Judgment aggregation in general logics

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Abstract. Within social choice theory, the new field of judgment aggregation aims to merge many individual sets of judgments on logically interconnected propositions into a single collective set of judgments on these propositions. Commonly, judgment aggregation is studied using standard propositional logic, with a limited expressive power and a problematic representation of conditional statements ("if P then Q") as *material* conditionals. In this methodological paper, I present a generalised model, in which most realistic decision problems can be represented. The model is not restricted to a particular logic but is open to several logics, including standard propositional logic, predicate calculi, modal logics and conditional logics. To illustrate the model, I prove an impossibility theorem, which generalises earlier results.

Key words: judgment aggregation, discursive dilemma, modelling methodology, formal logics, impossibility theorem

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1 Introduction

The traditional model of social choice theory, *preference aggregation*, defines a collective decision problem as the problem of forming collective preferences over a given set of alternatives (actions, policies, candidates, states of society etc.). By contrast, the newly arising model of *judgment aggregation* defines a collective decision problem as the problem of forming collective judgments (acceptance or rejection) on a given set of logically interrelated propositions. As a simple example, suppose the population of a country disagrees on whether the following propositions hold:

a : The birth rate is too low to guarantee long-term economic stability.
b : More immigration is needed.
$a \to b$: *If* the birth rate is too low to guarantee long-term economic stability, *then* more immigration is needed.

Reaching collective judgments on logically interrelated propositions is non-trivial. Suppose that in the example the population is split into three camps of equal size. As

	a	$a \to b$	b
1/3 of the population	True	True	True
1/3 of the population	True	False	False
1/3 of the population	False	True	False
Majority	True	True	False

Table 1: A discursive dilemma

shown in Table 1, each camp holds a logically consistent set of judgments; for instance the first camp accepts a and $a \to b$, and accordingly accepts b. Yet the propositionwise majority judgments are logically inconsistent: a and $a \to b$ are accepted, but b is rejected. Such situations are known as *discursive dilemmas* (e.g. Pettit 2001).

How can the group reach consistent decisions? Two aggregation rules have received particular attention; let me define them for our example. Under the *premise-based procedure*, the group takes majority votes only on a and $a \to b$ (the *premises*) and decides b (the *conclusion*) by logical entailment from the decisions on a and $a \to b$. So, in Table 1, a and $a \to b$ are accepted, and hence b is accepted. Under the *conclusion-based procedure*, the group takes a majority vote only on b and ignores the majority verdicts on a and $a \to b$. So, in Table 1, b is rejected and no collective judgment is made on a and $a \to b$.

The *propositions* in judgment aggregation can be atomic (such as a or b) or compound (such as $a \to b$), and they can express for instance *beliefs* (e.g. "pollution creates global warming"), *desires* (e.g. "global warming is undesirable") or *act preferences* (e.g. "measure X against pollution should be taken"). The judgment aggregation model is close to real decision situations. First, individuals are not required to rank many complex alternatives, but only to have opinions on different issues. Second, real decision situations often take indeed the form of accepting or rejecting different propositions, which are interconnected.

In this paper, I argue that judgment aggregation allows one to study a wide range of realistic collective decision problems. However, to do so the model should be extended beyond standard propositional logic. While standard propositional logic can adequately represent decision problems whose propositions involve only the logical operators "and", "or" and "not", many real problems do not have this form and require more expressive logics. Problems involving conditional statements, such as $a \to b$ above, usually require a *conditional logic*, although they were so far represented in standard propositional logic by using the (problematic) *material* conditional, as explained later. Other realistic decision problems may be represented in a *predicate logic*, in which atomic propositions are not taken as primitives but are constructed from constants, variables, functions and relations (just as in common language sentences are not primitives). This allows for instance to embed preference aggregation into judgment aggregation. Many decision problems can be represented using *modal logics*, in which one can express propositional attitudes such as "it is desirable that p", "it is ethically required/allowed that p", "it is probable that p" etc.

The good news is that a unified model of judgment aggregation is still possible, despite of the differences between the logics necessary for the various applications. I introduce a model in "general logics" that is not restricted to any specific logic but contains most practically relevant logics as special cases. Several previous results, if suitably restated, can be shown to hold in general logics; many others are yet to be derived.

In Section 2, I introduce the model, which is based on a set of mild conditions on the logic. In Section 3, I illustrate the model by discussing several types of collective decision problems, all of which can be represented within the model. In Section 4, I prove an impossibility theorem, which generalises earlier results to general logics. In Section 5, I provide a list of simple tools that can be used to prove results in general logics; these tools underlie many existing proofs in judgment aggregation, which shows that one can conveniently work in general logics. In Section 6, I mention that it can also be interesting to derive results restricted to a particular logic – though often not standard propositional logic –, which leads me to contrast judgment aggregation in general logics with judgment aggregation in a specific logic. In Section 7, I summarise and conclude the paper.

On a less formal basis, judgment aggregation has been discussed already for a while, partly focussing on the distinction between premise-based and conclusion-based decision-making (e.g. Kornhauser and Sager 1986, 2004, Chapman 1998, 2002, Pettit 2001, Bovens and Rabinowicz 2004). List and Pettit (2002) formalise judgment aggregation using standard propositional logic, and prove a first social-choice-theoretic impossibility result. This sparked a series of contributions. Pauly and van Hees (2004), Dietrich (2004), Gärdenfors (2004), Nehring and Puppe (2004) and van Hees (2004) prove several impossibility theorems, whose main message is that, given certain logical connections between the propositions under decision, propositionwise aggregation (satisfying some mild conditions) is impossible. To escape impossibilities, one may for instance restrict the domain of the aggregation rule (List 2003), restrict the independence condition to *premises* (Dietrich 2004), use *fusion operators* (Pigozzi 2004), or use *sequential* decision rules (List 2004b and Dietrich and List 2005). The models of Pauly and van Hees (2004) and van Hees (2004) allow for *degrees of acceptance*, a significant generalisation of the informational input and output of decision procedures, but still with the same limited expressive power of propositions as in the standard model. The probability of "correct" collective judgments is analysed by Bovens and Rabinowicz (2004) and List (2004a). Gärdenfors (2004) and Gekker (2003) question the requirement that judgments must be complete. Strategy-proof judgment aggregation is analysed in Dietrich and List (2004). Characterisations of the class of aggregation rules satisfying various collective rationality conditions (and other conditions) are provided in Nehring and Puppe (2004), Dietrich and List (2005) and Nehring (2005). List and Pettit (2004) discuss the connection to preference aggregation.

2 A judgment aggregation model in general logics

2.1 General logics

In its most general form, a *logic* (*with negation operator* $\neg$) is a pair $(\mathbf{L}, \models)$ consisting of:

• some non-empty set $\mathbf{L}$ of formal expressions[1], such that $p \in \mathbf{L}$ implies $\neg p \in \mathbf{L}$; $\mathbf{L}$ is the *language*, its elements are the *propositions*;

• some binary relation $\models$ $(\subseteq \mathcal{P}(\mathbf{L}) \times \mathbf{L})$ between sets $A \subseteq \mathbf{L}$ and propositions $p \in \mathbf{L}$; $\models$ is the *entailment relation*, and $A \models p$ is read A *(logically) entails* p, or p *is a (logical) consequence of* A; I write $p_1, ..., p_k \models p$ for $\{p_1, ..., p_k\} \models p$.

The language $\mathbf{L}$ tells what sentences can be formed, a purely syntactic notion. The entailment relation $\models$ tells how the propositions are interrelated, a semantic notion. The other important semantic notion, (in)consistency, can be defined out of entailment.

[1] A formal expression is simply a concatenation of symbols.

Definition 1 *A set* $A \subseteq \mathbf{L}$ *is* inconsistent *if there is a* $p \in \mathbf{L}$ *such that* $A \models p$ *and* $A \models \neg p$, *and* consistent *otherwise.*[2]

This abstract notion of a logic is very flexible, as examples will illustrate. All basic notions of judgment aggregation can be defined for a general logic $(\mathbf{L}, \models)$ in the familiar way: agendas, judgment sets, rationality conditions (e.g. deductive closure), aggregation rules, and conditions on aggregation rules (anonymity, independence, etc.). To obtain interesting results such as (im)possibility theorems, some conditions on the logic $(\mathbf{L}, \models)$ are of course inevitable. The good news is that three mild conditions are often sufficient (and necessary):

L1: For any $p \in \mathbf{L}$, $p \models p$ (*self-entailment*).
L2: For any $p \in \mathbf{L}$ and $A, B \subseteq \mathbf{L}$, if $A \models p$ and $A \subseteq B$ then $B \models p$ (*monotonicity*).
L3: The empty set $\emptyset$ is consistent, and each consistent set $A \subseteq \mathbf{L}$ has a consistent superset $B \subseteq \mathbf{L}$ containing a member of each pair $p, \neg p \in \mathbf{L}$ (*completability*).

By L1-L3, any proposition entails itself, any entailment $A \models p$ is preserved by adding new premises, and consistent sets can be extended to complete consistent sets.

