Y}$. For any $(g,z)\in\mathbb{R}^\ell_+\times\mathcal{Y}$ with $U_a(g,z)\geq U_a(f(a),y)=U_a(x(a,z),z)$ we have

$$p(z)\cdot g+V(a,z)=p(z)\cdot g+p(z)\cdot w(a)-p(z)\cdot x(a,z)\geq p(z)\cdot w(a), \tag{5}$$

since $p(z)\cdot g\geq p(z)\cdot x(a,z)$ by the definition of $x(a,z)$.
Let $z\in\mathcal{Y}$. By Condition (iii), as shown above,

$$\sum_{a\in A}V(a,z)\leq p(z)\cdot c(z).$$

By assumption, $\overline{w}\gg c(z)$. Thus, since $p(z)>0$, it follows that

$$p(z)\cdot\overline{w}>p(z)\cdot c(z)\geq\sum_{a\in A}V(a,z).$$

Therefore, there exists an $a\in A$ with $p(z)\cdot w(a)>V(a,z)$. For any $\breve{g}\in\mathbb{R}^\ell_+$ such that $p(z)\cdot\breve{g}+V(a,z)<p(z)\cdot w(a)$, it follows from (5) that $U_a(\breve{g},z)<U_a(x(a,z),z)$. Since every $\hat{g}\in\mathbb{R}^\ell_+$ with

$$p(z)\cdot\hat{g}+V(a,z)=p(z)\cdot w(a)$$

is the limit of a sequence $(\hat{g}_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^\ell_+$ with

$$p(z)\cdot\hat{g}_n+V(a,z)<p(z)\cdot w(a),$$

we must have

$$U_a(\hat{g},z)\geq U_a(x(a,z),z)=U_a(f(a),y).$$

Since z was chosen arbitrarily, $(f(a), y)$ maximizes U_a on

$$\left\{(g,z) \in \mathbb{R}^\ell_+ \times \mathcal{Y} \mid p(z) \cdot g + V(a,z) = p(z) \cdot w(a)\right\}.$$

By strict monotonicity of preferences, this implies that $p(z) \gg 0$. Since z was chosen arbitrarily, this holds for every $z \in \mathcal{Y}$.

Consider now an agent $a \in A$ and a project $z \in \mathcal{Y}$ such that $p(z) \cdot w(a) = V(a,z)$. Then $p(z) \cdot x(a,z) = 0$ by the definition of $V(a,z)$. Since $p(z) \gg 0$, we have $x(a,z) = 0$. By part (ii) of the essentiality condition and $U_a(f(a), y) = U_a(x(a,z), z) = 0$, we have $f(a) = 0$. Therefore, $x(a,z) = 0$ for every $z \in \mathcal{Y}$. Thus, U_a is maximized by $(f(a), y)$ on

$$\{(0,z) \,|z \in \mathcal{Y}\} = \left\{(g,z) \in \mathbb{R}^\ell_+ \times \mathcal{Y} \mid p(z) \cdot g + V(a,z) = p(z) \cdot w(a)\right\}.$$

This shows condition (ii). □

4 Continuity of the price system

Next we consider under what conditions the price system $p\colon \mathcal{Y} \to S^{\ell-1}$ of a valuation equilibrium can be chosen to be a continuous function. This seems to be a very natural requirement once the set of public projects is endowed with a topological structure.

We will assume that $\mathcal{Y}$ is endowed with a topology generated by some well-chosen metric $d\colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}^\ell_+$, and that the cost function c is continuous with respect to this topology. We first state our continuity theorem for a truncated economy.

