# 1 INTRODUCTION

There has been extensive analysis of the case of two parties, $i$ and $j$, competing in a policy space $Z$ for electoral votes. The two parties (or candidates) are assumed to pick policy positions $z_1, z_2$ both in $Z$, which they present as manifestos to a large electorate. Suppose that each member of the electorate votes for the position that the voter truly prefers. When $Z$ involves three or more dimensions, then under general conditions, developed by Plott (1967), McKelvey and Schofield (1986) and many others, there will exist no Condorcet point unbeaten under majority rule. That is to say, whatever position is picked by $z_i$, there always exists a point $z_j$ that will give party $j$ a majority over $z_i$.

However, the existence of a Condorcet point has been established in those situations where the policy space is one dimensional. In this case the parties can be expected to converge to the position of the median voter (Downs 1957). When $Z$ has two or more dimensions, it is known that a Condorcet point exists when electoral preferences are represented by a spherically symmetric distribution of electorally preferred points. Even when the distribution is not spherically symmetric, a Condorcet point can be guaranteed as long as the decision rule requires a sufficiently large majority (Caplin and Nalebuff, 1988).

One difficulty with the application of these two types of results in real-world politics has been the extreme nature of the predictions. The McKelvey-Schofield results seem to suggest that two-party political competition is extremely unstable, so that political outcomes are dependent essentially on random events. The Downsian results on the other hand imply that political competition gives rise to complete convergence in party platforms.

Schofield and Tovey (1992) have recently attempted to integrate the two classes of results by constructing a model where electoral preferences are obtained by sampling from a continuous distribution of preferred voter policy points. They show, under certain conditions on the distribution and decision rule, that the probability of a Condorcet point approaches one, as the electoral sample size approaches infinity. This conclusion corresponds to the intuition that voting instability may well be a significant phenomenon of small committees rather than of large electorates.

However it is still the case that the existence of a Condorcet point seems to imply a strong form of political convergence that is not generally observed. A number of attempts have been made to model the game between the candidates more fully, so as to give rise to a weaker form of convergence. For example the various interpretations made above assume that the candidates are solely interested in winning and gain no utility from implementing a particular policy point. A number of authors (Petry 1982; Wittman 1983) have argued, on the contrary, that if candidates are concerned with policy, then they will choose a compromise between their preferred position and one that they deem electorally superior. The difficulty with this argument is that when there is no electoral uncertainty a party is able to implement a particular policy only by winning the election. In a situation of electoral uncertainty, on the other hand, it is reasonable to suppose that each party will choose a platform or policy position that is the best compromise given beliefs about electoral preferences and its opponents' strategies.

Suppose then that there is uncertainty vis à vis the electoral response to candidate positions. For each profile of candidate positions $\{z_i, z_j\}$ define probabilities $p_i(z_i, z_j)$, $p_j(z_i, z_j)$ such that $i$ (respectively, $j$) wins. Generally it is assumed that $p_i + p_j = 1$ and that $i$ implements $z_i$ if it wins.

Then the *outcome* is a lottery, $f(z_i, z_j) = \{\{a_i(z_i), p_i(z_i, z_j)\}, \{a_j(z_j), p_j(z_i, z_j)\}\}$ where $a_i(z_i)$ is the outcome implemented when $i$ wins the election. Cox (1984) used standard fixed point arguments to obtain conditions under which an equilibrium exists.

A general difficulty with all the two party models described above concerns the credibility of the candidates. If candidate $i$, say, wins the election then it is difficult to see why $i$ need implement its declared policy when it has some hidden policy preferences of its own. Its credibility depends then on its perception of the degree to which the electorate can punish at some time in the future. This naturally will affect the electoral calculation and thus the form of the win probabilities, $p_i, p_j$.

The difficulty implicit in modelling two-party competition has until recently somewhat inhibited attempts to model a *multiparty* game of party competition, where there are at least three parties and coalitions between the parties are generally necessary to form majority governments. What work has been done has principally concentrated on one of the following partial models.

(i) The electorate is represented by a distribution of preferred points. Candidates are uninterested in policy *per se* and are concerned only with maximising the number of votes. Analyses of this model by Eaton and Lipsey (1975) and Shaked (1975) suggest that a Nash equilibrium is unlikely.

A number of variants of this vote/seat maximizing model are reviewed in Shepsle (1991). However there is a fundamental difficulty with all models of this kind. In the two-candidate case, maximizing the number of seats is essentially the same as winning the perquisites of office. In a multiparty situation, maximizing the seats or votes obtained need not be highly correlated with taking office. In particular small parties enter into coalition government on a regular basis in European multiparty systems.

(ii) The parties are concerned with policy outcomes, and declare their true positions in a policy space of two dimensions. Baron (1989, 1991) considers a three-party situation where the electoral responses have already been obtained and each two party coalition is a possible majority government. He assumes that for each two-party coalition, $M$, there can be associated a probability $\rho_M$ of its formation. His concern is to compute, for each $M$, a compromise position $z_M$ in $\Re^2$ that is functionally dependent on the party preferences and coalition probabilities.

Thus, in Baron's model the outcome function is a lottery

$$f(z) = \{(z_M, \rho_M) : | M | = 2\}.$$

Note that $f$ is functionally dependent on the exogenously chosen positions and probabilities.

One question that Baron did not initially consider was incentive compatibility: is it necessarily the case when $f$ is "common knowledge" that it is rational for each party to declare its true preferred position? Baron (1993) has recently attempted to resolve this question by considering the behavior of parties who represent an endogenously determined subset of all voters.

(iii) Austen-Smith and Banks (1988) examine a model in one-dimension, where the three parties are uninterested in policy, but are motivated to form governments because of the perquisites of office. Once the party positions $\{z_1, z_2, z_3\}$ are chosen, each coalition $\{i, j\}$ chooses a policy point that is in equilibrium with the other coalition policy points. Only one of the coalitions forms. It is assumed that voters know how party positions and seat strengths map to an outcome, and, with respect to this knowledge, optimally choose a party to vote for. It has not proved possible to extend this model to more than one dimenson. Moreover the coalition that forms need not contain the median, or center, party. This conclusion seems at odds with the typical understanding on the nature of coalition formation when only one policy dimension is relevant (see Laver and Schofield, 1990).

(iv) Schofield (1993a) considers a model with parties $\{1, ..., n\}$. Party $i$ has true preferences on $Z$ which are unknown to other parties and the electorate. It chooses a policy position $z_i$ from a space, $Z$, of at least two dimensions, and $z_i$ is declared as a platform or *manifesto* to all participants. Call $z = (z_1, ..., z_n) \in Z^N$ the policy (or manifesto) profile. The electorate responds by allocating seats $\{e_1(z), ..., e_n(z)\}$.

Call $e(z) = (e_1(z), ..., e_n(z))$ the seat profile. The seat profile defines a family of *decisive* (or winning) coalitions $\mathcal{D}(z)$. If $M$ belongs to $\mathcal{D}(z)$ then this coalition of parties is able to form a government controlling the required majority of seats. The policy and seat profiles define a set of possible outcomes $H(\mathcal{D}(z)) \subset Z$ called the *heart* of the game, which is contained within the convex hull of $\{z_1, ..., z_n\}$. The outcome of the game is a point $f(z) \in H(\mathcal{D}(z))$ where $f : Z^N \to Z$ is a continuous selection from the heart correspondence. The preference correspondences $P_i : Z \to Z$ are assumed to be well-behaved (namely continuous and convex-valued). If the induced preference correspondences $f^*(P_i) : Z^N \to Z$ are also well-behaved, then there will exist a mixed strategy Nash equilibrium in the choice of manifestos.

The purpose of this paper is to construct a general model of *multiparty* competition where parties are simultaneously interested in policy outcomes and in the perquisites (or private benefits) from office. The intention is to use this model to understand multiparty competition of the kind found in European democracies (Schofield 1993b; Laver and Schofield, 1990).

The previous model (Schofield, 1993a) is developed by making specific assumptions regarding the relationship between the policy profile, $z$, and political outcomes. To explore this in a theoretically tractable model we first of all assume that $N = \{1, 2, 3\}$ and that $\mathcal{D}(z)$ is fixed and defined to be the family consisting of three different two party coalitions. Given $\{z \in Z^N\}$, we assign a probability $\rho_M(z)$ to each to each two party coalition, $M$. In particular we assume that for each $z$, $\rho_{12}(z) + \rho_{13}(z) + \rho_{23}(z) = 1$. Subject to this constraint we also assume that $\rho_{ij}(z)$ is inversely proportional to the square of the distance between the manifestos $\{z_i, z_j\}$. If coalition $M = \{i, j\}$ forms, then the policy point for that coalition is chosen to be $z_M = \frac{z_i+z_j}{2}$. Player $i$, receives a private benefit or perquisite $\sigma_{ij}$ when coalition $\{ij\}$ forms.

Finally we suppose that the preferences of each party are separable in policy and perquisites. In particular if coalition $M = \{ij\}$ forms then the utility for party $i$ is $-\frac{1}{2} \parallel o_i - z_M \parallel^2 + \sigma_{ij}$. On the other hand if coalition $K = \{j, k\}$ forms then $i'$s utility is $-\frac{1}{2} \parallel o_i - z_K \parallel^2$ . Here $o_i$ is the preferred, or bliss point, of party $i$. Since the outcome defined by the policy profile $z = (z_1, z_2, z_3)$ is a finite lottery given by the fixed scheme $\Gamma = \{\sigma_{ij}\}$ of private benefits, it is possible to compute the von Neumann Morgenstern expected utility $U_i(z)$ for each party. Because the expected utility functions are smooth, there will exist mixed strategy Nash equilibria. Let $\tilde{Z}^N$ stand for the space of Borel probability measures over the joint strategy space $Z^N$. The simplifying assumptions that we have made allow us to examine the nature of the Nash equilibrium correspondence

$$\mathcal{E}_\Gamma : Z^N \rightarrow \tilde{Z}^N$$

given by the scheme of perquisites, $\Gamma$. Here $\mathcal{E}_\Gamma$ maps the vector $o = (o_1, o_2, o_3)$ of bliss points to the set $\{(z_1^*, z_2^*, z_3^*)\}$ of mixed strategy Nash equilibria. Theorem 1 specifies that there is always an *MSNE*. More importantly if the perquisites defined by the scheme are sufficiently large then for each $o \in Z^N$ the set $\mathcal{E}_\Gamma(o)$ consists only of pure strategy Nash equilibria. Developing the model analytically shows that there are two open domains, $X_1$ and $X_2$ in $Z^N$ such that for $o \in X_1 \cup X_2$, then $\mathcal{E}_\Gamma(o)$ gives a unique, pure strategy Nash equilibrium. In particular $X_1$ is the set of vectors of bliss points that are "close" to colinear. For $o \in X_1$ the Nash equilibrium $z^*$ is "convergent" in the sense that $\| z_i^* - z_j^* \| < \| o_i - o_j \|$ for each pair $i, j$.

On the other hand the domain $X_2$ comprises bliss points that are close to "symmetric" in the sense that $\| o_1 - o_2 \| \simeq \| o_1 - o_3 \| \simeq \| o_2 - o_3 \|$. For $o \in X_2$, the unique pure strategy Nash equilibrium $z^*$ is divergent in the sense that $\| z_i^* - z_j^* \| > \| o_i - o_j \|$ .