Most realistic judgment aggregation problems can be formalised in logics of type L1-L3 (see Section 2). While many arguments commonly used in proofs work perfectly for a general logic $(\mathbf{L}, \models)$ of type L1-L3, some arguments require additional properties, which are also satisfied by the logics of most realistic aggregation problems:

L4: For any $A \subseteq \mathbf{L}$ and $p \in \mathbf{L}$, if $A \cup \{\neg p\}$ is inconsistent then $A \models p$ (*non-paraconsistency*).
L5: For any $p \in \mathbf{L}$ and $A \subseteq \mathbf{L}$, if $A \models p$ there $B \models p$ for some finite subset $B \subseteq A$ (*compactness*).

In summary, the conditions L1-L3 (plus perhaps L4,L5) are sufficiently weak for representing many real-world aggregation problems, and sufficiently strong for deriving many interesting results; they are therefore a possible framework to study judgment aggregation in general logics. All this is argued more carefully later.

2.2 The basic notions of judgment aggregation

I now define the familiar notions of judgment aggregation for a general logic $(\mathbf{L}, \models)$ of type L1-L3 (and perhaps L4 and L5). Consider a group of n individuals denoted $1, ..., n$ $(n \geq 2)$, having to make collective judgments on interrelated propositions.

The agenda. The *agenda* (containing the propositions to be decided) is any non-empty set $X \subseteq \mathbf{L}$ that (i) contains no double-negated propositions $(\neg\neg p)$ and (ii) is the union of proposition-negation pairs $\{p, \neg p\}$. The starting example has the agenda $X = \{a, \neg a, a \rightarrow b, \neg(a \rightarrow b), b, \neg b\}$, where $\rightarrow$ should be a *subjunctive* conditional and $(\mathbf{L}, \models)$ a *conditional logic*, as explained later.

Judgment sets. A *judgment set* (held by an individual or the collective) is a subset $A \subseteq X$, where $p \in A$ means "p is accepted". A judgment set A is

[2] A propositions $p \in \mathbf{L}$ is *inconsistent* (or a *contradiction*) if $\{p\}$ is inconsistent, *consistent* if $\{p\}$ is consistent, a *tautology* if $\{\neg p\}$ is inconsistent, and *contingent* if $\{p\}$ and $\{\neg p\}$ are consistent.

- *complete* if it contains at least one member of each pair $p, \neg p \in X$;
- *weakly consistent* if it contains at most one member of each pair $p, \neg p \in X$;
- *consistent* if (see Definition 1) there is no $p \in \mathbf{L}$ with $A \models p$ and $A \models \neg p$;
- *deductively closed* if, for each $p \in X$, if $A \models p$ then $p \in A$;[3]
- *fully rational* if A satisfies completeness and consistency (and so all other rationality conditions by Proposition 1).

For instance, for the above example agenda X, the judgment set $A = \emptyset$ is consistent and deductively closed but incomplete, the judgment set $A = \{a, a \to b, \neg b\}$ is complete but only weakly consistent and not deductively closed, and the judgment set $A = \{a, \neg(a \to b), b\}$ is fully rational (since $\to$ is a subjunctive conditional, as defined below).

The various rationality conditions are interrelated as follows:

Proposition 1 *Let L1-L3 hold. For any judgment set $A \subseteq X$,*

(a) consistency implies weak consistency, and both are equivalent given deductive closure and L4;

(b) full rationality implies the conjunction of completeness, weak consistency and deductive closure, and both are equivalent given L4.

Proof. The proof uses tools derived in Section 5 and hence is given there. ■

Aggregation rules. A *profile* is an n-tuple $(A_1, ..., A_n)$ of (individual) judgment sets. A *(judgment) aggregation rule* is a function F assigning to each profile $(A_1, ..., A_n)$ in a given set of admissible profiles a (collective) judgment set $F(A_1, ..., A_n) = A$. The set of admissible profiles is called the *domain* of F, written *Domain*(F). All common requirements on aggregation rules (anonymity, universal domain, etc.) can easily be stated in our general framework, as they do not appeal to any particular logic. For instance:

Universal Domain. The domain of F, *Domain*(F), is the set of all profiles $(A_1, ..., A_n)$ of fully rational judgment sets.

Collective Rationality. The collective judgment set $F(A_1, ..., A_n)$ is fully rational for every profile $(A_1, ..., A_n) \in$ *Domain*(F).

Independence. For any proposition $p \in X$ and profiles $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in$ *Domain*(F), if [for all individuals i, $p \in A_i$ if and only if $p \in A_i^*$] then [$p \in F(A_1, \ldots, A_n)$ if and only if $p \in F(A_1^*, \ldots, A_n^*)$].

Of these requirements, the first one ensures that F always produces a decision (provided individuals are fully rational), and the second one ensures that the decision is always fully rational. The third (more controversial) one is analogous to Arrow's *independence of irrelevant alternatives* in preference aggregation, and prescribes *propositionwise* aggregation/voting: the collective judgment on any given proposition

[3] An alternative definition of deductive closure is: $A \subseteq X$ is deductive closed in case A contains every proposition $p \in X$ entailed by *some consistent subset* of A. Proposition 1 holds for both definitions.

$p \in X$ should be determined solely by the individual judgments on *this* propositions. Let me mention three aggregation rules, each defined for all profiles $(A_1, ..., A_n)$ in the universal domain.

• *Majority rule*: $F(A_1, ..., A_n) = \{p \in X$: more than half of the persons i have $p \in A_i\}$; this rule satisfies independence, but violates collective rationality for many agendas, as seen in the introduction.

• *Dictatorship by person j*: $F(A_1, ..., A_n) = A_j$, the dictator's judgment set; this rule satisfies both independence and collective rationality, but is undemocratic.

• *Premise-based procedure* for the above agenda $X = \{a, \neg a, a \to b, \neg(a \to b), b, \neg b\}$ and an odd group size n: $F(A_1, \ldots, A_n)$ contains

- a *premise* $p \in \{a, \neg a, a \to b, \neg(a \to b)\}$ if and only if more than half of the persons i have $p \in A_i$ (majority voting on premises);
- a *conclusion* $c \in \{b, \neg b\}$ if and only if $P \models c$, where P is the set of premises in $F(A_1, ..., A_n)$ (no vote taken on conclusions!).

Unlike majority rule, this rule generates consistent judgment sets, but it violates independence because of the decision method used for conclusions.

3 Decision problems and the logics to represent them

I now present several types of judgment aggregation problems, and define particular logics $(\mathbf{L}, \models)$ that can be used to represent them. As these logics satisfy L1-L5, judgment aggregation in general logics covers all these aggregation problems. The reader might be struck by the large variety of different logics, and their complexity compared to the general conditions L1-L5. Judgment aggregation in general logics is not affected by the complexity of particular logics, as it is based solely the conditions L1-L3 (plus perhaps L4,L5). So some readers might want to skip the details of the definitions of particular logics.

3.1 Decision problems with conjunctions and disjunctions: representable in standard propositional logic

Standard propositional logic (used so far in judgment aggregation) can represent decision problems that involve only the logical operators "and", "or" and "not". For example, the supervisory board of a loss-making Western European company might debate the following propositions:

a : A factory should be closed down.
b : A new factory should be created in Eastern Europe.
$a \wedge b$: A factory should be closed down *and* a new one created in Eastern Europe.

The agenda is thus $X = \{a, \neg a, b, \neg b, a \wedge b, \neg(a \wedge b)\}$, which belongs to a standard propositional logic $(\mathbf{L}, \models)$ defined as follows.

Standard propositional logic (with $\neg, \wedge, \vee$). Define the language $\mathbf{L}$ as the (smallest) set such that (i) $\mathbf{L}$ contains each given *atomic proposition* $a, b, ...$, and (ii) if $\mathbf{L}$ contains p and q, then $\mathbf{L}$ also contains $\neg p$ ("not p"), $(p \wedge q)$ ("p and q"), and

$(p \vee q)$ ("p or q"). For simplicity, for this and all following logics I often drop brackets when there is no ambiguity, e.g. I write $(a \wedge b \wedge c) \to d$ for $((a \wedge (b \wedge c)) \to d)$.

To define entailment $\models$ on $\mathbf{L}$, let an *interpretation* be a ("truth") function $v : \mathbf{L} \to \{T, F\}$ that assigns to each proposition a truth value T ("true") or F ("false"), such that, for any propositions $p, q \in \mathbf{L}$,

($\neg$) $v(\neg p) = T$ if and only if $v(p) = F$,

($\wedge$) $v(p \wedge q) = T$ if and only if $v(p) = T$ and $v(q) = T$,

($\vee$) $v(p \vee q) = T$ if and only if $v(p) = T$ or $v(q) = T$.

Each interpretation v stands for "one way the world could be". By definition, $A \subseteq \mathbf{L}$ entails $p \in \mathbf{L}$ ($A \models p$) if, for every interpretation v such that $v(q) = T$ for all $q \in A$, we have $v(p) = T$. Informally, A entails p in case p is true *whenever* each $q \in A$ is true. For instance, $p, q \models p \wedge q$ (since $v(p) = v(q) = T$ implies $v(p \wedge q) = T$ by ($\wedge$)), $a \models \neg\neg a$ (since $v(a) = T$ implies $v(\neg a) = F$ by ($\neg$), which implies $v(\neg\neg a) = T$ by ($\neg$)), $a \wedge b \models a$ (since $v(a \wedge b) = T$ implies $v(a) = T$ by ($\wedge$)), etc. Note that A is consistent (i.e. entails no pair $p, \neg p \in \mathbf{L}$) if and only there is an interpretation under which each $p \in A$ is true. Informally, A is consistent in case its members *can* be simultaneously true. For instance, $\{\neg a, a \vee b\}$ is consistent (take $v(a) = F$ and $v(b) = T$), but $\{a, \neg a\}$ is inconsistent.