Theorem 4.1 *Let $k \in \mathbb{N}$ be some positive integer such that $k > \overline{w}_j$ for every $j \in \{1, \ldots, \ell\}$. Let $\mathbb{E}^k$ be a truncated economy such that the following conditions are satisfied:*

- *(i) The set of public projects $\mathcal{Y}$ is endowed with a topology generated by a metric $d\colon \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$.*
- *(ii) For all $a \in A$, U_a is continuous, quasi-concave and strictly monotonic on $\mathbb{R}^\ell_+$.*
- *(iii) The cost function c is continuous on $(\mathcal{Y}, d)$ and $c(z) \ll \overline{w}$ for all $z \in \mathcal{Y}$.*
- *(iv) The set of public projects $\mathcal{Y}$ satisfies the essentiality condition.*
- *(v) The consumption space of every agent $a \in A$ is restricted to $\mathbb{R}^\ell_+ \cup \{z \in \mathbb{R}^\ell \mid z \leq ke\}$, where $e = (1, \ldots, 1) \in \mathbb{R}^\ell_+$ denotes the vector with unity in every component.*

Then every Pareto efficient allocation in $\mathbb{E}^k$ can be supported as a valuation equilibrium with a continuous price system.

PROOF
In the sequel we denote by K the set $\{z \in \mathbb{R}^\ell \mid z \leq ke\}$.
We show that there exists a price system $p\colon \mathcal{Y} \to S^{\ell-1}$ which has the same properties as the one constructed in the proof of Theorem 3.5 and is, in addition, continuous with respect to the metric space $(\mathcal{Y}, d)$.
Let (f, y) be a Pareto efficient allocation in $\mathbb{E}^k$ and let $a \in A$ and $z \in \mathcal{Y}$ be arbitrary. Define

$$F^k(a,z) := \left\{g \in \mathbb{R}^\ell_+ \cap K \mid U_a(g,z) > U_a(f(a),y)\right\}.$$

Claim 1. The correspondence $\sum_{a\in A} F^k(a,z)$ has a closed graph.

Proof of Claim 1.
Note that $F^k(a,\cdot)$ has a closed graph for all $a \in A$ because $U_a(g,\cdot)$ is a continuous function on $\mathcal{Y}$, for all $a \in A$. Further, because of the boundedness of the range of F^k, there exists a function $h : A \to \mathbb{R}$ such that $\| F^k(a,z) \| < h(a)$ for every $a \in A$. By, e.g., Proposition D.8 of Hildenbrand (1974, page 73) it follows that $\sum_{a\in A} F^k(a,z)$ is closed at z, for all $z \in \mathcal{Y}$. Since $z \in \mathcal{Y}$ is arbitrary, Claim 1 is shown. QED.

As in the proof of Theorem 3.5, define, for all $z \in \mathcal{Y}$,

$$F^k(z) \equiv \sum_{a\in A} F^k(a,z) + c(z) - \overline{w}.$$

By Claim 1 and the continuity of the function c on $\mathcal{Y}$, the correspondence F^k has a closed graph. Define now the correspondence $T\colon \mathcal{Y} \to \mathbb{R}^\ell$ by

$$T(z) \equiv \left\{p \in \mathbb{R}^\ell \mid p \cdot F^k(z) > 0\right\}.$$

The correspondence T assigns to every potential public project the set of *all* normal vectors of separating hyperplanes of $F^k(\cdot)$ and $\mathbb{R}^\ell_-$. Following the proof of Theorem 3.5 we know that any selection in T corresponds to a valuation equilibrium supporting the Pareto efficient allocation (f, y). We now show that there exists a continuous selection in T, thus establishing the assertion.
Note that by application of Minkowski's separation theorem (see the proof of Theorem 3.5) (i) $T(z)$ is a non-empty convex cone, and by monotonicity (ii) $T(z) \subset \mathbb{R}^\ell_+$ for all $z \in \mathcal{Y}$.
We now adapt a result from Aubin (1993, pages 291, 338):

Claim 2. If $T(z)^- \equiv \{q \in \mathbb{R}^\ell_+ \mid p \cdot q \leq 0 \text{ for all } p \in T(z)\}$ has a closed graph, then T is lower semi-continuous.