Theorems 2, 3, 4 spell out the relationship between the scheme of benefits, $\Gamma$, the structure of bliss points and the nature of the Nash equilibrium.

This first model concentrates on political negotiation when the electoral response is unimportant. Although it is a special case, the reader may view it as an attempt to model three party coalition in a country such as Germany. In general no one party can expect to gain a majority, and each two party coalition is a real possibility. In such a situation it is natural to suppose that two parties whose declarations are very close to each other would be very likely to form a government.

The simple three party model suggests that when only one dimension is really relevant, then "Downsian" (1957) convergence is likely. On the other hand, if two or more dimensions are relevant then parties will, in equilibrium, maintain quite distinct policy positions. This, of course, is observed in European multiparty situations.

It is a simple matter to extend the simple three party model to the general case of $n$ parties. We suggest that a similar pattern of convergence or divergence, determined by the location of bliss points, will again occur.

Finally, we propose a general model based on Schofield (1993a) where the policy profile, $z$, determines a lottery over electoral responses, and thus the decisive structure $\mathcal{D}(z)$. Assuming that for each $\mathcal{D}(z)$ the outcome is a lottery over various coalition possibilities, again there will exist a mixed strategy Nash equilibrium. We conjecture that in these models there are open domains in the joint strategy space characterized by a fixed number of pure strategy Nash equilibria.

# 2 THE FORMAL MODEL

We now set up the general form of our model. For a political game involving the set $N = \{1, \ldots, n\}$ of parties, on a compact, convex set $W \subset \Re^w$ of outcomes, each party $i$, has a true preference correspondence $P_i : W \to W$, where for $a \in W$, $P_i(a)$ is a convex set of outcomes strictly preferred to $a$. We use $W$ here as distinct from the policy space $Z$, since $W$ may include both private and policy outcomes. We shall also assume that $P_i$ can be represented by a strictly pseudo concave utility function $u_i : W \to \Re$. That is $u_i$ is differentiable and has a unique critical point at which $u_i$ is maximized.

Let $\tilde{W}$ be the space of (Borel) probability measures on $W$, endowed with the topology of weak convergence (see Fudenberg and Tirole, 1991 for example). In particular $\tilde{W}$ contains the space $\tilde{W}_o$, of all finite lotteries over $W$, where a finite lottery is a collection $\{a_j;\ p(a_j)\}_J$ of outcomes $a_j \in W$ and probabilities $p(a_j)$ satisfying $\sum p(a_j) = 1$. Individual preference is extended over $\tilde{W}$ giving $\tilde{P}_i : \tilde{W} \to \tilde{W}$. We shall assume $\tilde{P}_i$ is represented by an affine Von Neumann-Morgenstern utility function $U_i : \tilde{W} \to \Re$. On $\tilde{W}_0, U_i$ is defined by

$$U_i\left(\{a_j,\ p(a_j)\}\right) = \sum_j p(a_j) u_i(a_j).$$

Each party has a strategy space, $Z_i \in \Re^q$ which is compact, convex. Each strategy *profile* of party choices $z = (z_1, \ldots, z_n) \in Z^N = \prod_N Z_i$ gives an outcome in $\tilde{W}_o$, namely a finite lottery $f(z) = \{a_j(z),\ p(a_j(z))\} \in \tilde{W}_o$. Thus $f : Z^N \to \tilde{W}_o$. The function $f$ induces a preference correspondence $\tilde{f}(P_i) : Z^N \to Z^N$ defined by $(z_1, \ldots, z_i, \ldots, z_n) \in \tilde{f}(P_i)(z'_1, \ldots, z'_i, \ldots, z'_n)$ iff $f(z_1, \ldots, z_i, \ldots, z_n) \in \tilde{P} f(z'_1, \ldots, z'_i, \ldots, z'_n)$ and $z_j = z'_j \; \forall\, j \neq i$.

Define $f^*(P_i) : Z^N \to Z_i$ by taking $f^*(P_i)(z)$ to be the $i^{th}$ projection of $\tilde{f}(P_i)(z)$ onto $Z_i$.

A **pure strategy Nash equilibrium** (*PSNE*) for this game is a strategy profile $z^* = (z_1^*, \ldots, z_n^*) \in Z^N$ such that $f^*(P_i)(z^*) = \emptyset$ for all $i \in N$.

A **mixed strategy Nash equilibrium** (*MSNE*) is a strategy profile $z^* \in (\tilde{Z})^N$ such that $f^*(P_i)(z^*) = \emptyset \ \forall\, i \in N$, where $f * (P_i)$ is extended over $(\tilde{Z})^N$, and $\tilde{Z}$ is the space of Borel probability measures over the pure strategy space $Z$. The *MSNE*, $z^*$, has finite support if $z^* \in (\tilde{Z}_o)^N$, the space of finite lotteries.

Existence of a *PSNE* can be guaranteed by the Fan (1961)-Bergstrom (1975, 1992) Theorem or by Glicksburg's (1952) Theorem when the $f^*(P_i)$ satisfy certain convexity and continuity properties.

**Definition 1.** The best response correspondence

$$R(P_i) : \prod_{j \neq i} Z_j \to Z_i.$$

is defined in the usual way by $z_i \in R(P_i)(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ iff $f^*(P_i)(z_1, \ldots, z_i, \ldots, z_n) = \emptyset$.

More generally, for the preference profile $P = \{P_1, \ldots, P_n\}$, define the joint best reaction correspondence

$$R(P) : Z^N \to Z^N$$

by $(z'_1, \ldots, z'_n) \in R(P)(z_1, \ldots, z_n)$ iff $z'_i \in R(P_i)(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$.

A *pure strategy Nash equilibrium* (*PSNE*) is a fixed point of $R(P)$. From Glicksburg (1952) if $R(P)$ is compact and convex-valued and upper hemi-continuous, then the Kakutani (1941) fixed point theorem gives a fixed point and hence a *PSNE*. A difficulty arises if $R(P_i)$ is non-single valued, since the joint reaction correspondence will not be convex-valued. However, assuming that each $\tilde{P}_i$ is represented by a Von Neumann-Morgenstern utility function, then convexifying the reaction correspondence

$$R(\tilde{P}) : (\tilde{Z})^N \to (\tilde{Z})^N$$

will give a fixed point in $(\tilde{Z})^N$ and thus a *mixed strategy Nash equilibrium* (*MSNE*).

It is possible that a *PSNE* exists but is unstable. Say a *PSNE*, $z$, is *stable* (*locally*) if for any neighborhood $U$ of $z$ there exists a neighborhood $V$ of $z$ with $V \subset U$ such that for any $z' \in V$, then $R(P)(z') \subset U$.

It is of interest to examine the relationship between the equilibrium outcome $f(z^*)$, if it exists, and the preferences $\{P_i\}$ of the players. To do this define the expectation operator

$$E : \tilde{W}_o \to W$$

by

$$E\left(\{a_j,\ p_j(a_j)\}\right) \ = \ \sum_j p_j(a_j)a_j,$$

where the right hand is the point in $W$ obtained by interpreting $\{p_j(a_j)\}$ as real coefficients.

In the political model examined here the outcome space $W$ is identified with $Z \times \Delta_N$ where $\Delta_N \ = \Delta$ is a (compact) simplex in $\Re^n$. $\Delta$ is to be thought of as the space of private benefits that accrue to the parties as a result of their activity. Given some *PSNE*, $z$, then $(E_i|\Delta)(f(z)) = E_i(z)$ is the expectation to $i$ of private benefits, while $(E/Z)(f(z))$ is the expectation in $Z$, resulting from the joint strategy, $z$.

**Plurality Maximizing in the two-party case.** A number of earlier competitive political models can be viewed as particular instances of this general model. For example Kramer (1978) considered the case of two parties choosing positions $z_1, z_2$. The outcome $f$ is given by the vote shares $f(z) \ = \ (e_1(z_1, z_2), e_2(z_1, z_2))$ such that $e_1 + e_2 \ = 1$. Party preferences are induced from a utility function of the form $u_i(z) = e_i(z) - e_j(z)$. Kramer showed the existence of a *MSNE*. As we have noted, Eaton and Lipsey (1975) consider an $n-$party competitive model, based on the electoral function $e : Z^N \ \rightarrow \ \Delta_N$, where $e_i(z)$ is party $i$'s share of the vote, given the strategy vector $z$. Party $i$'s utility is given by its vote share, which is known with certainty once $z$ is chosen. It was difficult to show existence of Nash equlibrium in this model because of the violation of both continuity and preference convexity, but Dasgupta and Maskin (1986) showed that *MSNE* would exist if the discontinuities were restricted to lower dimensional "strata" in the joint strategy space. As we noted in the introduction, Cox (1984) has developed a model of two-party competition under electoral risk. That is, the outcome $f(z_1, z_2)$ is a lottery

$$\{a_1(z_1), p_1(z)\}, \{a_2(z_2), (1 - p_1(z))\}$$

where $a_i(z_i)$ is the outcome that party $i$ wins the election, implements the policy $z_i$ and gains a perquisite $\sigma_i$. This outcome occurs with probability $p_i(z)$. Assuming the party utility is additive in the policy outcome and perquisite means that the best response for party 1 is obtained by maximizing the expected utility function

$$U_1(f(z)) = U_1(z_1, z_2) = p_1(z)(u_1(z_1) + \sigma_1) + (1 - p_1(z))(u_1(z_2)).$$

Standard assumptions on the induced preference correspondences were needed to obtain existence of Nash equilibria.

The model we develop is a natural extension of Cox's in the sense that instead of the "win" probabilities we use probabilities that winning coalitions will form. Instead of implicit assumptions on the induced preference correspondences, we first of all make specific structural assumptions concerning the parameters of the model and show existence of equilibria. We then argue that a more general $n-$party structural model will also exhibit equilibria.

**Multiparty Competition and Coalition Government with Electoral Certainty.** We consider a general form of the multiparty game for $N = \{1, \ldots, n\}$, where party $i$ selects a strategy $z_i \in Z_i$, and each $Z_i = Z$, a fixed policy space. As in the Eaton-Lipsey model, we assume first that there is electoral certainty. That is there exists a continuous electoral function $e : Z^N \to \Delta_N$, which for each profile $z \in Z^N$ gives the vector of electoral strength $e(z) = (e_1(z), \ldots, e_n(z))$ satisfying $\sum_N e_i(z) = 1$. The decisive structure defined by $z$ is the family $\mathcal{D}(z) = \{M \subset N : \sum_{i \in M} e_i(z) > \frac{1}{2}\}$. The outcome is a lottery across coalition events as follows.

We assume in Model 1 that each coalition $M \in \mathcal{D}(z)$ implements a policy outcome $z_M = \frac{1}{|M|} \sum_M z_i$ in $Z$, forming a government with probability $\rho_M(z)$, which is inversely proportional to the variance of $\{z_i : i \in M\}$.

**Model 1.** To illustrate the nature of the model, suppose that there are three parties $N = \{1, 2, 3\}$, and that $\mathcal{D}(z)$ is fixed and equal to $\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. Let $Z$ be a compact convex subset of $\Re^2$.