3.2 Decision problems with conditional statements: representable in conditional logics

Now consider aggregation problems involving (bi)conditional statements $p \to q$ ("if p then q") or $p \leftrightarrow q$ ("p if and only if q"), such as the standard examples for judgment aggregation in the literature. So, let us extend the language by adding the connectives $\to$ and $\leftrightarrow$. $\mathbf{L}$ is now defined as the (smallest) set such that (i) $\mathbf{L}$ contains each given *atomic proposition* $a, b, ...$, and (ii) if $\mathbf{L}$ contains p and q, then $\mathbf{L}$ also contains $\neg p$, $(p \wedge q)$, $(p \vee q)$, $(p \to q)$, and $(p \leftrightarrow q)$.

How should the entailment relation $\models$ on $\mathbf{L}$ be defined? Let me briefly discuss three alternative semantics. The first one is that of standard propositional logic and interprets $\to$ as a *material* conditional (problematic); the second one is a simple modal logic and interprets $\to$ as a *strict* conditional; the third one is a conditional logic and interprets $\to$ as a fully-fleshed *subjunctive* conditional.

Standard propositional logic (with $\neg, \wedge, \vee$ and material (bi)conditional $\to, \leftrightarrow$). For the above language $\mathbf{L}$, define an interpretation as a (truth) function $v : \mathbf{L} \to \{T, F\}$, assigning a truth value to each proposition, such that,, for any $p, q \in \mathbf{L}$, we have ($\neg$), ($\wedge$), ($\vee$), and

($\to_{\text{material}}$) $v(p \to q) = T$ if and only if $v(p) = F$ or $v(q) = T$;

($\leftrightarrow$) $v(p \leftrightarrow q) = T$ if and only if $v(p \to q) = T$ and $v(q \to p) = T$.

So, $a \to b$ is declared equivalent to $\neg a \vee b$ ("not-a or b"). Entailment is defined as in Section 3.1: $A \models p$ holds if and only if, for every interpretation v such that $v(q) = T$ for all $q \in A$, we have $v(p) = T$. By ($\to_{\text{material}}$), we have $p, p \to q \models q$ ("modus ponens"); also, $\neg p \models p \to q$ and $q \models p \to q$ (the "paradoxes of the material conditional").

Material conditionals raise well-known problems and misrepresent the intended meaning of most conditional statements in normal language (e.g. Priest 2001). The statement "if it's the 15th century then people drive cars" is true as a material conditional because it's *not* the 15th century (and also because people drive cars). This clash between our intuition and the material conditional is due to the fact that, usually, by "if a then b" one intends a statement not about the *actual* truth values of a and b, but about b's truth value in other (perhaps non-actual) worlds in which a holds, e.g. worlds where it's the 15th century. The statement "if it's the 15th century then people drive cars" does not mean "either it's *not* the 15th century *or* we drive cars" (which is true), but it means "in a world of the 15th century, people drive cars" (which is false).

Also in judgment aggregation, the relevant conditionals are usually not material, as argued also in Dietrich (2005). For instance, the conditional

$a \rightarrow b$: "*if* the birth rate is too low *then* more immigration is needed"

does not mean

$\neg a \vee b$: "the birth rate is *not* too low *or* more immigration is needed",

but it means "in the case (world) where the birth rate is too low, more immigration is needed". Under the latter reading, it is perfectly consistent to reject a without accepting $a \rightarrow b$; by contrast, in the above logic $(\mathbf{L}, \models)$, $\neg a$ entails $a \rightarrow b$ by $(\rightarrow_{\text{material}})$.

By removing the restriction to standard propositional logic, it becomes possible to represent conditional statements more adequately. The two logics $(\mathbf{L}, \models)$ defined below are based on possible-worlds semantics, which goes back to Kripke (1963) and others and is now widely used; for reference, e.g. Priest (2001).

The modal logic $S5$ (with $\neg, \wedge, \vee$ and strict (bi)conditional $\rightarrow, \leftrightarrow$). I now endow $\mathbf{L}$ with a new entailment relation $\models$. While for standard propositional logic an interpretation is given by a single truth function, an $S5$-*interpretation* is a pair $(W, (v_w)_{w \in W})$, where:

- W is a non-empty set of objects called *(possible) worlds*;
- $(v_w)_{w \in W}$ is a family of ("truth") functions $v_w : \mathbf{L} \rightarrow \{T, F\}$, assigning to each proposition $p \in \mathbf{L}$ its truth value $v_w(p)$ in world $w \in W$, such that, for each world $w \in W$ and any propositions $p, q \in \mathbf{L}$, the truth function $v = v_w$ satisfies $(\neg)$, $(\wedge)$, $(\vee)$, $(\leftrightarrow)$, and

$(\rightarrow_{\text{strict}})$ $v_w(p \rightarrow q) = T$ if and only if $v_{w'}(q) = T$ for each world $w' \in W$ with $v_{w'}(p) = T$,

Condition $(\rightarrow_{\text{strict}})$ defines *strict* conditionals: $p \rightarrow q$ holds in a world just in case q holds in *every* world in which p holds. So the truth value of $p \rightarrow q$ depends not just on the actual world, but on all worlds in W. Interpretations differ in what worlds are considered possible, and what propositions are true in them. If the only relevant aspect of the world is the season, then, not knowing what season it is, all seasons are possible, say $W = \{Sp, Su, Au, Wi\}$; but knowing that it is not winter, only three seasons are possible, say $W = \{Sp, Su, Au\}$. Presumably, in our immigration example the possible worlds include ones with a bright economic future as well as ones with long-term economic instability.

By definition, $A \subseteq \mathbf{L}$ *entails* $p \in \mathbf{L}$ ($A \vDash p$) in case, for every $S5$-interpretation $(W, (v_w)_{w \in W})$ and every world $w \in W$ such that $v_w(q) = T$ for all $q \in A$, we have $v_w(p) = T$. Informally, A entails p if, whenever each $q \in A$ is true, p is true. The strict conditional still satisfies $a, a \rightarrow b \models b$ ("modus ponens"), but does not satisfy

$\neg a \models a \to b$ and $b \models a \to b$ (the "paradoxes of the material conditional"). Note also that A is consistent (i.e. entails no pair $p, \neg p$) if and only if *some* world of *some* $S5$-interpretation makes all $q \in A$ simultaneously true.

Strict conditionals avoid the two most striking problems of material conditionals, since neither $\neg a$ nor b entails $a \to b$. In judgment aggregation, it makes a considerable difference whether strict or material conditionals are used. To illustrate this, consider again our immigration example with agenda $X = \{a, \neg a, a \to b, \neg(a \to b), b, \neg b\}$, and let us see which judgment sets $A \subseteq X$ are declared fully rational (i.e. consistent and complete) under the two logics.

• if X belongs to standard propositional logic, the only fully rational judgment sets are $\{a, b, a \to b\}$, $\{\neg a, b, a \to b\}$, $\{\neg a, \neg b, a \to b\}$, $\{a, \neg b, \neg(a \to b)\}$;

• if X belongs to the modal logic $S5$, there are three additional fully rational judgment sets, namely $\{a, b, \neg(a \to b)\}$, $\{\neg a, b, \neg(a \to b)\}$, $\{\neg a, \neg b, \neg(a \to b)\}$.

Nevertheless, strict conditionals – historically the first attempt to formalise non-material conditionals – face other, more subtle, problems, which suggest that they do still not fully faithfully represent the intended meaning of many conditional statements. Often, only *subjunctive* conditionals are considered fully adequate. For, "if a then b" often means not that b holds in *every* world where a holds (strict conditional), but that b holds in worlds similar to the actual world except that a is true (subjunctive conditional). Thus the meaning of "if a then b" is often: if a *were* true – if the actual world were modified so that a becomes true *ceteris paribus* – then b would be true. One might even interpret "if a then b" as "in the closest world(s) in which a holds, b holds".

Subjunctive conditionals were formalised by D. Lewis (1973) using *conditional logics* and have become well-established. Let me introduce a standard version of conditional logic. It leads to the same fully rational judgment sets as $S5$ for many agendas, including the above agenda $X = \{a, \neg a, a \to b, \neg(a \to b), b, \neg b\}$.