Proof of Claim 2.
Consider a sequence $z_n \to z$ and take some $p \in T(z)$. Suppose that $\Pi_{T(z_n)}$ is the orthogonal projector onto $T(z_n)$. It suffices to show that $p_n \equiv \Pi_{T(z_n)}(p)$ converges to p. Then $\pi_n = p - p_n$

is the projector of p onto $T(z_n)^-$. Thus $\pi_n \cdot p_n = 0$ and $\| \pi_n \| \leq \| p \|$. Consequently, since $\mathbb{R}^\ell_+$ is finite-dimensional, a subsequence (keeping the notation π_n) converges to $\bar{\pi}$. This element $\bar{\pi}$ belongs to $T(z)^-$, since the graph of $T(z)^-$ is closed by hypothesis, and satisfies $\bar{\pi} \cdot (p - \bar{\pi}) = \lim_{n\to\infty} \pi_n \cdot p_n = 0$. Thus $\| \bar{\pi} \|^2 = \bar{\pi} \cdot p \leq 0$ since $\bar{\pi} \in T(z)^-$ and $p \in T(z)$. Consequently, $\bar{\pi} = 0$ and $p_n \to p = p - \bar{\pi}$. QED.

Claim 3. The correspondence $T(\cdot)^-$ has a closed graph.

Proof of Claim 3.
Take a sequence $(z_n, q_n) \to (z, q)$ with $z_n \in \mathcal{Y}$, $q_n \in T(z_n)^-$ for all n. This means that $p_n \cdot q_n \leq 0$ for all $p_n \in T(z_n)$ by the definition of $T(\cdot)^-$. We must show that $p \cdot q \leq 0$ for all $p \in T(z)$.
Suppose the contrary holds. Then there exists $p \in T(z)$ such that $p \cdot q > 0$. Since $q_n \to q$, $p_n \cdot q \leq 0$ for all $p_n \in T(z_n)$, we have that $p_n \cdot q_n \leq 0$. Therefore, for n large enough, $p \notin T(z_n)$. But $p \in T(z)$. This implies that for large enough n, there exists $x_n \in F^k(z_n)$ such that $p \cdot x_n \leq 0$. However, $p \cdot F^k(z) > 0$. The sequence (x_n) is bounded (because of the definition of the range of F^k), therefore it has a converging subsequence. Keeping notation and switching to the subsequence, $x_n \to x$.
By the closedness of the correspondence F^k, which, as noted, follows from Claim 1, $x \in F^k(z)$. Therefore, $p \cdot x > 0$. But we also have $p \cdot x_n \leq 0$ for large enough n, which is a contradiction. QED.

By combination of Claims 2 and 3 it follows that the correspondence T is lower semi-continuous. Hence, we may apply Michael's selection theorem (see, e.g., Klein and Thompson 1984, Theorem 8.1.8). The proof of Theorem 3.5 shows that every selection p from T is a price system corresponding to a valuation equilibrium. Applying Michael's selection Theorem, we conclude that there exists a continuous selection in T. By repeating the proof of Theorem 3.5 for this continuous selection we establish the assertion, i.e., the allocation (f, y) indeed can be supported by a continuous price system $p \colon \mathcal{Y} \to S^{\ell-1}$ as a valuation equilibrium. □

The extension of Theorem 4.1 to a nontruncated economy is evident as k above becomes large enough. We refer to Debreu (1959) for the formal techniques to be used. The proof of the following corollary is therefore omitted.

Corollary 4.2 *Let $\mathbb{E}$ be an economy such that $\mathcal{Y}$ is endowed with a topology generated by a metric d, for all agents $a \in A$ preferences are continuous, quasi-concave and strictly monotonic on $\mathbb{R}^\ell_+$, the cost function is continuous on $(\mathcal{Y}, d)$, and $\mathcal{Y}$ satisfies the essentiality condition. Then every Pareto efficient allocation in $\mathbb{E}$ can be supported as a valuation equilibrium with a continuous price system.*

5 Cost Sharing and the Core

Following Mas-Colell (1980) we now turn to the question whether we are able to give a plausible extension of the notion of a cost share equilibrium to our setting, in such a way as to preserve core equivalence. A *cost share* equilibrium is a special kind of valuation equilibrium, in which the valuation function is simply a distribution of the costs of the public project over all agents in the economy; subsidies are not allowed. It turns out that the plausible extension of cost share equilibrium is immediate, and all cost share equilibria are in the core, but the core may be strictly larger than the set of cost share equilibria. Hence, the core equivalence results in Mas-Colell (1980) and Weber and Wiesmeth (1991) depend crucially on the assumption that there is only one private good.