If a strategy profile $\{z_1, z_2, z_3\}$ is chosen, then the cost of bargaining, $c_{ij}$, for coalition $\{i, j\}$ is proportional to $\| z_i - z_j \|^2$ and the probability that coalition $\{i, j\}$ forms is proportional to $\frac{1}{c_{ij}}$.

Moreover when coalition $\{i, j\}$ forms there is a private benefit $\sigma_{ij}$ to party $i$. Call $\Gamma = \{\sigma_{ij}\}$ the scheme of private benefits ("perquisites"). Finally, suppose that each party has a Euclidean utility function $u_i : Z \to \Re$ defined on $Z$ by

$u_i(z) = -\frac{1}{2} \parallel z - o_i \parallel^2$ where $o_i$ is the *ith* bliss point, belonging to the interior of $Z$. Overall utility for each party is additively separable in policy outcome and perquisite. Thus the von Neumann-Morgenstern utility to player 1 for example from the profile $z$ can be written

$$U_1(z) = \rho_{12}(z_{12})(u_1(z_{12}) + \sigma_{12}) + \rho_{13}(z_{13})(u_1(z_{13}) + \sigma_{13}) + \rho_{23}(z_{23})(u_1(z_{23})).$$

**Theorem 1.** If $Z$ is compact, convex, then for each vector of bliss points $o = \{o_1, o_2, o_3) \in Z^N$, and scheme $\{\Gamma = (\sigma_{ij}) : i,j \in \{1,2,3\}, i \neq j\}$ of private benefits, there exists a mixed strategy Nash equilibrium. For each $o \in Z^N$, there exists $\sigma^* > 0$ such that whenever $\sigma_{ij} > \sigma^*$ for each $\sigma_{ij}$ in $\Gamma$, then there exists a stable *PSNE*.

The proof of this and the following theorems can be found in the Appendix. We must note first that any profile $z = (z_1, z_2, z_3)$ satisfying $z_1 = z_2 = z_3$ is a Nash equilibrium, irrespective of the nature of preferences. This is clear since no individual can effect the outcome by changing $z_i$ to $z_i'$. In general such an equilibrium will be unstable. Since we wish to examine stable Nash equilibria, we perform (in the Appendix) a smooth perturbation of the probabilities when the parties are within an $\epsilon$-ball of each other. This perturbation does not affect the structure of the model, but it does eliminate such degenerate, unstable Nash equilibria from consideration.

We show in the Appendix that the utility functions $U_i$ need not be quasi-concave in the strategy variable, $z_i$. Nonetheless the $U_i$ are continuous (indeed differentiable) in the strategy variables and thus *MSNE* exist. We also show in the Appendix that when the private benefits are sufficiently large, then for each $i$, the best reaction correspondence is single-valued and continuous in the strategies of players other than $i$. This implies that the joint reaction $R(P)$ has a fixed point, which corresponds to a stable *PSNE*. Moreover specific configurations of bliss points give rise to *unique*, stable *PSNE*.

Two different cases giving unique *PSNE* are considered in detail.

($A$) Consider first the symmetric case in which $\parallel o_i - o_j \parallel = K$ constant for all pairs $\{i, j\}$. For convenience suppose $Z$ is a disc $D$ centered at the barycenter of $\{o_1, o_2, o_3\}$ with radius >> $K$. If private benefits are zero, then the best response correspondence of each player is single-valued, and the fixed point $z^*$ satisfies $\parallel z_i^* - z_j^* \parallel = 2 \parallel \sigma_i - \sigma_j \parallel$ for each pair $\{i, j\}$. Since $\parallel z_i^* - z_j^* \parallel > \parallel o_i - o_j \parallel$ for each pair, say *divergence* occurs. (See Figure 2 in the Appendix.) Note in particular, in this case, that no profile where each player chooses the same policy can be stable.

($B$) In the second case, suppose the bliss points are colinear. Write the bliss points as $\sigma_i = (0, y_i)$ and suppose $y_1 > y_3 > y_2 > 0$. Assuming zero private benefits, the best response by 3 is to choose $y_3^*$ closer to the mid-point $\frac{1}{2}(y_1 + y_2)$. On the other hand the best response by both 1 and 2 is to move closer to $y_3$. In this second, degenerate, case we find, as we might expect, that convergence occurs. By convergence, we mean that $\| z_i^* - z_j^* \| < \| o_i - o_j \|$ for each pair $i, j$.

In this co-linear case, there is an attractor $(z^*, z^*, z^*)$ of the best response function, where each player chooses the same policy. By definition this is a stable *PSNE*. Note that this is a form of Downsian convergence.

To solve the general case, we consider the problem of the best response by party 3 to $z_1 = (0, \frac{r_1}{2})$, $z_2 = (0, \frac{-r_2}{2})$. The best response in $x$ is essentially a function of $(r_1 + r_2)$, while the best response in $y$ is a function of $(r_1 - r_2)$. (See Figure 1 of the Appendix.) Either the response equation in $x$ or the equation in $y$ will dominate, giving either divergence or convergence. Full mathematical analysis of the general equations has not been possible, but computer simulation indicates that for almost any assignment of bliss points, a stable pure strategy Nash equilibrium occurs in the interior of the space.

Even in the symmetric case $A$, if there are private returns from coalition membership, and if these private benefits are sufficiently large, then the Nash equilibrium $z^*$ is convergent.

The results obtained in the Appendix on the relationship between bliss points, private benefits and Nash equilibria are described in Theorems 2, 3, and 4. We first need generalizations of the notions colinear and symmetric.

**Definition 2.** Say three points $\{z_i, z_j, z_k\}$ are $\epsilon-$*bounded in linearity* if

$$min_{\lambda_j, \lambda_k \in \Re} \ \{ \| z_i - \lambda_j z_j - \lambda_k z_k \| \} \leq \epsilon.$$

Say three points $\{z_i, z_j, z_k\}$ are $\epsilon$- *bounded in symmetry* if

$$max_{i,j,k} \ | \ \| z_i - z_k \| - \| z_j - z_k \| \ | \leq \epsilon,$$

where $max_{i,j,k}$ means across all permutations of $i, j, k$.

In the case $\epsilon = 0$, say simply that the points are symmetric.

Note of course that if three points are $\epsilon-$bounded in linearity, for $\epsilon \simeq 0$, then the degree of symmetry they exhibit will be low. Thus these two definitions attempt to capture the difference between the extreme cases $A$ and $B$.

**Theorem 2.** Suppose in Model 1 that private benefits are zero. There exists $\epsilon^* > 0$ such that if the bliss points are $\epsilon$-bounded in linearity, for any $\epsilon < \epsilon^*$, there exists a unique, stable Nash equilibrium which is convergent. That is $\{z_1^*, z_2^*, z_3^*\}$ all lie within the convex hull of $\{o_1, o_2, o_3\}$. Moreover, the Nash equilibrium strategies are also $\epsilon-$bounded in linearity.

**Theorem 3.** Suppose in Model 1 that private benefits are zero and $Z$ is the disc, $D$, as above. Then there exists $\epsilon^* > 0$ such that if the bliss points are $\epsilon-$bounded in symmetry, for $\epsilon < \epsilon^*$, then there exists a unique stable pure strategy Nash equilibrium $z^*$ in the interior of $Z$ which satisfies

$$\frac{\| z_i^* - z_j^* \|}{\| o_i - o_j \|} = b_{ij}(\epsilon).$$

Here $b_{ij}(\epsilon) > 1$. In the case $\epsilon = 0$, then $b_{ij}(\epsilon) = 2$, for each pair $\{i, j\}$.

Theorem 3 shows that the Nash equilibrium $z^*$ is *divergent* in the sense that $\| z_i^* - z_j^* \| > \| o_i - o_j \|$ for each pair. Moreover, if the bliss points are symmetrically located ($\epsilon = 0$), then $z^*$ is symmetric, *i.e.*,

$$\| z_1^* - z_2^* \| = \| z_2^* - z_3^* \| = \| z_1^* - z_3^* \| .$$

Note that the Nash equilibrium positions do not lie in the Pareto set of the parties, namely the convex hull of $\{o_1, o_2, o_3\}$.

However notice, in the symmetric case, that the expectation $(E|Z)(f(z^*))$ is precisely the center of the distribution of bliss points, namely

$$\frac{1}{3}(o_1 + o_2 + o_3).$$

See Figure 2 of the Appendix for an illustration of the divergence result.

In the non-symmetric case, it is intuitively clear that the expectation can be written as $\sum \lambda_i o_i$. The closer the bliss points are to colinearity the higher will be the coefficient of the median party. When the bliss points are far from colinear, then the coefficients will reflect the proximity of the parties. Thus, as expected, the further is the bliss point of a party from the "center of mass" of the bliss points, the less it will affect the lottery outcome. The effect of perquisites is captured by the following result.

**Theorem 4.** In Model 1, if private returns are non-zero and constant ($\sigma_{ij} = \sigma$ for all $i, j$), then for each $\{o_1, o_2, o_3\}$, which is $\epsilon-$bounded in symmetry, there exists a unique, stable Nash equilibrium $z^*$ which satisfies

$$\| z_i^* - z_j^* \| = b_{ij}(\epsilon, \sigma) \| o_i - o_j \| .$$

Here $b_{ij}(\epsilon, \sigma)$ decreases as $o$ increases (for each $\epsilon$). In particular, if the bliss points are symmetric ($\epsilon = 0$), then $b_{ij}(0, \sigma) = b(0, \sigma)$ for each pair, so that the Nash equilibrium will be symmetric. There is a bound $\sigma^* < \frac{4}{25} \| o_i - o_j \|^2$ such that $b(0, \sigma) < 1$ for all $\sigma > \sigma^*$.

See Figure 6 in the Appendix for an illustration of the convergence result.

Theorem 4 shows that if the bliss points are symmetric, and the private benefits are sufficiently high, then there exists a stable symmetric, convergent Nash equilibrium which is uniquely determined by the parameters $\{o_1, o_2, o_3, \Gamma\}$. Clearly $\{z_1^*, z_2^*, z_3^*\}$ all lie in the convex hull of the bliss points.

Again note that in the symmetric case the expectation $(E|Z)(f(z^*))$ is the center of the distribution of bliss points. Since the probability associated with each coalition is $\frac{1}{3}$, the expectation of private benefits of each party is $\frac{2}{3}\sigma$.

These four theorems all use the properties of the reaction functions, together with Fort's Theorem (1950), to assert that the Nash equilibrium mapping

$$\mathcal{E}_\Gamma \ : \ Z^N \ \to \ \tilde{Z}^N$$

defined by the scheme $\Gamma$ of private benefits, and which maps the bliss points to the Nash equilibria, is a continuous function on specific open domains in $Z^N$.

Because utilities are characterized by bliss points, the appropriate topology on $Z^N$ is simply the Euclidean topology. Say a property of a model is *generic* if it is true for an open dense set $\mathcal{O}$ of profiles in $Z^N$.