The conditional logic C^+ (with $\neg, \wedge, \vee$ and subjunctive (bi)conditional $\to, \leftrightarrow$). Still for the same language $\mathbf{L}$, a C^+-*interpretation* is defined as a triple $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$, where:

• W is again a non-empty set of *(possible) worlds*;

• $(R_p)_{p \in \mathbf{L}}$ is a family of binary relation on W (wR_pw' is interpreted as "world w' is similar to world w, and p is true in w'"), such that, for any $w, w' \in W$ and $p \in \mathbf{L}$, (i) if wR_pw' then $v_{w'}(p) = T$ (an obvious requirement given the interpretation of wR_pw') and (ii) if $v_w(p) = T$ then wR_pw (since w is similar to itself);

• $(v_w)_{w \in W}$ is a family of ("truth") functions $v_w : \mathbf{L} \to \{T, F\}$, assigning to each proposition $p \in \mathbf{L}$ its truth value $v_w(p)$ in world $w \in W$, such that, for any $w \in W$ and $p, q \in \mathbf{L}$, the truth function $v = v_w$ satisfies ($\neg$), ($\wedge$), ($\vee$), ($\leftrightarrow$) and

($\to_{\text{subjunctive}}$) $v_w(p \to q) = T$ if and only if $v_{w'}(q) = T$ for each world $w' \in W$ with wR_pw'.

By ($\to_{\text{subjunctive}}$), $p \to q$ is true in world w just in case q is true in every world w' similar to w and with true p. This captures the above intuition for subjunctive conditionals. For instance, "if the earth falls on the sun then we freeze" is plausibly false in the actual world, because we do *not* freeze in those worlds similar to the actual world except that the earth falls on the sun.

Entailment is defined as for $S5$, but now relative to C^+-interpretations: $A \subseteq \mathbf{L}$ entails $p \in \mathbf{L}$ ($A \models p$) in case, for any world w of any C^+-interpretation, if each $q \in A$ is true in w then p is true in w. For instance, like in $S5$, modus ponens holds but the paradoxes of the material conditional do not hold. Again, this implies that A is consistent if and only if for *some* world of *some* C^+-interpretation each $q \in A$ is true.

3.3 Decision problems with modal statements: representable in modal logics

Modal operators are used to represent phrases in front of propositions such as "it is desirable that", "it is ethically required that", "it is in our interest that", "it is feasible that", "it is probable that", "it is known that" etc. For instance, "it is desirable that p" does not say whether p is true or false, but that p is desirable. There are various ways in which modal operators can be relevant in judgment aggregation. Let me give two examples.

Non-separable decisions on acts. Many decision problems consist in deciding collective acts. Consider act-describing propositions, such as "income taxes are raised", "indirect taxes are raised", "the budget deficit is reduced". Certain acts may be non-separable from certain other acts: whether the former should be taken depends on whether the latter are taken. To represent this, one may use a modal operator S standing for "it is desirable that", and consider an agenda X that contains the following two types of propositions (and their negations):

• propositions of the form $S(p)$, where p is an act-describing proposition; $S(p)$ could be "*it is desirable that* income taxes are raised", in short "income taxes *should* be raised";

• conditional statements of the form $p \to S(q)$, where p and q are propositions describing acts that are not separable from each other; $p \to S(q)$ could be "*if* income taxes are raised *then* the budget deficit *should* be reduced".

A judgment set $A \subseteq X$ then states that certain acts should (not) be taken, unconditionally or conditionally on other acts. Note that this approach differs from the way preference aggregation handles non-separability.

Probabilistic statements. In a private communication, R. Gekker drew my attention towards the importance of probabilistic statements of the form "it is probable that p" (in short: "probably p"), where p is a factual proposition such as "carbon dioxide emissions are a cause of global warming" or "global warming will continue over the next 10 years". If may be interesting to use a modal operator P for "probably" and to consider an agenda X containing:

• propositions of the form $P(p)$, where p is a factual proposition; $P(p)$ could be "*Probably* carbon dioxide emissions are a cause of global warming";

• conditional statements of the form $p \to P(q)$, where p and q are factual propositions; $p \to P(q)$ could be "*if* carbon dioxide emissions are a cause of global warming *then probably* global warming will continue over the next 10 years".

A judgment set $A \subseteq X$ expresses probabilistic beliefs about the world. These probabilistic beliefs are less exact than those expressed in a probability function (how probable is "probably"?); however, it is more realistic that a person can submit a judgment set $A \subseteq X$ than a full probability function.

Of course, each type of modal operator requires a particular semantics, and hence a particular logic $(\mathbf{L}, \models)$. For instance, the above operator S should plausibly satisfy the entailment $S(p), S(q) \models S(p \wedge q)$ (if p and q are desirable, so is $p \wedge q$), but the above operator P should not satisfy $P(p), P(q) \models P(p \wedge q)$ (if p and q are each probable, $p \wedge q$ need not be probable). While there are many alternative ways to formalise modal operators, many of them lead to logics satisfying the conditions L1-L5 of the present model of judgment aggregation. Many formalisations share a common feature: they are based on possible-worlds semantics, like the formalisation of conditional statements discussed earlier. The reason is that a modal operator in front of a proposition p can often be interpreted as stating that p holds in *every* (or in *some*) *possible world*, under an appropriate notion of possibility: "it is desirable that p" means "p holds in every world respecting our desires"; "it is ethically required that p" means "p holds in every (ethically) permissible world"; "it is in our interest that p" means "p holds in every world respecting our interests", etc. (There are exceptions, notably regarding the operator "it is probable that"; see for instance Gekker 2003).

One may also use a multi-modal logic (with more than one modal operator), or a logic with a modal operator *and* a subjunctive conditional $\rightarrow$, as might be appropriate in the two examples above.

Focussing on simple cases, let me briefly define two modal logics, each with a single modal operator $\Box$ representing some type of modal necessity such as those mentioned above. Intuitively, $\Box p$ means that p holds *necessarily*, i.e. *in every possible world*, where a world is possible if it respects desires, or interests, or morality, or budget constraints, or is compatible with our information, etc. So, let the language $\mathbf{L}$ now be the (smallest) set such that (i) $\mathbf{L}$ contains each given *atomic proposition* $a, b, ...$, and (ii) if $\mathbf{L}$ contains p and q, then $\mathbf{L}$ also contains $\neg p$, $(p \wedge q)$, $(p \vee q)$, and $\Box p$ ("necessarily p", under the relevant notion of necessity).

How should entailment $\models$ on $\mathbf{L}$ be defined? The simplest possible-worlds semantics, appropriate only for some modal operators, is that of the logic $S5$. This logic was defined above for a language without operator $\Box$ but with $\rightarrow, \leftrightarrow$; I now define $S5$ for the present language $\mathbf{L}$.

The modal logic $S5$ (with $\neg, \wedge, \vee$ and necessity operator $\Box$). For the language $\mathbf{L}$ defined above, an $S5$-*interpretation* is a pair $(W, (v_w)_{w \in W})$, where

- W is a non-empty set of *(possible) worlds*;
- $(v_w)_{w \in W}$ is a family of ("truth") functions $v_w : \mathbf{L} \rightarrow \{T, F\}$, assigning to each proposition $p \in \mathbf{L}$ its truth value $v_w(p)$ in world $w \in W$, such that, for each world $w \in W$ and any propositions $p, q \in \mathbf{L}$, the truth function $v = v_w$ satisfies $(\neg)$, $(\wedge)$, $(\vee)$, and

$(\Box_{S5})$ $v(\Box p) = T$ if and only if $v_{w'}(p) = T$ for *every* world $w' \in W$.

(One can define a strict conditional $p \rightarrow q$ by $\Box(\neg p \vee q)$, and $\diamond p$ ("it is possible that p") by $\neg\Box\neg p$ ("it is not necessary that not p"); for instance, if $\Box$ represents moral necessity, $\diamond$ represents moral permissibility.)

Again, $A \subseteq \mathbf{L}$ *entails* $p \in \mathbf{L}$ $(A \vDash p)$ in case, for every world w of any $S5$-interpretation, if each $q \in A$ is true in w then p is true in w. For instance, for each $p \in \mathbf{L}$ we have $\Box p \models p$ (since if p is true in every world, it is true in the actual world) and $\Box p \models \diamond p$ (since if p is true in every world it is true in some world). Again, it follows that A is consistent (i.e. entails no pair $p, \neg p \in \mathbf{L}$) if and only if *some* world of *some* $S5$-interpretation makes all $q \in A$ true.

While $S5$ is the simplest formalisation of a modality, it far from represents all forms of modality adequately. Assume $\Box$ stands for "it is desirable that". Then the entailment $\Box p \models p$, which holds in $S5$, is obviously inappropriate for many p: "it is desirable that all humans live in harmony" does not entail "all humans live in harmony". Similarly, the entailment $\Box p \models p$ is problematic if $\Box$ is "it is ethically required that": being ethically required does not entail being fulfilled. What causes the problem is that $S5$ interprets $\Box p$ as meaning that p holds in *all* worlds, rather in *certain* worlds. This suggests that one needs a notion of conditional possibility: relative to a given world w, only certain worlds should be possible. Writing wRw' if w' is a possible world relative to world w, we can then define $\Box p$ to hold in a world w in case p holds in every world w' with wRw', i.e. in every possible world *relative to* w. R establishes a binary relation R on W. But not *any* binary relation R on W can represent relative possibility: R has to satisfy certain properties that depend on the type of modality to be represented. For many types of modality, one or more of the following properties are often imposed on R: *reflexivity* (wRw for all $w \in W$), *symmetry* (if wRw' then $w'Rw$, for all $w, w' \in W$), *transitivity* (if wRw' and $w'Rw''$ then wRw'', for all $w, w', w'' \in W$), and *extensibility* (for all $w \in W$ there is a $w' \in W$ such that wRw'). For each set S of properties of R, a corresponding modal logic can be defined, denoted K_S after Kripke (1963), one of the founders of possible-words semantics.