Definition 5.1 *A feasible allocation $(f, y) \in \Phi$ is a* ***cost share equilibrium*** *in $\mathbb{E}$ if there exist a price system $p\colon \mathcal{Y} \to S^{\ell-1}$ and a non-negative valuation function $V\colon A \times \mathcal{Y} \to \mathbb{R}_+$ such that*

(i) *there is no surplus, i.e., $\sum_{a\in A} V(a, y) = p(y) \cdot c(y)$;*

(ii) *for every agent $a \in A$, $(f(a), y)$ maximizes U_a on the budget set*

$$\left\{(g, z) \in \mathbb{R}^{\ell}_{+} \times \mathcal{Y} \mid p(z) \cdot g + V(a, z) = p(z) \cdot w(a)\right\}$$

and for every $z \in \mathcal{Y}$: $V(a, z) \leq p(z) \cdot w(a)$;

(iii) *y maximizes $\sum_{a\in A} V(a, z)\, d\mu - p(z) \cdot c(z)$ over $z \in \mathcal{Y}$.*

A cost share equilibrium is a valuation equilibrium in which there are no subsidies. In the sequel we will use the abbreviation CSE to indicate cost share equilibria.

We now define the core for our framework. In this setting a core allocation is defined straightforwardly as an extension of the one proposed by Mas-Colell (1980). In Roberts (1974), Wooders (1989), Vasil'ev, Weber and Wiesmeth (1992), and Gilles and Diamantaras (1994) alternative formulations of the core are proposed and core equivalence theorems are established. From these studies it follows that such equivalence results hold only for a modified core concept. In this paper we show that there is no equivalence of the core and the set of CSE, a not unexpected result. This shows that the core equivalence results of Mas-Colell (1980) and Weber and Wiesmeth (1991) for finite economies rely solely on the assumption that $\ell = 1$.

Definition 5.2 *A feasible allocation $(f, y) \in \Phi$ is in the* ***core*** *of $\mathbb{E}$ if there are no coalition $E \subset A$, public project $z \in \mathcal{Y}$, and allocation of private goods $g\colon E \to \mathbb{R}^{\ell}_{+}$ such that the following requirements are satisfied:*

(i) (g,z) *is feasible for E, i.e.,*

$$\sum_{a\in E} g(a) + c(z) = \sum_{a\in E} w(a), \quad \textit{and}$$

(ii) *for all members a of E:*

$$U_a(g(a),z) > U_a(f(a),y).$$

The core of the economy $\mathbb{E}$ will be indicated by $\mathcal{C}(\mathbb{E}) \subset \Phi$.

Our first result extends the insight that CSE always are in the core — as shown in Mas-Colell (1980) — to our framework.

Theorem 5.3 *Let for all agents $a \in A$ the utility function U_a be monotone. Then every cost share equilibrium is in the core of $\mathbb{E}$.*

PROOF

Suppose that there is a coalition $E \subset A$, a public project $z \in \mathcal{Y}$, and an allocation of private goods $g: E \to \mathbb{R}^{\ell}_{+}$ such that

$$\sum_{a\in E} g(a) + c(z) = \sum_{a\in E} w(a),$$

and, for all all $a \in E$,

$$U_a(g(a),z) > U_a(f(a),y).$$

By condition (ii) of the definition of CSE, for all $a \in E$,

$$p(z)\cdot g(a) + V(a,z) > p(z)\cdot w(a).$$

It follows that

$$p(z)\cdot \sum_{a\in E} g(a) + \sum_{a\in E} V(a,z) > p(z)\cdot \sum_{a\in E} w(a). \tag{6}$$