We have established that there exist two open domains

1. $X_1 \ = \ \{o \in Z^N : \ o$ is $\epsilon-$bounded in linearity$\}$
2. $X_2 \ = \ \{o \in Z^N : \ o$ is $\epsilon-$bounded in symmetry$\}$

such that $\mathcal{E}(o)$ is single-valued for $o \in X_1 \ \cup \ X_2$. Computer simulation indicates that in a domain $X_{ij}$ say, where the bliss points of $i$ and $j$ are significantly closer to each other than to $k$, then again there exists a unique stable Nash equilibrium such that $i$ and $j$ converge but $k$ diverges. In other words, the Nash equilibrium satisfies

$$\| \ z_i^* \ - \ z_j^* \ \| \ < \ \| \ o_i \ - \ o_j \ \|$$

but

$$\| \ z_k^* \ - \ z_i^* \ \| \ > \ \| \ o_k \ - \ o_i \ \|$$
$$\| \ z_k^* \ - \ z_j^* \ \| \ > \ \| \ o_k \ - \ o_j \ \| \ .$$

Even when the private benefits are zero, computer simulation indicates that the set of bliss points in $Z^N$ such that the best reaction correspondence will not be single-valued is a lower-dimensional stratum. It is consistent with our simulation that there exists an open dense set $\mathcal{O}$ of profiles such that the Nash equilibrium set $\mathcal{E}(o)$, for all $o \in \mathcal{O}$, consists only of *PSNE*.

The Appendix gives a reason for inferring, for a fixed profile of bliss points, that the degree of convergence in best responses increases as the private benefit, $\sigma$, increases. This implies that if $\sigma$ is sufficiently high then the best reaction correspondence is a contraction. This, in turn, implies that the *PSNE* will be unique.

We propose the following conjecture.

**Conjecture 1.** There generically exists a stable pure strategy Nash equilibrium in Model 1 (when private benefits are zero). If private benefits are sufficiently high, then generically there is a dense set $\mathcal{O} \in Z^N$ such that $\mathcal{E}_\Gamma$ is single-valued on $\mathcal{O}$ (that is for each $o \in \mathcal{O}^N$, there is a unique stable *PSNE* $\mathcal{E}_\Gamma(o)$).

The model for three parties can easily be extended to $n$ parties. For Model 2 we obtain a mixed strategy Nash equilibrium result as before, and suggest that Conjecture 1 is also valid for this model.

**Model 2.** Now let $N = \{1, \cdots, n\}$ and let $\mathcal{D}$ be a fixed family of decisive coalitions. Any coalition $M \in \mathcal{D}$ has a probability $\rho_M(z)$ of forming, which is inversely proportional to the variance of $\{z_i : i \in M\}$, and $\sum_{\mathcal{D}} \rho_M(z) = 1$. Every member of coalition $M$ receives a private benefit $\sigma > 0$ when $M$ forms. Preferences, as before, are represented by the profile $\{o_i : i = 1, \cdots, n\} \subset Z^N$ of bliss points. (Without loss of generality we suppose each $i$ in $N$ belongs to at least one coalition in $\mathcal{D}$.)

**Theorem 5.** In Model 2, for each $\sigma$ and profile $\{o_i : i \in N\}$, there exists a mixed strategy Nash equilibrium.

Note that Theorem 5 only asserts the existence of a mixed strategy Nash equilibrium. It is possible, when $n \geq 4$, that a party whose bliss point is centrally located has multiple best responses. However, it is also plausible that this can only occur when there is a strong degree of symmetry in the bliss points. The best reaction correspondences are determined by smooth induced utility functions. This further suggests that transversality arguments can be used to argue that the Nash equilibrium is generically a pure strategy equilibrium. This suggests that Conjecture 1 is also valid for the $n-$party case of Model 2.

Model 2 deals with the case where the family of decisive coalitions is fixed. We can readily extend the model by incorporating risk in the electoral response to the profile, $z$, of policy declarations.

**Model 3.** Let $\{\mathcal{D}_t : t = 1, \cdots, T\}$ be the family of all possible decisive structures on the set $N$. For each $z \in Z^N$ let $p_t(z)$ be the probability that the electoral response to the profile of declared positions results in the decisive structure $\mathcal{D}_t$. Suppose that this "electoral map"

$$p \ : \ Z^N \to \Delta_T$$

is smooth. (Note that $p_t(z)$ defines the degree of electoral risk, while $\{\rho_M(z) : M \in \mathcal{D}_t\}$ defines the degree of coalitional risk in the event that the coalitional structure $\mathcal{D}_t$ occurs.) Then for any profile, $z \in Z^N$, of declared positions, the induced utility function of party $i$ is smooth and has the form

$$U_i(z) \ = \ \sum_t p_t(z)\,[U_{it}(z, \sigma_t)]$$

where $U_{it}(z, \sigma_t)$ is the induced utility function (as in Model 2) defined by the decisive structure $\mathcal{D}_t$.

Since nothing fundamental has changed in this model there will again exist *MSNE*. We suggest that Conjecture 1 is also valid for Model 3. To provide some justification for this inference let us consider a very general form of Model 3.

**Model 4.** To generalize Model 3, assume each strategy space $Z$ is a compact subset of $\Re^w$. As in Model 3 we assume at profile $z \in Z^N$ that $U_i(z) = \sum_t p_t(z)[U_{it}(z, \sigma_{it})]$. Here $U_{it}(z, \sigma_{it})$ is the utility obtained by party $i$ in the event of decisive structure $\mathcal{D}_t$ occurring: that is $U_{it}(z, \sigma_{it})$ is the Von-Neumann utility obtained by $i$ from the lottery across coalition events $\{\rho_M(z), z_M, \sigma_{iM} : M \in \mathcal{D}_t\}$. In this case $\rho_M(z)$, $z_M$ are arbitrary smooth functions of $z$, and $\sigma_{iM}$ is the private benefit obtained by $i$ in coalition $M$. Assume that utility for $i$ is additively separable and determined on $Z$ by a smooth function $u_i : Z \to \Re$. Fix the profile $u = (u_1, \ldots, u_n)$, and the private benefit scheme $\Gamma = \{\Gamma_t : t = 1, \ldots, \Gamma\}$. (Note that the scheme $\Gamma_t$ may be taken to be smoothly dependent on the electoral return $e(z)$ and thus on $z$.) Then for fixed $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) = (z_{-i})$, let

$$R_\Gamma(u_i)(z_{-i}) \ = \ \{z_i \in Z : dU_i(z_i, z_{-i}) = 0\}.$$

It is intuitively obvious that the set of points

$$Crit(u_i) \ = \ \{(z_i, z_{-i}) \ \in \ Z^N \ : \ z_i \ \in \ R_\Gamma(u_i)(z_{-i})\}$$

will generally be a smooth geometric object of dimension $(n-1)w$. If we put the Whitney topology on the class of profiles, now labelled $U(Z)^N$, then the Thom Transversality Theorem (Golubitsky and Guillemin, 1973 and Hirsch, 1976) shows that there exists an open dense set $\mathcal{O} \subset U(Z)^N$ such that for each profile $u = (u_1, \ldots, u_n) \in \mathcal{O}$, the set $Crit(u_i)$, of critical points, is a smooth submanifold of $Z^N$ of dimension $(n-1)w$. Moreover, for any profile $u \in \mathcal{O}$, the submanifolds $Crit(u_i)$, $Crit(u_j)$, for $i \neq j$, intersect transversally. Thus

$$dim\ [Crit(u_i) \ \cap \ Crit(u_j)] \ = \ (n-2)w.$$

where this intersection is also a smooth submanifold of $Z^N$. By induction the set

$$\Sigma_\Gamma(u) \ = \ \cap_{i=1}^n \ Crit(u_i)$$

is a smooth submanifold of dimension zero, that is a set of isolated points.

Note that the map $\Sigma_\Gamma \ : \ U(Z)^N \ \to \ Z^N$ gives information on the Nash equilibrium correspondence $\mathcal{E}_\Gamma \ : \ U(Z)^N \ \to \ (\tilde{Z})^N$, since for each $u$, a point $z \in \Sigma_\Gamma(u)$ satisfies the first order optimality conditions. Moreover if $\Sigma_\Gamma(u)$ is single-valued at some profile $u$, then there exists a neighborhood $V$ of $u$ in $U(Z)^N$ such that $\Sigma_\Gamma$ is single-valued and continuous on $V$. This suggests that $\mathcal{E}_\Gamma$ is also single-valued and continuous on $V$.

Our second conjecture sums up these inferences on the structure of the Nash equilibrium map.

**Conjecture 2.** In Model 4, given a private benefit scheme, $\Gamma$, the stable Nash equilibrium map $\mathcal{E}_\Gamma \ : \ U(Z)^N \ \to \ (\tilde{Z})^N$ has the following form: there exists a family of disjoint open subsets $\{X_s \ : \ s = 1, \ldots, S\}$ in $U(Z)^N$ such that the number of stable pure strategy Nash equilibria is constant and equal to $s$ on $X_s$. For all $u \in X_1$, there exists a unique pure strategy stable Nash equilibrium. If $u \notin \cup_j X_j$, then $\mathcal{E}_\Gamma(u)$ is a mixed strategy equilibrium.

The simulation exercises that we have examined strongly suggest that when the electoral response is generated by a symmetric electoral distribution and when $\rho_M(z)$ and $z_M$ have structural properties such as in Model 3, then for a sufficiently high scheme of benefits, $\Gamma$, the set $X_1$ is open dense.

# 3 CONCLUSION

The model analyzed here has taken a specific, and fairly natural structural form for the coalition probabilities and outcomes. With this assumption, the three party case exhibits intuitively expected properties, namely convergence to a median in the degenerate one-dimensional case. More generally, when two dimensions are involved, then divergence of party declarations occurs. However, the assumption of risk aversion in utility, together with uncertainty over coalition outcomes or risk over electoral responses would reduce the degree of divergence in equilibrium declarations. Moreover, if private benefits of the same order of magnitude as policy payoffs are introduced into the model, then a weak form of convergence of party declarations is observed. Only if private benefits completely dominate policy benefits would Downsian convergence to a single policy point generally occur. The model appears to be robust in the structural assumptions that are utilized. It is conjectured that identical properties will be observed in a general model of $n-$party competition incorporating similar structural assumptions. Now that a general form of the pre-election declaration game has been modelled, it should prove possible to analyze in more detail the question of post-election committment to declared policies. Note in particular that the model focusses on Nash equilibrium rather than strong (coalitional) Nash equilibrium. However if coalitional contracts are non-binding, then the nature of the equilibrium should induce a degree of commitment to the declared policies (see Schofield 1993a for some discussion on the degree of commitment). Since the model attempts to generalize both the two-party model under electoral risk (Cox, 1984) and the multiparty models of Eaton-Lipsey and Baron, it may provide the basis for comparison of two party political systems with multiparty, coalition systems.