The modal logic K_S (with $\neg, \wedge, \vee$ and necessity operator $\Box$). For the language $\mathbf{L}$ defined above, a *K_S-interpretation* is a triple $(W, R, (v_w)_{w \in W})$, where

- W is a non-empty set of *(possible) worlds*;
- R is a binary relation on W satisfying the conditions in S; R is the *relative possibility* relation or *accessibility* relation, and wR_pw' is interpreted as "relative to world w, world w' is possible";
- $(v_w)_{w \in W}$ is a family of ("truth") functions $v_w : \mathbf{L} \to \{T, F\}$, assigning to each proposition $p \in \mathbf{L}$ its truth value $v_w(p)$ in world $w \in W$, such that, for any $w \in W$ and $p, q \in \mathbf{L}$, the truth function $v = v_w$ satisfies $(\neg)$, $(\wedge)$, $(\vee)$, and

$(\Box_{K_S})$ $v(\Box p) = T$ if and only if $v_{w'}(p) = T$ for *every* world $w' \in W$ with wRw'.

So $\Box p$ is true in w in case it is true in those worlds possible relative w.

(Strict conditional $\to$ and possibility $\diamond$ can be defined from $\Box$ as in $S5$ above.)

Again, $A \subseteq \mathbf{L}$ *entails* $p \in \mathbf{L}$ ($A \vDash p$) in case, for every world w of any K_S-interpretation, if each $q \in A$ is true in w then p is true in w. Again, it follows that A is consistent if and only if *some* world of *some* K_S-interpretation makes all $q \in A$ true.

Each set S of conditions on the relative possibility relation R generates different properties of entailment $\models$. For instance, the characteristic property of the reflexivity condition is that $\Box p \models p$ for all $p \in \mathbf{L}$ (since if p holds in every possible world then, as the actual world is possible, p holds in the actual world). The characteristic feature of the extensibility condition is that $\Box p \models \diamond p$, i.e. if p is necessary then p is possible (since if p holds in all possible worlds then, as there exists a possible world, it holds in some possible world).

It is debatable which set of conditions S is appropriate for the different modal operators. If $\Box$ represents "it is known that" (i.e. "the available information implies that"), S should contain the reflexivity condition since we want $\Box p \models p$ (if p is known to be true then p must be true). If $\Box$ represents "it is desirable that" then S should *not* contain the reflexivity condition since otherwise we obtain $\Box p \models p$ (a problematic entailment, as seen above). Similarly, if $\Box$ represents "it is ethically required that", S should not contain the reflexivity condition; it is often argued that S should only contain the extensibility condition, which guarantees that $\Box p \models \Diamond p$, in accordance with the principle "ought implies can".

As one easily checks, the logics $S5$ and K_S coincide (i.e. have the same entailment relation $\models$) if S contains only the *universality* condition that wRw' for all $w, w' \in W$ (each world is possible relative to each world).

3.4 The formation of collective preferences: representable in a predicate logic

Suppose that, as in preference aggregation, a group has to establish a collective preference relation of set of alternatives $C = \{c_1, c_2, ..., c_k\}$ ($k \geq 2$), based on individual preference relations. Let me present two ways to model this decision problem as an instance of judgment aggregation in general logics. Following List and Pettit (2004), I will represent preferences and rationality conditions as propositions of predicate calculus.

We consider the predicate language given by the set of constants $C = \{c_1, ..., c_k\}$, the set of variables $V = \{v_1, v_2, ...\}$, the two binary predicates R ("is at least as good as") and $=$ ("is equal to"), and the standard operators $\neg$ ("not"), $\wedge$ ("and"), $\vee$ ("or"), $\rightarrow$ ("if-then") and $\forall$ ("for all"). Formally, the *atomic propositions* are the expressions xRy and $x = y$, where $x, y \in C \cup V$, and the set of *all* propositions, $\mathbf{L}$, is defined as the (smallest) set such that (i) $\mathbf{L}$ contains each atomic proposition and (ii) if $\mathbf{L}$ contains p and q then $\mathbf{L}$ also contains $\neg p$, $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$, and $(\forall v)p$ for each variable $v \in V$. Then:

(i) Each preference relation $\succeq$ on C can be represented as a set of propositions

$$A_{\succeq} := \{cRc' : c, c' \in C \text{ and } c \succeq c'\} \cup \{\neg cRc' : c, c' \in C \text{ and } c \not\succeq c'\} \subseteq \mathbf{L},$$

reflecting all pairwise rankings under $\succeq$.

(ii) Any rationality condition on preferences can be expressed as a proposition in $\mathbf{L}$; for instance, the completeness condition is $\forall v_1 \forall v_2 (v_1 R v_2 \vee v_2 R v_1)$, and the transitivity condition is $\forall v_1 \forall v_2 \forall v_3 ((v_1 R v_2 \wedge v_2 R v_3) \rightarrow v_1 R v_3)$. Let $\mathcal{R} \subseteq \mathbf{L}$ be some set, interpreted as the set of desirable rationality conditions, for instance those of a weak order, or those of a linear order, or those of an acyclic and reflexive partial order.

(iii) The *exclusiveness* of the options can be expressed by the proposition $\wedge_{1 \leq j < j' \leq k} \neg(c_j = c_{j'})$ (stating that $c_1, ..., c_k$ are pairwise distinct), and the *exhaustiveness* of the options can be expressed by the proposition $\forall v_1 (\vee_{1 \leq j \leq k} v_1 = c_j)$ (stating that there are no options except $c_1, ..., c_k$); here, "$\wedge_{1 \leq j < j' \leq k}$" and "$\vee_{1 \leq j \leq k}$" are shorthands, e.g. "$\vee_{1 \leq j \leq k} v_1 = c_j$" stands for $v_1 = c_1 \vee ... \vee v_1 = c_k$. Let $\mathcal{E}$ be the set of these two conditions.

Exogenous rationality conditions. Assume first that, as in standard preference aggregation, the rationality conditions in $\mathcal{R}$ are *exogenously imposed*, i.e. not subject to a decision. Then the agenda should be defined as the set $X := X_C := \{cRc', \neg cRc' : c, c' \in C\}$. To make the rationality conditions and the exclusiveness and exhaustiveness conditions true by definition, let me turn them into axioms of the logic. Specifically, I consider the *set of axioms* $\mathcal{A} = \mathcal{R} \cup \mathcal{E}$, containing the rationality, exclusiveness and exhaustiveness conditions, and I define a *preference logic* by the language is $\mathbf{L}$ together with the entailment relation given by:

$$A \vDash p \text{ if and only if } \begin{array}{l} A \cup \mathcal{A} \text{ logically entails } p \text{ in the} \\ \text{standard sense of predicate logic.}^4 \end{array} \quad (1)$$

In this logic, a set $A \subseteq \mathbf{L}$ is consistent (i.e. entails no pair $p, \neg p \in \mathbf{L}$) if and only if $A \cup \mathcal{A}$ is consistent in the standard sense of predicate logic. By consequence, the preference-theoretic notion of *rationality* translates into the logical notion of *consistency*:

• *A preference relation $\succeq$ on C satisfies all rationality conditions in $\mathcal{R}$ if and only if the corresponding judgment set $A_{\succeq}$ (defined above) is consistent.*

Hence, the preference-theoretic problem of deriving rational preferences becomes the judgment-theoretic problem of deriving consistent judgments.

Endogenous rationality conditions. Judgment aggregation also allows one to study the interesting decision problem, in which the (amount and type of) collective rationality is itself subject to a decision. To this end, let us augment the agenda by the rationality conditions, i.e. let us consider the agenda $X := X_{C,\mathcal{R}} = \{cRc', \neg cRc' : c, c' \in C\} \cup \{r, \neg r : r \in \mathcal{R}\}$. Rather than building the rationality conditions as axioms into the logic, I now consider the smaller *set of axioms* $\mathcal{A} := \mathcal{E}$, containing only the exclusiveness and exhaustiveness axioms, and define the logic as the language $\mathbf{L}$ endowed with the entailment relation $\models$ defined by (1) using now the new set of axioms $\mathcal{A}$.[5]

4 An impossibility theorem in general logics

After having given multiple examples of particular logics to represent particular decision problems, we are now back to full generality. Let $(\mathbf{L}, \models)$ be *any* logic of type L1-L3. I prove an impossibility theorem, generalising earlier results to general logics. The theorem is the first one to apply to the standard examples of judgment aggregation under an adequate representation of conditional statements.