Now we have the following chain:

$$\begin{aligned}
0 &= \sum_{a\in A} V(a,y) - p(y)\cdot c(y) \\
&\geq \sum_{a\in A} V(a,z) - p(z)\cdot c(z) \\
&\geq \sum_{a\in E} V(a,z) - p(z)\cdot c(z) \\
&> p(z)\cdot \sum_{a\in E} w(a) - p(z)\cdot \sum_{a\in E} g(a) - p(z)\cdot c(z).
\end{aligned}$$

In this chain the first equality follows from condition (i) of the definition of CSE and the first weak inequality follows from condition (iii) of the same definition. The second weak inequality follows from $V(\cdot,\cdot) \geq 0$, and the strict inequality follows from (6) above. However, the first and last items in the chain contradict the feasibility of improving by coalition E because $p(z) > 0$ by monotonicity. □

The failure of the converse to hold in economies with multiple private commodities is shown next by example. The example also illustrates how to compute a valuation equilibrium and how there may not be much choice in the price system and valuation functions that support a given efficient allocation.

Theorem 5.4 *There exists well-behaved economies $\mathbb{E}$ such that the core of $\mathbb{E}$ is strictly larger than the set of cost share equilibria of $\mathbb{E}$.*

PROOF

Let $A = \{a,b\}$, $\ell = 2$, $\mathcal{Y} = \{y,z\}$, $w(a) = (3,0)$, $w(b) = (0,3)$, $c(y) = (1,1)$, and $c(z) = (0,2)$. The utility functions are $U_a(f_1,f_2,y) = U_b(f_1,f_2,y) = 3\sqrt{f_1}+\sqrt{f_2}$ and $U_a(f_1,f_2,z) = U_b(f_1,f_2,z) = \sqrt{f_1} + \sqrt{f_2}$ for every commodity bundle $f = (f_1,f_2) \in \mathbb{R}^2_+$.

Consider the allocation (f,y) with $f(a) = f(b) = (1,1)$. It is a feasible allocation, because $f(a) + f(b) + c(y) = (3,3) = w(a) + w(b)$.

We claim that (f,y) is a core allocation. First, it is clearly Pareto efficient. Indeed, as long as y is chosen, clearly there can be no improvement for any agent without harming the other; if z is chosen, one of the agents must become worse off relative to (f,y). Second, no agent can unilaterally improve on (f,y): agent a cannot undertake any project alone because he does not have any amount of private good 2, while agent b can produce z but this leaves her with a utility of 1, less than the 4 attained at (f,y).

There is a price system and valuation functions that turn the allocation (f,y) into a valuation equilibrium: $p(y) = (3/4, 1/4)$, $p(z) = (1/2, 1/2)$, and $V(a,y) = 5/4$, $V(b,y) = -1/4$, $V(a,z) = 1/2$, $V(b,z) = 1/2$. It is easy to check that these make (f,y) a valuation equilibrium, and it is clear that it is not a cost share equilibrium, since $V(b,y) < 0$. The question then is whether it is possible to find another price and valuation system supporting this allocation, in which no valuation is negative.

Note that $p(y)$ must remain the same in any valuation equilibrium, because the utility functions are smooth and must be supported by $p(y)$ at $(1,1)$. Let us try to construct a valuation equilibrium with $V(b,y) \geq 0$. This inequality implies $V(a,y) \leq 1$. From a's budget we see that this implies $3/4 f_1(a) + 1/4 f_2(a) \geq 5/4$; now this implies that agent a obtains a utility equal to at least $2\sqrt{5}$ by demanding a vector $(g_1(a), g_2(a)) = (x,x)$ with $x \geq 5/4$. This means that the allocation (f,y) is not a valuation equilibrium. □

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