# 4 APPENDIX: Analysis of the Model and Proof of the Theorems.

To solve the response problem for 3 in Model 1, choose coordinates such that $z_1 = (0, \frac{r_1}{2}), z_2 = (0, \frac{-r_2}{2}), z_3 = (x, y)$, and let party 3 have a Euclidean utility function $u_3(x, y) = -\frac{1}{2}[(x-L)^2 + y^2]$. That is party 3 has a bliss point, $o_3$, at $(L, 0)$. Now $\| z_1 - z_2 \| = \frac{1}{2}(r_1 + r_2)$. Define $s_i = \| z_3 - z_i \|$ where $z_3 \in Z$, as the strategy of party 3. For $i = 1, 2$, let $\rho_i = \rho_{i3}(z)$ be the probability that coalition $\{i3\}$ forms and define $\rho_3 = 1 - \rho_1 - \rho_2$. We also assume that if coalition $\{i3\}$ forms, then party 3 receives a private reward of $\sigma_{3i} \geq 0$. Thus given $(z_1, z_2, z_3)$, the outcome for 3 is a lottery:

1. policy $\frac{z_1+z_3}{2} = \left(\frac{x}{2}, \frac{y}{2} + \frac{r_1}{4}\right)$ and bonus $\sigma_{31}$, with probability $\rho_1$
2. policy $\frac{z_2+z_3}{2} = \left(\frac{x}{2}, \frac{y}{2} - \frac{r_1}{4}\right)$ and bonus $\sigma_{32}$ with probability $\rho_2$, and
3. policy $\frac{z_1+z_2}{2} = \left(0, \frac{r_1-r_2}{4}\right)$ and no bonus with probability $1 - \rho_1 - \rho_2$.

Note we do not consider the formation of the grand coalition $\{1, 2, 3\}$. Write $U(x, y)$ for $U_3(z_1, z_2, z_3)$ and $u$ for $u_3$. Then the response problem for 3 is to maximize

$$U(x,y) = \rho_1 \left\{ u\left(\frac{z_1+z_3}{2}\right) + \sigma_{31} \right\} + \rho_2 \left\{ u\left(\frac{z_2+z_3}{2}\right) + \sigma_{32} \right\} + (1-\rho_1-\rho_2)\left\{ u\left(\frac{z_1+z_2}{2}\right)\right\}.$$

Note of course that $\rho_i$ are both functions of $s_1, s_2$ and $\| z_1 - z_2 \|$. By assumption

$$\rho_i = \left(\frac{1}{s_i^2} + \frac{1}{s_j^2} + \left(\frac{2}{r_1+r_2}\right)^2\right)^{-1} \frac{1}{s_i^2}.$$

Note that $\frac{\partial \rho_i}{\partial s_i} = \frac{-2}{s_i}\left(\rho_i - \rho_i^2\right)$ and $\frac{\partial \rho_i}{\partial s_j} = \frac{2\rho_i\rho_j}{s_j}$. Since $s_1^2 = x^2 + \left(\frac{r_1}{2} - y\right)^2$ and $s_2^2 = x^2 + (\frac{r_2}{2} + y)^2$ we find that $\frac{\partial s_i}{\partial x} = \frac{x}{s_i}$, $\frac{\partial s_i}{\partial y} = \left(y + (-1)^i(\frac{r_i}{2})\right)\frac{1}{s_i}$. Then

$$d\rho = \begin{pmatrix} \frac{\partial s_1}{\partial x} & \frac{\partial s_2}{\partial x} \\ \frac{\partial s_1}{\partial y} & \frac{\partial s_2}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial \rho_1}{\partial s_1} & \frac{\partial \rho_2}{\partial s_1} \\ \frac{\partial \rho_1}{\partial s_2} & \frac{\partial \rho_2}{\partial s_2} \end{pmatrix} = 2\begin{pmatrix} x & x \\ y - \frac{r_1}{2} & y + \frac{r_2}{2} \end{pmatrix} \begin{pmatrix} \frac{-\rho_1(1-\rho_1)}{s_1^2} & \frac{\rho_1\rho_2}{s_1^2} \\ \frac{\rho_1\rho_2}{s_2^2} & \frac{-\rho_2(1-\rho_2)}{s_2^2} \end{pmatrix}. (Eq.1)$$

Differentiating Eq. 1 gives:

$$\begin{pmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\rho_1+\rho_2)(L - \frac{x}{2}) \\ \frac{-\rho_1}{2}(\frac{y}{2} + \frac{r_1}{4}) + \frac{-\rho_2}{2}(\frac{y}{2} - \frac{r_2}{4}) \end{pmatrix} + (d\rho)\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}. \qquad (Eq.2.)$$

Here

$$\delta_1 = \sigma_{31} + u\left(\frac{x}{2}, \frac{y}{2} + \frac{r_1}{4}\right) - u\left(0, \frac{r_1 - r_2}{4}\right)$$

$$\delta_2 = \sigma_{32} + u\left(\frac{x}{2}, \frac{y}{2} - \frac{r_2}{4}\right) - u\left(0, \frac{r_1 - r_2}{4}\right).$$

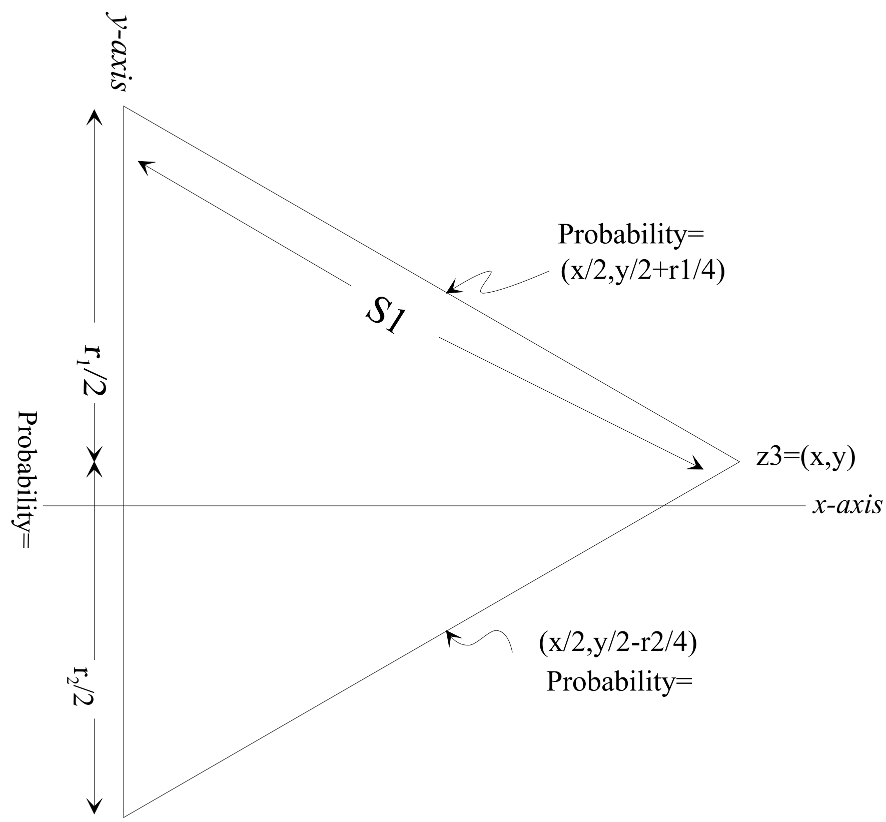


Figure 1: Lottery across three outcomes at profile $(z_1, z_2, z_3)$.

The first term on the right hand side of Eq. 2 involves the differential $(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = \frac{1}{2}(L - \frac{x}{2}, -(\frac{y}{2} + \frac{r_1}{4}))$, *etc.*

It is relatively easy to solve Eq. 2 in the symmetric case when $\sigma = \sigma_{31} = \sigma_{32}$ and $r_1 = r_2$ (so, $s_1 = s_2$). To consider the symmetric case ($A$), where the equation in $x$ dominates, assume that $L >> 0$ and $x \simeq L$. Then the equation $\frac{\partial U}{\partial y} = 0$ immediately gives $y^* = 0$.

Substituting in the equation $\frac{\partial U}{\partial x} = 0$ then gives $\rho(L - \frac{x}{2}) - \frac{2x\rho(1-2\rho)\delta}{s^2} = 0$, where $\delta = \delta_1 = \delta_2$.

We thus obtain

$$\rho\left\{\left(L - \frac{x}{2}\right) - \frac{2x\rho^2}{r^2}\left(\sigma + Lx - \frac{x^2}{4} - \frac{r^2}{16}\right)\right\} = 0. \qquad (Eq.3)$$

Assume, until specified below, that $\sigma = 0$.

To solve this equation, consider the case $L = \beta r, x = \alpha r$. We obtain

$$\rho\left\{\left(\beta - \frac{\alpha}{2}\right) - 2\alpha\rho\left(\alpha\beta - \frac{\alpha^2}{4} - \frac{1}{16}\right)\right\} = 0. \qquad (Eq.4)$$

Now $\rho = \frac{r^2}{2r^2+s^2} = \frac{1}{\frac{9}{4}+\alpha^2}$ since $s^2 = (\frac{r}{2})^2 + x^2$. Assuming $\rho \neq 0$, gives a quadratic expression in $\alpha, \beta$ with solution

$$\alpha = \frac{1}{2}\left(-\frac{1}{\beta} \pm \sqrt{\frac{1}{\beta^2} + 9}\right). \qquad (Eq.5)$$

Thus if $L \gg r$, so that $\beta \gg 1$, then $x^* = \frac{3r}{2}$. Notice that there is only one positive solution, so that $U(x, 0)$ is concave on the positive $x-$axis.

Examination of $\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right)$ in a neighborhood $(x^*, 0)$ shows that $(x^*, 0)$ maximizes $U$. (This is because the determinant of $d\rho$ is a function of $\rho_1\rho_2(1-\rho_1-\rho_2) > 0$ and has negative diagonal terms.) The negative solution for $x$ in Eq. 5 corresponds to a minimum. Note also that as $x \to \pm\infty$, then $\rho \to 0$ and the outcome approaches $(0, 0)$.

Clearly, if $r_1 = r_2 \neq 0$, the (maximum) solution to Eq. 2 is unique and gives a global maximum for $U$, for each fixed $z_1, z_2$. Moreover, the solution to Eq. 2 is continuous in $(z_1, z_2)$. If the bliss points satisfy the symmetry condition $\| o_1 - o_2 \| = \| o_2 - o_3 \| = \| o_1 - o_3 \|$, then the conditions of the Glicksburg theorem are satisfied and a stable, pure-strategy Nash equilibrium exists.

Note however that if $x \simeq \frac{3r}{2}$, then $s^2 = \left(\frac{r}{4}\right)^2 + x^2 > r^2$. Thus best response produces a divergence of the position of party 3 from the positions of parties 1 and 2. To examine the symmetric case further, suppose that $\beta \simeq 1$. It is easy to show that if $\beta = \frac{\sqrt{5}}{2}$, then $\alpha = \beta$. In particular, if $\beta \in \left(\frac{\sqrt{5}}{2}, \infty\right)$, then $\alpha \in \left(\frac{\sqrt{5}}{2}, \frac{3}{2}\right)$ with $\alpha < \beta$, while if $\beta \in \left(0, \frac{\sqrt{5}}{2}\right)$ then $\alpha > \beta$, but $\alpha \in \left(0, \frac{\sqrt{5}}{2}\right)$. If $\beta = \frac{1}{\sqrt{3}}$, then $\alpha = \frac{\sqrt{3}}{2}$, and $s^2 = r^2$.