[4] Entailment in predicate logic is defined in most standard logic textbooks, e.g. Mendelssohn 1979

[5] There is an additional subtlety. Intuitively, it is perfectly consistent to reject, say, the completeness condition while holding a complete preference relation by pure coincidence. Indeed, rejecting the completeness condition is not claiming that preferences *must* be incomplete, but that they *may* be incomplete. However, assume a person holds a complete preference relation $\succeq$. As one can check, the corresponding set of ranking judgments $C_{\succeq}$ logically entails the completeness condition, i.e. $C_{\succeq} \models \forall v_1 \forall v_2 (\neg v_1 R v_2 \rightarrow v_2 R v_1)$; so, if the person holds a fully rational judgment set $A \subseteq X$, where $C_{\succeq} \subseteq A$, then A must contain the completeness condition by deductive closure. Similar remarks apply to all rationality conditions. To make it possible to reject a rationality condition even if it happens to be satisfied by the actual preferences, one could add a modal necessity operator $\Box$ in front of each rationality condition – which requires using a modal predicate logic instead.

By contrast to Arrow's Theorem in preference aggregation, here an independence condition on aggregation rules as defined earlier (together with other mild conditions) does *not* generally lead into dictatorship; it does so only under rather demanding agenda assumptions. This is why many impossibility theorems (including ones by List and Pettit's 2002, Pauly and van Hees 2004 and Nehring and Puppe 2004) are based on a more demanding condition than independence, namely on:

Systematicity. For any propositions $p, p^* \in X$ and profiles $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in Domain(F)$, if [for all individuals i, $p \in A_i$ if and only if $p^* \in A_i^*$] then $[p \in F(A_1, \ldots, A_n)$ if and only if $p^* \in F(A_1^*, \ldots, A_n^*)]$.

Taking $p = p^*$ yields exactly the independence condition. Systematicity is equivalent to the existence of a function $M : \{0,1\}^n \to \{0,1\}$ (the "universal decision method") such that, for every proposition $p \in X$ and profile $(A_1, \ldots, A_n) \in Domain(F)$, $F(A_1, ..., A_n)(p) = M(A_1(p), ..., A_n(p))$. Here, for any $A \subseteq X$, $A(p)$ is defined as 1 if $p \in A$ and as 0 if $p \notin A$. Systematic rules include majority rule ($M(t_1, ..., t_n) = 1$ if and only if $t_1 + ... + t_n > n/2$), unanimity rule ($M(t_1, ..., t_n) = 1$ if and only if $t_1 = ... = t_n = 1$), dictatorial rules ($M(t_1, ..., t_n) = t_j$, where j is the *dictator*), inverse dictatorial rules ($M(t_1, ..., t_n) = 1 - t_j$, where j is the *inverse dictator*), oligarchic rules, inverse oligarchic rules, etc. Systematicity is the conjunction of independence and a neutrality condition: collective judgments should be made by an independent vote on each proposition (as $F(A_1, ..., A_n)(p)$ depends exclusively on $A_1(p), ..., A_n(p)$), where each proposition is treated the same (as M does not depend on p).

For the below result to hold in general logics, the agenda X should not be required to contain particular propositions (that could be formed only in certain logics), but rather to display certain *logical relations between* its members. It turns out the following agenda type is appropriate for the theorem (recall that a set $Y \subseteq \mathbf{L}$ is *minimal inconsistent* if it is inconsistent and every proper subset of Y is consistent):

Definition 2 *The agenda X is "weakly connected" if*
(i) there is an inconsistent set $Y \subseteq X$ such that $\{\neg p : p \in Y\}$ is consistent,
(ii) there is a minimal inconsistent set $Y \subseteq X$ such that $|Y| \geq 3$, and
(iii) there is a minimal inconsistent set $Y \subseteq X$ such that $(Y \backslash Z) \cup \{\neg z : z \in Z\}$ is consistent for some subset $Z \subseteq Y$ of even size.

This definition is not *ad hoc*: parts (i) and (ii) are indispensable for the below impossibility to hold[6], and part (iii) is only a mild addition[7]. The class of weakly connected agendas includes the agendas of all standard example of judgment aggregation, where conditional statements are modelled either with material conditionals (problematic) or with strict or subjunctive conditionals (adequate). For instance, the agenda $X = \{a, \neg a, b, \neg b, a \wedge b, \neg(a \wedge b)\}$ is weakly connected (take $Y = \{\neg a, a \wedge b\}$ in (i) and $Y = \{a, b, \neg(a \wedge b)\}$ in (ii),(iii)); and the agenda

[6] If (i) is violated, *inversely dictatorial* rules (defined by $F(A_1, ..., A_n) = X \backslash A_j$ for some person j) satisfy collective rationality. If (ii) is violated, majority rule satisfies collective rationality, provided that n is odd (and X is finite or L5 holds).

[7] Without the requirement that Z has even size, (iii) would hold for every agenda X (take any minimal inconsistent set $Y \subseteq X$ and any singleton $Z \subseteq Y$).

$X = \{a, \neg a, b, \neg b, a \to b, \neg(a \to b)\}$ is weakly connected, whether $\to$ is a material conditional (take $Y = \{\neg(a \to b), b\}$ in (ii) and $Y = \{a, a \to b, \neg b\}$ in (ii),(iii)) or a strict/subjunctive conditional (take $Y = \{a, a \to b, \neg b\}$ in (i)-(iii)).

Let us call an aggregation rule F *regular* if it satisfies universal domain and collective rationality (as defined in Section 2.2).

Theorem 1 *For a weakly connected agenda X, a regular aggregation rule F is systematic if and only if it is a dictatorship.*

This generalises a theorem by Pauly and van Hees (2004), which in turn genralises List and Pettit's (2002) original impossibility theorem. Nehring and Puppe (2004) prove a related theorem, which actually continues to hold in general logics (of type L1-L3): under an even weaker agenda assumption (only part (ii) of weak connectedness is needed), F is systematic *and monotone* if and only if it is dictatorial.

Proof. Several times, I will implicitly use L1-L3 and Proposition 2. Let X be weakly connected, and let F be regular. Put $N := \{1, ..., n\}$. If F is dictatorial, F is obviously systematic. Now assume F is systematic. Then there is a set $\mathcal{C}$ of ("winning") coalitions $C \subseteq N$ such that, for every $p \in X$ and every $(A_1, ..., A_n) \in Domain(F)$, $F(A_1, ..., A_n) = \{p \in X : \{i : p \in A_i\} \in \mathcal{C}\}$. For every consistent set $Z \subseteq X$, let A_Z be some consistent and complete judgment set such that $Z \subseteq A_Z$. Recall that X contains no doubly-negated propositions ($\neg\neg p$) and is a union of pairs $\{p, \neg p\}$. For simplicity, for any negated proposition $q = \neg p \in X$, let $\neg q$ stand for p rather than $\neg\neg p$.[8] (p and $\neg\neg p$ are essentially identical in the sense of Proposition 2(f), but only p is contained in X.)

Claim 1. $N \in \mathcal{C}$, and, for every coalition $C \subseteq N$, $C \in \mathcal{C}$ if and only if $N \backslash C \notin \mathcal{C}$.

The second part of the claim follows from collective rationality together with universal domain. Now assume $N \notin \mathcal{C}$. Let $Y \subseteq X$ as in part (i) of the definition of weakly connected agendas. Then $Y^{\neg} := \{\neg z : z \in Y\}$ is consistent. Let $(A_1, ..., A_n)$ be the profile for which $A_i = A_{Y^{\neg}}$ for all $i \in N$. As $N \notin \mathcal{C}$, $F(A_1, ..., A_n)$ contains no element of $Y^{\neg}$. So $Y \subseteq F(A_1, ..., A_n)$, violating the consistency of $F(A_1, ..., A_n)$.

Claim 2. For any coalitions $C, C^* \subseteq N$, if $C \in \mathcal{C}$ and $C \subseteq C^*$ then $C^* \in \mathcal{C}$.

Let $C, C^* \subseteq N$ with $C \in \mathcal{C}$ and $C \subseteq C^*$. Assume for contradiction that $C^* \notin \mathcal{C}$. Then $N \backslash C^* \in \mathcal{C}$. Let Y be as in part (iii) of the definition of a weakly connected agenda. So there exists a subset $Z \subseteq Y$ such that Z has even size and $(Y \backslash Z) \cup \{\neg z : z \in Z\}$ is consistent. Let $Z \subseteq Y$ be minimal satisfying these two properties. We have $Z \neq \emptyset$, since otherwise Y would equal $(Y \backslash Z) \cup \{\neg z : z \in Z\}$, and hence be consistent. So, as Z has even size, there are two distinct propositions $p, q \in Z$. Since Y is minimal inconsistent, $(Y \backslash \{p\}) \cup \{\neg p\}$ and $(Y \backslash \{q\}) \cup \{\neg q\}$ are each consistent. This and the consistency of $(Y \backslash Z) \cup \{\neg z : z \in Z\}$ allow us to define a profile $(A_1, ..., A_n) \in Domain(F)$ as follows. Putting $C_1 := C^* \backslash C$ and $C_2 := N \backslash C^*$ (note that $\{C, C_1, C_2\}$ is a partition of N), let

$$A_i := \begin{cases} A_{(Y \backslash \{p\}) \cup \{\neg p\}} & \text{if } i \in C \\ A_{(Y \backslash Z) \cup \{\neg z : z \in Z\}} & \text{if } i \in C_1 \\ A_{(Y \backslash \{q\}) \cup \{\neg q\}} & \text{if } i \in C_2. \end{cases} \quad (2)$$

[8] More precisely, when I use the negation symbol $\neg$ I mean a modified negation symbol $\sim$, where $\sim p := \neg p$ if p is unnegated and $\sim p := q$ if $p = \neg q$ for some q.

By (2), we have $Y\backslash Z \subseteq F(A_1, ..., A_n)$ as $N \in \mathcal{C}$. Also by (2), we have $q \in F(A_1, ..., A_n)$ as $C \in \mathcal{C}$, and $p \in F(A_1, ..., A_n)$ as $C_2 = N\backslash C^* \in \mathcal{C}$. In summary, writing $Z^* := Z\backslash\{p, q\}$, we have (*) $Y\backslash Z^* \subseteq F(A_1, ..., A_n)$. First assume $C_1 \notin \mathcal{C}$. Then $C \cup C_2 = N\backslash C_1 \in \mathcal{C}$. So $Z^* \subseteq F(A_1, ..., A_n)$ by (2), which together with (*) implies $Y \subseteq F(A_1, ..., A_n)$. But then $F(A_1, ..., A_n)$ is inconsistent, a contradiction. Hence $C_1 \in \mathcal{C}$. So $\{\neg z : z \in Z^*\} \subseteq F(A_1, ..., A_n)$ by (2). This together with (*) implies that $(Y\backslash Z^*) \cup \{\neg z : z \in Z^*\} \subseteq F(A_1, ..., A_n)$. So $(Y\backslash Z^*) \cup \{\neg z : z \in Z^*\}$ is consistent. As Z^* also has even size, the minimality condition in the definition of Z is violated.

Claim 3. For any coalitions $C, C^* \subseteq N$, if $C, C^* \in \mathcal{C}$ then $C \cap C^* \in \mathcal{C}$.

Consider any $C, C^* \in \mathcal{C}$. Let $Y \subseteq X$ be as in part (ii) of the definition of weakly connected agendas. As $|Y| \geq 3$, there are pairwise distinct propositions $p, q, r \in Y$. As Y is minimally inconsistent, each of the sets $(Y\backslash\{p\}) \cup \{\neg p\}$, $(Y\backslash\{q\}) \cup \{\neg q\}$ and $(Y\backslash\{r\}) \cup \{\neg r\}$ is consistent. This allows us to defined a profile $(A_1, ..., A_n) \in Domain(F)$ as follows. Putting $C_0 := C \cap C^*$, $C_1 := C^*\backslash C$ and $C_2 := N\backslash C^*$ (note that $\{C_0, C_1, C_2\}$ is a partition of N), let

$$A_i := \begin{cases} A_{(Y\backslash\{p\})\cup\{\neg p\}} & \text{if } i \in C_0 \\ A_{(Y\backslash\{r\})\cup\{\neg r\}} & \text{if } i \in C_1 \\ A_{(Y\backslash\{q\})\cup\{\neg q\}} & \text{if } i \in C_2. \end{cases} \tag{3}$$

By (3), $Y\backslash\{p, q, r\} \subseteq F(A_1, ..., A_n)$ as $N \in \mathcal{C}$. Again by (3), we have $q \in F(A_1, ..., A_n)$ as $C_0 \cup C_1 = C^* \in \mathcal{C}$. As $C \in \mathcal{C}$ and $C \subseteq C_0 \cup C_2$, we have $C_0 \cup C_2 \in \mathcal{C}$ by claim 2. So, by (3), $r \in F(A_1, ..., A_n)$. In summary, $Y\backslash\{p\} \subseteq F(A_1, ..., A_n)$. As Y is inconsistent, $p \notin F(A_1, ..., A_n)$, and hence $\neg p \in F(A_1, ..., A_n)$. So, by (3), $C_0 \in \mathcal{C}$.

Claim 4. There is a dictator.

Consider the intersection of all winning coalitions, $\widetilde{C} := \cap_{C\in\mathcal{C}} C$. By claim 3, $\widetilde{C} \in \mathcal{C}$. So $\widetilde{C} \neq \emptyset$, as by claim 1 $\emptyset \notin \mathcal{C}$. Hence there is a $j \in \widetilde{C}$. As j belongs to every winning coalition $C \in \mathcal{C}$, j is a dictator: indeed, for each profile $(A_1, ..., A_n) \in Domain(F)$ and each $p \in X$, if $p \in A_j$ then $\{i : p \in A_i\} \in \mathcal{C}$, so that $p \in F(A_1, ..., A_n)$; and if $p \notin A_i$ then $\neg p \in A_i$, so that $\{i : \neg p \in A_i\} \in \mathcal{C}$, implying $\neg p \in F(A_1, ..., A_n)$, and hence $p \notin F(A_1, ..., A_n)$. ■

As an application, consider the agendas introduced in Section 3.4 to represent a preference aggregation problem with set of options $C = \{c_1, ..., c_k\}$ and set of desirable rationality conditions $\mathcal{R}$. I have distinguished between the agenda $X_C := \{cRc', \neg cRc' : c, c' \in C\}$, where the rationality conditions are exogenously given (as in standard preference aggregation), and the agenda $X_{C,\mathcal{R}} := \{cRc', \neg cRc' : c, c' \in C\} \cup \{r, \neg r : r \in \mathcal{R}\}$, where the rationality conditions are endogenous, i.e. also under decision (the logic $(\mathbf{L}, \models)$ of each agenda is also defined in Section 3.4). For instance, suppose that $\mathcal{R}$ is the set of conditions defining a weak order *or* a linear order:

$$\begin{array}{l} \mathcal{R} = \{r_1, r_2\} \text{ (weak order) or } \mathcal{R} = \{r_1, r_2, r_3\} \text{ (linear order), where} \\ r_1 \text{ is } \forall v_1 \forall v_2 (v_1 R v_2 \vee v_2 R v_1) \text{ (completeness),} \\ r_2 \text{ is } \forall v_1 \forall v_2 \forall v_3 ((v_1 R v_2 \wedge v_2 R v_3) \rightarrow v_1 R v_3) \text{ (transitivity),} \\ r_3 \text{ is } \forall v_1 \forall v_2 ((v_1 R v_2 \wedge v_2 R v_1) \rightarrow v_1 = v_2) \text{ (asymmetry).} \end{array} \tag{4}$$

Then, if C contains more than two options, the agenda X_C is weakly connected: take $Y = \{\neg c_1 R c_1\}$ in (i), and $Y = \{c_1 R c_2, c_2 R c_3, \neg c_1 R c_3\}$ in (ii)-(iii). Interestingly, the agenda $X_{C,\mathcal{R}}$ is weakly connected even for two options: take $Y =$

$\{\neg c_1 R c_2, \neg c_2 R c_1, \forall v_1 \forall v_2 (v_1 R v_2 \vee v_2 R v_1)\}$ in (i)-(iii). So Theorem 1 has the following implication.

Corollary 1 *Let the set of rationality conditions $\mathcal{R}$ satisfy* (4)*.*

(a) (exogenous rationality) If there are $|C| \geq 3$ options, a regular aggregation rule for the agenda X_C is systematic if and only if it is a dictatorship.

(b) (endogenous rationality) If there are $|C| \geq 2$ options, a regular aggregation rule for the agenda $X_{C,\mathcal{R}}$ is systematic if and only if it is a dictatorship.

Part (a) is not much surprise in the light of Arrow's theorem (see also theorems by List and Pettit 2004 and Nehring 2003).[9] Part (b) shows that endogenising the rationality conditions does not help overcoming the impossibility: on the contrary, it extends the impossibility to the case of only two options.

5 Why judgment aggregation in general logics works

I now state what I take to be the basic technical tools in order to easily derive judgment aggregation results in general logics. Many past results are essentially based on these tools, and so they continue to hold in general logics. In Section 5.1, I focus on the framework L1-L3 and argue that it is appropriate if the only relevant rationality conditions are consistency and completeness. In Section 5.2, I discuss the slightly less general framework L1-L4, and argue that it may become relevant when the rationality condition of deductive closure is analysed.

5.1 Judgment aggregation in L1-L3: rationality as completeness and consistency

Suppose that the only rationality criteria of interest are *completeness* and *consistency* (and weak consistency), whose conjunction defines *full rationality.* Thus the remaining rationality condition, *deductive closure*, is not considered. The reason may be either that one considers only fully rational judgment sets, which are automatically deductively closed (see Proposition 1), or that one is simply not interested in deductive closure, whether or not all judgment sets are fully rational. Under this premise, the results that one may want to derive – such as results on the (im)possibility of fully rational aggregation or characterisations of consistent aggregation – can usually be based on a general logic of type L1-L3 (plus perhaps L5); L4 plays no role and need not be assumed.