Thus a *symmetric* Nash equilibrium $\| z_1^* - z_2^* \| = \| z_1^* - z_3^* \| = \| z_2^* - z_3^* \|$ can occur. The equilibrium is related to the bliss points $\{o_1, o_2, o_3\}$ by the condition

$$\frac{\| z_i^* - z_j^* \|}{\| o_i - o_j \|} = 2$$

for each pair $i, j$. See Figure 2.

In the non-symmetric case with $r_1 \neq r_2$, consider first a perturbation of the above situation with $r_1 \simeq r_2$, and $\rho_1 \simeq \rho_2$. In Eq. 2, note that $\delta_1 \simeq \delta_2 \simeq \delta$.

Then Eq. 3 becomes

$$\frac{\rho_1 + \rho_2}{2}\left(L - \frac{x}{2}\right) - 2x\left(\frac{\rho_1^2 + \rho_2^2}{\overline{r}^2}\right)\left(Lx - \frac{x^2}{4} - \xi\right) = 0 \ (Eq.6)$$

where $\overline{r} = \frac{r_1 + r_2}{2}$ and $\xi \simeq \frac{r^2}{16}$.

Equation $\frac{\partial U}{\partial y} = 0$ gives

$$y(\rho_1 + \rho_2) = -\frac{1}{2}(\rho_1 r_1 - \rho_2 r_2) - 8y\left(\frac{\rho_1^2}{\overline{r}^2} + \frac{\rho_2^2}{\overline{r}^2}\right)\delta + 4\left(\frac{r_1\rho_1^2}{\overline{r}^2} - \frac{r_2\rho_2^2}{\overline{r}^2}\right)\delta \ (Eq.7).$$

Clearly, for $r_1 > r_2$, the optimal choice $y^* < 0$, since $\rho_1, \rho_2$ are dominated by the choice in $x$. Thus $y^*$ is essentially a function of $r_1 - r_2$. Hence for $r_1 \simeq r_2$, $y^* \simeq 0$. For this case let us say that the $x-$solution dominates the $y-$solution.

We have shown above that if $r_1 = r_2$ then Eq. 3 is a quadratic expression in $x$, with one positive root. For general $r_1, r_2$, Eq. 6 involves higher powers of $x$. Nonetheless the quadratic terms dominate and the critical point in the domain $x > 0$ always corresponds to a global maximum of $U$. Figures 3 to 5 explore the behavior of $U(x, y)$ for various values of $r_1, r_2$. Figure 3 shows a symmetric case where $o_1 = (0, 5), o_2 = (0, -5)$ and $o_3 = (5\sqrt{3}, 0)$. If we assume that initially parties 1 and 2 choose $o_1, o_2$ respectively, then $U$ has a maximum at $(x^*, 0) = (10.3, 0)$ and a minimum at a point $(x^{**}, 0)$ for $x^{**} < 0$. The Nash equilibrium can be calculated

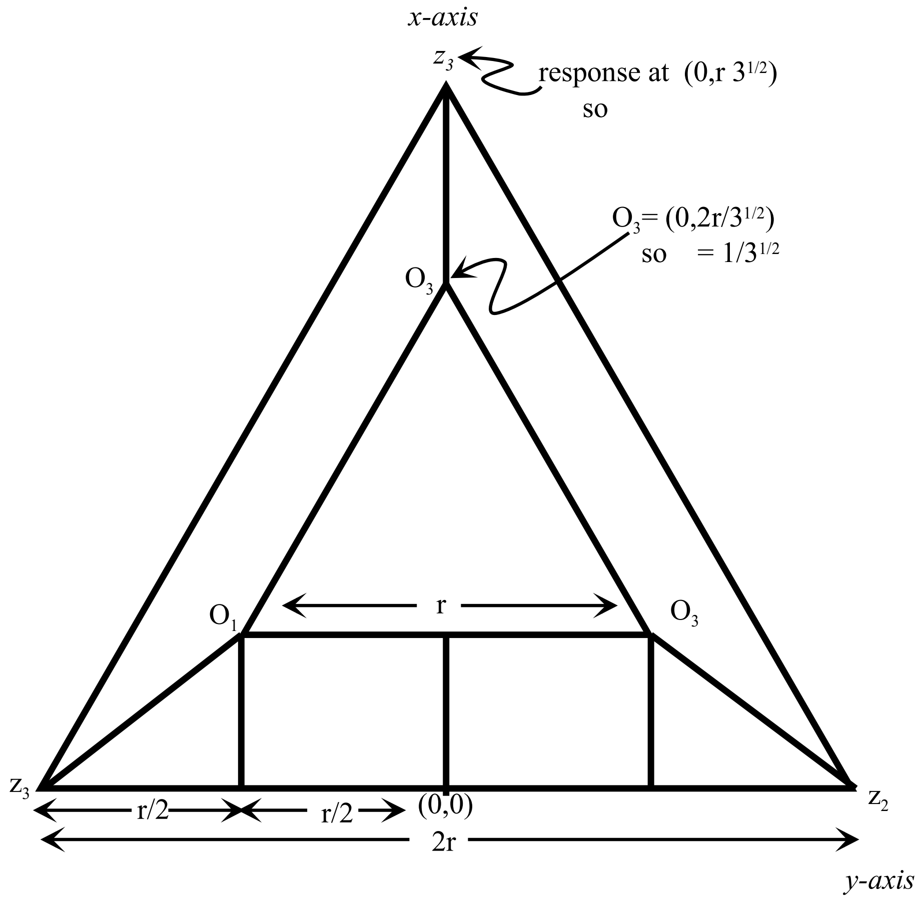


Figure 2: Divergent Nash Equilibria with no private benefits.

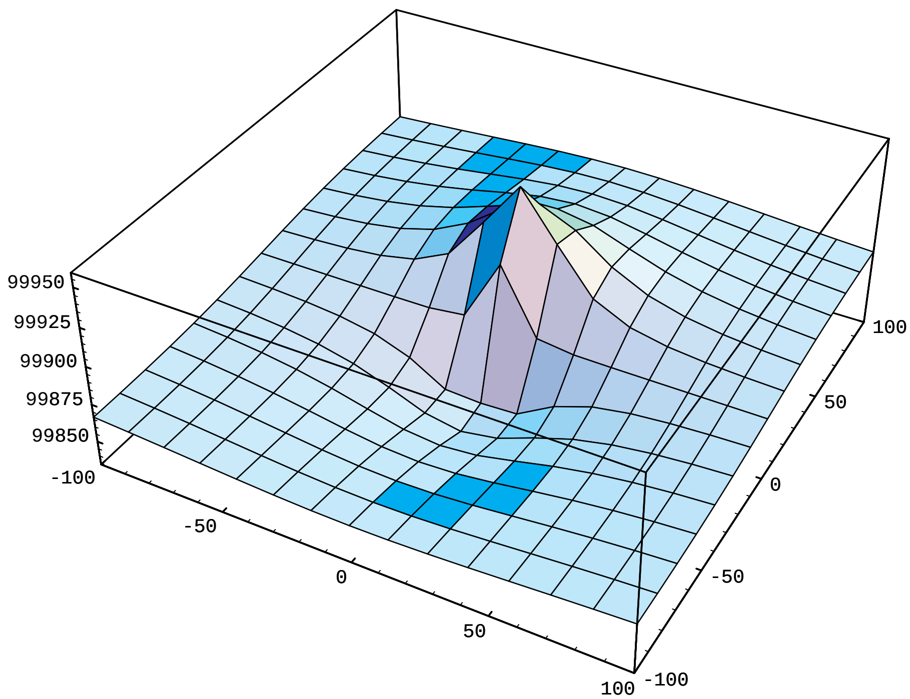


Figure 3: Induced Utility Function, M.3, where $O_3 = (5\sqrt{3}, 0)$, $z_1 = (0, 5), z_2 = (0, -5)$, and zero private benefits.

to be $z_1^* = (-2.866, 10)$,
$z_2^* = (-2.866, -10)$, and $z_3^* = (14.433, 0)$, so $\| z_i^* - z_j^* \| = 20$ for each pair, $i, j$.

Figure 4 shows an asymmetric case with $o_1 = (0, 5)$, $o_2 = (0, -20)$ and $o_3 = (5\sqrt{3}, 0)$. Assuming again that parties 1 and 2 choose $o_1, o_2$, then $U$ has a local minimum at a point $(x^{**}, y^{**})$ with $x^{**} < 0$ and $y^{**} < 0$. The global maximum is at a point $(x^*, y^*)$ with $y^* \simeq 0$ and $x^* > 10.3$.

Finally Figure 5 illustrates the case with $o_1 = (0, 4)$, $o_2 = (0, 5)$ and $o_3 = (5\sqrt{3}, 0)$. Again there is a global minimum at a point $x^{**} < 0$, $y^{**} > 0$, and a global maximum at $(x^*, y^*)$, with $y^* \simeq 0$ and $x^* > 10.3$.

To deal with the situation where the bliss points are non-symmetric, consider the colinear case (B) when the third party has bliss point $(0, 0)$ and the two other parties are positioned at $z_1 = (0, \frac{r_1}{2})$, $z_2 = (0, \frac{-r_2}{2})$ as before. Let us first consider the response by 3 on the $y$-axis.

As a first order approximation suppose $r_1 \simeq r_2$, with $r_1 > r_2$. Then from Equation 7, at $(x, y) = (0, 0)$ we obtain

$$\frac{dU}{dy} \simeq \frac{1}{8}(\rho_1 r_1 - \rho_2 r_2) \simeq -\frac{1}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) > 0.$$

Note here that $\rho_i = \frac{4}{r_i^2}$.

Thus the best response is at a point $y^* > 0$. In fact it can be shown that $y^* \simeq \frac{r_1 - r_2}{12}$, so the optimal response is to partially equalize the distances to $z_1$ and $z_2$. We can also determine the optimal response by 3 when both $z_1$ and $z_2$ lie on the same side of the origin. To illustrate, suppose $r_1 = 1, 0; r_2 = -0.8$. Then it is easy to show that $(\rho_1, \rho_2, \rho_3) = (.04, .06, .9)$.

For $y = 0$, Equation 7 can be rewritten as Equation 8:

$$\frac{\partial U}{\partial y} = -\frac{1}{8}(\rho_1 r_1 - \rho_2 r_2) + \frac{\rho_1 r_1}{s_1^2}((1 - \rho_1)\delta_1 - \rho_2 \delta_2) + \frac{\rho_2 r_2}{s_2^2}(\rho_1 \delta_1 - \delta_2(1 - \rho_2))$$

In this case $\delta_i$ can be regarded as the utility gain from having an outcome at $\frac{1}{2}(z_3 + z_1)$ rather than $\frac{1}{2}(z_1 + z_2)$. Easy computation shows that $\delta_1 \simeq 0.07$, $\delta_2 \simeq .08$.

Thus at $y = 0$,

$$\frac{dU}{dy} = -\frac{1}{8}((.04) + (.06)(.8)) + .027 = .016 > 0.$$

Thus the optimal response is to move to a position $y^*$ closer to both $z_1$ and $z_2$.

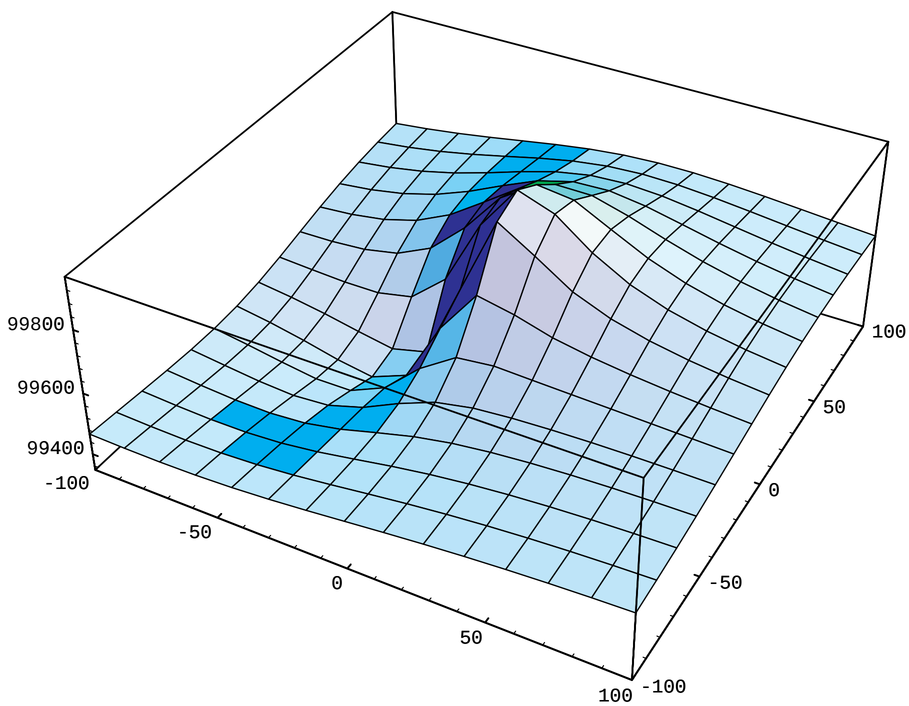


Figure 4: Induced Utility Function, M.3, where $O_3 = (5\sqrt{3}, 0)$, $z_1 = (0, 5), z_2 = (0, -20)$, and zero private benefits.

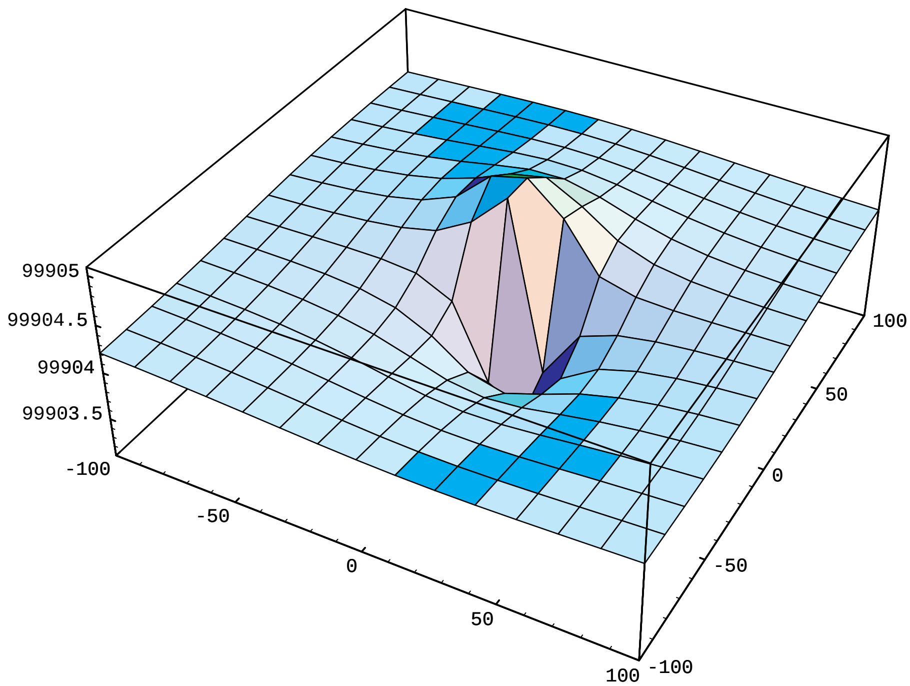


Figure 5: Induced Utility Function, M.3, where $O_3 = (5\sqrt{3}, 0)$, $z_1 = (0, 4), z_2 = (0, 5)$, and zero private benefits.

To determine the $x-$coordinate behavior in this case, if we substitute $L = 0$ in Equation 6, the maximum solution is $x^* = 0$. We can also determine the optimal behavior if the bliss points are close to colinear. Suppose $r = r_1 = r_2$ and $L << r$. Then from Equation 4 we can see that $x^* > L$ but $x^* \simeq L$. For the case $L \simeq 0$, the previous analysis for $y^*$ is still valid. In this case we can say that the $y-$solution dominates the $x$ solution. If the bliss points of the parties are close to colinear, then the best response of each party gives a unique outcome. The three best responses will also be close to colinear and will give convergent strategies.

**Preliminaries to a Proof of the Theorems.** Proof of existence of a *PSNE* in any spatial model must deal with problems of non-convexity and failure of continuity (Dasgupta and Maskin, 1986).

A failure in continuity of the probability functions $\rho_{ij}$ (the probability associated with coalition $\{i,j\}$) can occur in the following way. Suppose that $z_1 = z_2$ and $z_3 \neq z_1$. Then it is natural to suppose that $\rho_{12} = 1$, and $\rho_{13} = \rho_{23} = 0$. However if $z_1 = z_2 = z_3$, then $\rho_{ij} = \frac{1}{3}$ for each $i,j$. Note that any such profile $(z_1, z_2, z_3)$ is a Nash equilibrium, since it is the best response for each party. This can be seen by noting that $\rho_{13} = \rho_{23} = 0$ implies that the appropriate optimality equations for party 3 are identically zero.

Note however that this Nash equilibrium is unstable since in any neighborhood there exists a profile $(z_1', z_2', z_3')$ which is not in equilibrium. We modify Model 1 by smoothing the probabilities in a neighborhood of such a point, by bounding the probabilities. That is, choose $\epsilon > 0$, small, and redefine $\rho_{ij}$ to be proportional to $\frac{1}{\epsilon^2}$ for all $z_i, z_j$ satisfying $\| z_i - z_j \| \leq \epsilon$. If $z_k$ also lies within the $\frac{\epsilon}{2}$-ball of $\frac{1}{2}(z_i + z_j)$, then redefine $\rho_{ij} = \rho_{ik} = \rho_{jk} = \frac{1}{3}$. Smooth $\rho_{ik}$ and $\rho_{jk}$ over the annulus.

$$\{z_k : \frac{\epsilon}{2} < \| z_k - \frac{1}{2}(z_i + z_j) \| < \epsilon\}.$$

With this redefinition, a point $(z, z, z)$ which is an unstable Nash equilibrium under the original definition, will no longer be a Nash equilibrium.

The second problem concerns the quasi-concavity of the induced utility functions. As we have noted in the previous examples, the induced utility functions are not quasi-concave. However as Figures 3, 4 and 5 illustrate, for each $o_3$ and for each $z_1, z_2$, the induced utility function $U_3$ is single-peaked in $z_3$. To see why $U_3$ is single-peaked in general, note that the optimality equations $\frac{\partial U}{\partial x} = \frac{dU}{\partial y} = 0$ are fundamentally quadratic equations, but involving higher order terms in $\rho_{13}^2$ and $\rho_{23}^2$. If the scheme $\Gamma = \{\sigma_{ij}\}$ involves benefits that are sufficiently large, then the coefficients of the higher order terms will be small, and each optimality equation will have only two roots, corresponding to a maximum and minimum. If we parametrise the joint best reaction correspondence, $R_\Gamma$, by $o = (o_1, o_2, o_3) \in Z^N$ when the scheme, $\Gamma$, of private benefits is sufficiently large, then

$$R_\Gamma(o) : Z^N \to Z^N$$

is a continous *function*, which will exhibit a fixed point.

**Proof of Theorem 1.** For a general scheme $\Gamma$ it is possible that some best reaction correspondences will not be single-valued. Convexification of the best reaction corresponence, and application of the Glicksburg Theorem gives existence of an *MSNE*.

However the previous argument implies that there exists $\sigma^*$ such that when $\sigma_{ij} > \sigma^*$, then each individual best response will be single-valued. Examination of $R_\Gamma(o) : Z^N \to Z^N$ shows that it is a continuous function. Thus there exists a fixed point, giving a *PSNE*. Moreover any such *PSNE*, $z^*$, simultaneously satisfies the optimality equations for each party, and is therefore stable. *Q.E.D.*

**Proof of Theorem 2.** Above we considered the extreme case of $\Gamma$ identically zero with colinear bliss points, and showed that best response by each player required convergence to a point on the bliss point axis. The resulting fixed point, $\mathcal{E}(o)$, of $R(o) : Z^N \to Z^N$ is unique and corresponds to a stable Nash equilibrium. Now $R(o)$ is a contraction on $Z^N$, in the sense that $\| R(o)(z_1) - R(o)(z_2) \| < \| z_1 - z_2 \|$ for any $z_1, z_2 \in Z^N$. Moroever there is a neighborhood $V$ of $o$ in $Z^N$ such that $R(o')$ is single-valued, as well as continuous on $V$. Thus there exists a neighborhood $V$ of $o$ in $Z^N$ such that for all $o' \in V, \mathcal{E}(o')$ is a unique stable Nash equilibrium. By Fort's Theorem (1950) there exists a neighborhood $V_1$ of $o$ in $Z^N$, with $V_1 \subset V$, and a neighborhood $V_2$ of $\mathcal{E}(o)$ in $Z^N$ such that $\mathcal{E}(o') \in V_2$ for all $o' \in V_1$.

Since "$o$" comprises colinear points, there exists $\epsilon^* > 0$ such that $o' \in V_1$ implies $o'$ is $\epsilon-$bounded in linearity for some $\epsilon < \epsilon^*$. Moreover since $\mathcal{E}(o)$ is convergent, $\mathcal{E}(o') \in V_2$ implies that $\mathcal{E}(o')$ is also $\epsilon-$bounded in linearity. *Q.E.D.*

**Proof of Theorem 3.** Suppose now that the profile of bliss points, $o$ is symmetric. Setting $\sigma = 0$ in Equation 3 shows that for the best response there is a unique scale relationship between the bliss points and the Nash equilibrium. As we have shown $\| z_i^* - z_j^* \| = 2 \| o_i - o_j \|$, for each pair $i, j$. Just as in the proof of Theorem 2, there exist neighborhoods $V_1$ of $o$ and $V_2$ of $\mathcal{E}(o)$ such that the Nash equilibrium correspondence is a function

$$\mathcal{E} \ : \ V_1 \ \rightarrow \ V_2.$$

Again there exists $\epsilon^* \ > \ 0$ such that any profile which is $\epsilon$-bounded in symmetry, for $\epsilon \ < \ \epsilon^*$, lies in $V_1$. Thus $\mathcal{E}(o') \ = \ (z_1^*, z_2^*, z_3^*)$ will satisfy

$$\| \ z_i^* \ - \ z_j^* \ \| = \ b_{i,j}(\epsilon) \ \| \ o_i' \ - \ o_j' \ \|$$

for $b_{i,j}(\epsilon)$ close to 2.

**Proof of Theorem 4.** Consider Equation (3) in the case $r_1 \ = \ r_2$, and $\sigma \neq 0$. If $\sigma \ = \ r^2$, instead of Equation 4 we obtain

$$\left(\beta - \frac{\alpha}{2}\right) - 2\alpha\rho \left(1 + \alpha\beta - \frac{\alpha^2}{4} - \frac{1}{16}\right) \ = \ 0,$$

with solution

$$\alpha \ = \ \frac{1}{2}\left(\frac{-3}{\beta} \pm 3\sqrt{\frac{1}{\beta^2} + 1}\right).$$

Again, there is only one positive root, so that if $\beta = \sqrt{3}$, then $\alpha \ = \ \frac{\sqrt{3}}{2}$. This implies that a symmetric Nash equilibrium exists, but satisfying

$$\frac{\| \ z_i^* - z_j^* \ \|}{\| \ o_i - o_j \ \|} \ < \ 1$$

for each pair $i, j$. Thus the equilibrium is convergent.

The computation presented in Figure 6 shows that the equilibrium satisfies the equation $\| \ z_i^* - z_j^* \ \|^2 = \ \frac{4}{25} \ \| \ o_i - o_j \ \|^2$ . It is evident that a unique symmetric equilibrium occurs if $\sigma \ = \ r^2 \ = \ \frac{4}{25} \ \| \ o_i - o_j \ \|^2$ . Thus there is some bound, $\sigma^* \ < \ \frac{4}{25} \ \| \ o_i \ - \ o_j \ \|^2$, such that there exists a unique symmetric convergent equilibrium, $z^*$, satisfying $\| \ z_i^* - z_j^* \ \| < \| \ o_i \ - \ o_j \ \|$, whenever $\sigma \ > \ \sigma^*$. A continuity argument, together with Fort's Theorem completes the proof. *Q.E.D.*

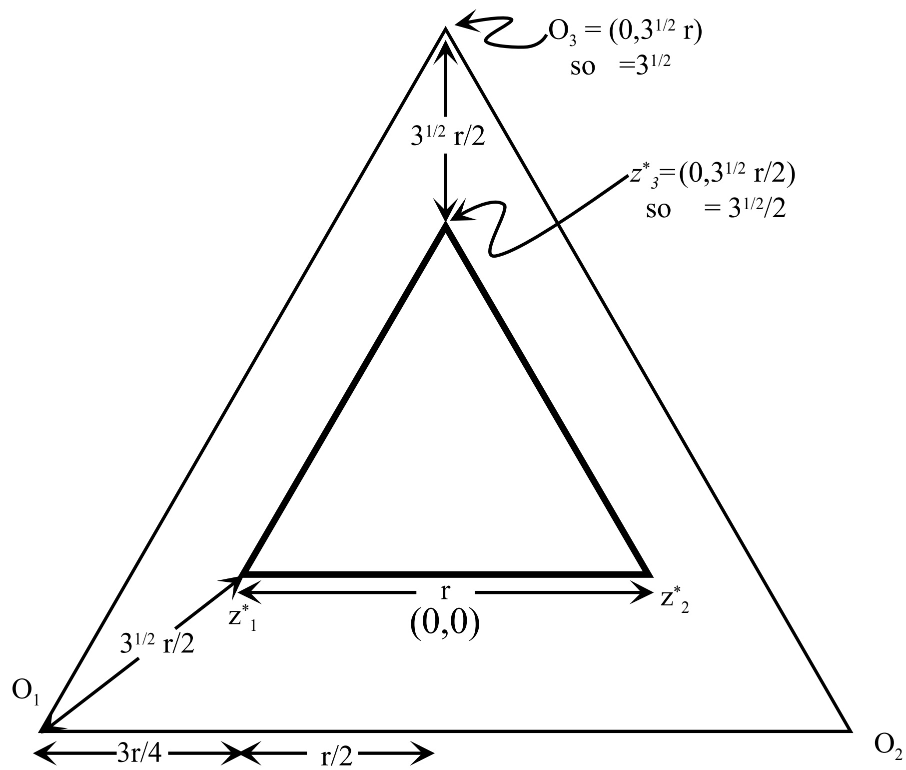


Figure 6: Convergent Nash equilibrium in the symetric case, $\sigma = r^2$.

**Proof of Theorem 5.** We smooth out the probabilities at proximate locations, as in the proof of Theorem 1. Now consider the response function of player *i*. This is given by maximizing the function

$$U_i(z:\sigma) \;=\; \sum_{i\in M, M\in\mathcal{D}} \rho_M\left(u_i(z_M) \;+\; \sigma\right) \;+\; \sum_{i\in M, M\notin\mathcal{D}} \rho_M\left(u_i(z_M)\right).$$

The response correspondence $R_\sigma(U_i) : \Pi_{j\neq i} Z_j \to Z_i$ given by $R_\sigma(U_i)(z_{-i}) \;=\; \{argmax_{z_i}\; U_i(z:\sigma)\}$ will be upper hemi-continuous, but in general need not be single-valued, and hence not convex-valued. However, convexifying $R(U_i)$ will give a convex-valued and upper hemi-continuous response correspondence. As for Theorem 1, a standard fixed point argument on the joint response correspondence

$$\mathcal{R}_\sigma(U) \;:\; \left(\tilde{Z}\right)^N \;\to\; \left(\tilde{Z}\right)^N.$$

gives a mixed strategy Nash equilibrium. *Q.E.D.*

We suggest in Conjecture 1 that $R(U_i)$ will generically be single-valued and thus that the Nash equilibrium is generically a pure strategy equilibrium.

If we consider, in Model 2, two private benefit schemes $\sigma^1, \sigma^2$ satisfying $\sigma^1 > \sigma^2$, then the response functions (when single-valued) will converge, in the sense that

$$\|\; R_{\sigma^1}(U_i)\,(z_i) \;-\; \frac{1}{n-1}\sum_{j\neq i} z_j \;\| < \|\; R(U_i)_{\sigma^2}(z_i) \;-\; \frac{1}{n-1}\sum_{j\neq i} z_j \;\|$$

Thus as $\sigma$ increases, the Nash equilibrium will converge to the center, $\frac{1}{n}\sum_i o_i$, of the distribution of bliss points.

**Footnotes.**

1. In Baron's model the compromise positions and probabilities are determined by an equilibrium process based on initial and exogeneous probabilities that each party will be a "formateur". In general $z_{ij}$ is linear in $\{z_i, z_j\}$ and $\rho_M$ is approximately inversely proportional to $\| z_i - z_j \|^2$ .

2. As in Cox's two party model, we implicitly assume each party has a preferred policy point but conceals this point from the electorate because of the electoral consequences. Implicitly we assume that the electorate believes each party, $i$, will attempt to implement the declared policy $z_i$. Call this assumption *credible commitment.*

3. Because of the constraint that the three coalition probabilities sum to one, the "constant of proportionality" is dependent on the distances between the manifestos. The appendix gives the precise form of each $\rho_M$. To deal with situations where $z_i = z_j$, a smoothing operation is performed near such discontinuities. The analysis of the appendix makes it clear that this operation does not affect the results.

4. Note that the assumptions on coalition outcomes and probabilities are structurally very similar to those of Baron's (1991) model. These simplifying assumptions are made so as to be able to differentiate a specific structural form of the model. An analytical form can generally not be obtained in Baron's model.

5. Note that we concentrate on Nash equilibrium rather than strong (or coalitional) Nash equilibrium. We implictly assume that binding contracts between the parties before the election are impossible.

6. Note that if $z^* = (z_1^*, z_2^*, z_3^*)$ is a stable *PSNE* then no party has any motivation to deviate from its *PSNE*. Consequenlty the credible commitment assumption (footnote 2) is justified.

7. Although the analysis is performed in $\Re^2$, it is evident that the analysis is valid if $Z \simeq \Re^q$ for any $q \geq 2$. In this case all party behavior will lie in the affine subspace of $Z$ generated by $\{o_1, o_2, o_3\}$.

8. Note that even with a *MSNE*, the contraction property means that the support of the *MSNE* will lie within the Pareto set of the parties (namely the convex hull of the bliss points).

9. It is usual in the literature to assume that $p$ is generated by individual voters who tend to the choose the party with the nearest manifesto. This may not be justified in the general model.

10. The nature of *MSNE* will depend, of course, on the nature of the electoral map. With electoral risks, rational voter behavior may involve an estimate of what policy each party will attempt to implement in the post-election situation. Since the electorate can be assumed to be large, it is plausible that $p$ will be smooth. It is hoped that later analyses will indicate how $p$ may be modeled.

    An important first step in determining $p$ can be found in Austen-Smith (1986) in the one-dimensional case. Just as in the model here, Austen-Smith assumes that each admissible coalition, $M$, forms with probability $\rho_M$, inversely proportional to its variances and adopts the mean policy point $z_M$. Admissible can mean "minimal winning". An obvious motivation in Austen-Smith's paper concerns the creation of parties (that is coalitions of autonomous and diverse candidates). It is plausible that the convergence phenomena found in our model may provide a theoretical framework for the coalescence of parties.

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EXISTENCE OF NASH EQUILIBRIUM
IN A SPATIAL MODEL OF n-PARTY COMPETITION


Abstract

In the model presented here, $n$ parties choose policy positions in a space $Z$ of dimension at least two. Each party has true preferences on $Z$ that are unknown to other agents. In the first version of the model the party declarations determine the lottery outcome of coalition negotiation. The lottery outcome function is common knowledge to the parties and is determined by probabilities of coalition formation inversely proportional to the variance of the declarations of coalition members.

It is shown that with this outcome function and with three parties there exists a stable, pure strategy Nash equilibrium for certain classes of policy preferences. The Nash equilibrium can be explicitly calculated in terms of the preferences of the parties and the scheme of private benefits from coalition membership.

In particular, convergence in equilibrium party positions is shown to occur if the party bliss points are close to colinear. Conversely, divergence in equilibrium party positions occurs if the bliss points are close to symmetric. If private benefits are sufficiently large (that is, of the order of policy benefits), then the variance in equilibrium party positions is less than the variance in bliss points.

The general model attempts to incorporate party beliefs concerning electoral responses to party declarations. A mixed strategy Nash equilibrium is shown to exist. It is conjectured that generically (with respect to party policy preferences) there exists a finite number of pure strategy Nash equilibria. Moreover if the scheme of private benefits is sufficiently large, then there is generically a $1:1$ correspondence between the profile of party preferences and the Nash equilibria. That is to say, generically there is a unique pure strategy Nash equilibirum for each profile of party preferences.

EXISTENCE OF NASH EQUILIBRIUM IN A SPATIAL MODEL OF n-PARTY COMPETITION

by

**NORMAN SCHOFIELD**

and

**ROBERT PARKS**

Center in Political Economy
Campus Box 1208
Washington University in St. Louis
St. Louis, Missouri 63130
(314) 935-4774

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