Let me justify this claim. Such results appeal to the inconsistency notion *but not* to the underlying entailment relation: they depend only on properties of the system $\mathcal{I}$ ($\subseteq \mathcal{P}(\mathbf{L})$) of inconsistent sets, regardless of the particular entailment relation $\models$ that generates these inconsistent sets. What properties of inconsistency do the results depend on? The properties listed in Proposition 2 turn out to allow one to prove many results, such as Theorem 1 above, Nehring and Puppe's (2004) characterisation of consistent *voting by committees*, Dietrich and List's (2004) *liberal paradox* for judgment aggregation, and Dietrich and List's (2005) characterisation of consistent

[9] While Arrow's *independence of irrelevant alternatives* is weaker than our systematicity, his impossibility theorem also requires a Pareto condition.

quota rules. So, as Proposition 2 requires only L1-L3 (plus perhaps L5), all of these results are valid in general logics of type L1-L3 (plus perhaps L5 if the agenda X is infinite). Condition L4 would not add anything: while L1-L3 guarantee essential properties of the system of inconsistent sets $\mathcal{I}$, L4 implies *no* additional property of $\mathcal{I}$.[10] (One may alternatively take inconsistency as a primitive notion, i.e. define a logic not as a pair $(\mathbf{L}, \models)$ but as a pair $(\mathbf{L}, \mathcal{I})$, at the expense of loosing the notion of entailment and hence the rationality condition of deductive closure.[11])

Proposition 2 *(properties of inconsistency) Assume L1-L3. For any $A \subseteq \mathbf{L}$ and $p \in \mathbf{L}$,*

(a) if $p, \neg p \in A$ then A is inconsistent;

(b) if A is inconsistent then so is any superset of A;

(c) if A is consistent then so is any subset of A;

(d) if A is consistent then $A \cup \{p\}$ or $A \cup \{\neg p\}$ is consistent;

(e) for every consistent judgment set $B \subseteq X$, there is a fully rational judgment set $C \subseteq X$ with $B \subseteq C$;

(f) A is consistent if and only if A^* is consistent, for any set A^* arising from A by replacing elements $q \in A$ by one or more q-variants;[12]

(g) every finite inconsistent set has a minimal inconsistent subset;

(h) given L5, every inconsistent set has a finite minimal inconsistent subset.

The experienced reader will have noticed that these properties are indeed the basic tools underlying many proofs in judgment aggregation.[13] By part (f), any double-negated proposition $\neg\neg p$ is essentially identical to p with respect to inconsistencies: both stand in exactly the same inconsistency relations with other propositions.

Proof. (a) If $p, \neg p \in A$ then by L1-L2 $A \models p$ and $A \models \neg p$; so A is inconsistent.

(b), (c): These claims follow from L2.

(d) If A is consistent, then by L3 A has a consistent superset $C \subseteq \mathbf{L}$ containing a member of each pair $p, \neg p \in \mathbf{L}$. As C is a superset of either $A \cup \{p\}$ or $A \cup \{\neg p\}$, either of the latter sets is consistent by (c).

(e) Let $B \subseteq X$ be a consistent judgment set. By L3 B has a consistent superset $D \subseteq \mathbf{L}$ containing a member of each pair $p, \neg p \in \mathbf{L}$. So the judgment set $C := D \cap X$ is complete and by (c) consistent, and it satisfies $B \subseteq C$.

[10] The assumptions L1-L4 on $\models$ have exactly the same implication for the system of inconsistent sets $\mathcal{I}$ as the weaker assumptions L1-L3: $\mathcal{I}$ will be of the type I1-I3 (see footnote 11).

[11] One then needs to impose the following conditions on $(\mathbf{L}, \mathcal{I})$: (I1) for each $p \in \mathbf{L}$, we have $\{p, \neg p\} \in \mathcal{I}$; (I2) for all $A, B \subseteq \mathbf{L}$, if $A \in \mathcal{I}$ and $A \subseteq B$ then $B \in \mathcal{I}$; (I3) $\emptyset \notin \mathcal{I}$, and every $A \subseteq \mathbf{L}$ with $A \notin \mathcal{I}$ has a superset $B \subseteq \mathbf{L}$ with $B \notin \mathcal{I}$ containing a member of each pair $p, \neg p \in \mathbf{L}$. The conditions I1-I3 on $(\mathbf{L},\mathcal{I})$ are equivalent to the conditions L1-L3 on $(\mathbf{L}, \models)$: for any language $\mathbf{L} \neq \emptyset$, a system $\mathcal{I}$ $(\subseteq \mathcal{P}(\mathbf{L}))$ satisfies I1-I3 *if and only if* it is the set of inconsistent sets generated by some entailment relation $\models$ satisfying L1-L3 (if we even require L1-L4, $\models$ is unique and given by [$A \models p$ if and only if $A \cup \{\neg p\} \in \mathcal{I}$]). While inconsistency can be defined in terms of entailment, the converse is impossible (without assuming L4): starting from $(\mathbf{L}, \mathcal{I})$, one cannot define $\models$ (without assuming L4), and hence deductive closure is undefinable.

[12] q^* is called a q-variant (and q a q^*-variant) if one of q and q^* is a k-fold negation of the other for some *even* number $k \in \{0, 2, 4, ...\}$. For instance, q and $\neg\neg q$ are q-variants.

[13] For instance, to show that an aggregation rule F violates collective rationality, it is by (a) and (b) sufficient to construct a profile $(A_1, ..., A_n) \in \mathit{Domain}(F)$ such that $F(A_1, ..., A_n)$ contains some pair $p, \neg p$ or has some inconsistent subset; each person i's (fully rational) judgment set A_i can often be constructed using (e).

(f) Let A^* be as specified. I show that any set $S \subseteq \mathbf{L}$ is consistent if and only if the set $\widetilde{S} := \{q \in \mathbf{L} : q$ is a variant of some $r \in S\}$ is consistent. This implies the claim, since $\widetilde{A} = \widetilde{A^*}$. So, consider any $S \subseteq \mathbf{L}$. If $\widetilde{S}$ is consistent, then so is S by $S \subseteq \widetilde{S}$ and (c). Now let S be consistent. Then, by L3, S has a consistent superset $T \subseteq \mathbf{L}$ containing a member of each pair $q, \neg q \in \mathbf{L}$. I show that $\widetilde{S} \subseteq T$, which by (c) implies that $\widetilde{S}$ is consistent, as desired. To show $\widetilde{S} \subseteq T$, I have to prove that, for any $q \in S$, T contains all q-variants, i.e. all $q^* \in \mathbf{L}$ such that for some even $k \in \{0, 2, 4, ...\}$ (i) q^* is the k-fold negation of q *or* q is the k-fold negation of q^*. I only show case (i), as case (ii) can be shown analogously. Of course $q \in B$. By $q \in B$, we have $\neg q \notin B$ by (a), and hence $\neg\neg q \in B$ since B contains one of $\neg q, \neg\neg q$. Repeating this argument, one finds $\neg\neg\neg\neg q \in B$, then $\neg\neg\neg\neg\neg\neg q \in B$, etc., as claimed.

(g) Let $B \subseteq \mathbf{L}$ be any finite inconsistent set. Among all inconsistent subsets of B, choose a minimal one (with respect to inclusion); there exists one since B is finite. This set is minimal inconsistent.

(h) Any inconsistent set has by L5 a finite inconsistent subset, and hence by (g) a finite minimal inconsistent subset. ■

5.2 Judgment aggregation in L1-L4: analysing deductive closure

In addition to completeness and consistency, it may be interesting to analyse the rationality condition of *deductive closure*; this analysis may require the additional assumption of L4 (non-paraconsistency). One motivation for studying deductive closure can be derived from the various impossibility theorems. Let me explain how. These theorems tell us that fully rational aggregation is often unrealistic. But if collective judgments cannot be fully rational, what weaker form of rationality should one aim at? There are (at least) two approaches:

• *Relaxing completeness, but keeping consistency and deductive closure.* Here, the collective abstains from a decision on certain pairs $p, \neg p$, but otherwise forms judgments that are not only logically consistent, but also deductively closed, i.e. whenever a proposition $p \in X$ follows from the collectively accepted propositions, p is also accepted. It is then interesting to analyse which aggregation rules generate consistent and deductively closed judgment sets. As seen in Proposition 3 below, some tools to analyse deductive closure hold for all logics of type L1-L3, but many others require a logic of type L1-L4. Dietrich and List's (2005) characterisation of the class of all consistent and deductively closed quota rules holds in logics of type L1-L4 (plus L5 if the agenda X is infinite).

• *Relaxing consistency to weak consistency, but keeping completeness and deductive closure.* Here, the collective accepts exactly one member of each pair $p, \neg p$, in a deductively closed but perhaps not consistent way. This form of restricted rationality exists only in logics violating L4, because, under L1-L4, by keeping completeness, weak consistency and deductive closure one actually keeps consistency (see Proposition 1). But the escape route may be an interesting option in logics violating L4, i.e. in paraconsistent logics with various degrees of consistency. In some real situations, relaxing consistency may even be the *only* feasible collective rationality relaxation, because completeness cannot be given up since a decision on each pair $p, \neg p \in X$ is strictly required. Obviously, the analysis of such rationality relaxations should not impose L4, hence cannot use the tools (d)-(f) in Proposition 3.

Proposition 3 *(properties of entailment) Assume L1-L3. For any* $A \subseteq \mathbf{L}$ *and* $p \in \mathbf{L}$*,*

(a) if $p \in A$ *then* $A \models p$*;*

(b) if $A \models p$ *then* $A \cup \{\neg p\}$ *is inconsistent; in particular,* $\emptyset$ *entails only tautologies;*

(c) if $A \models p$ *and* A *is consistent then* $A \cup \{p\}$ *is consistent;*

(d) given L4, each inconsistent set entails any proposition;

(e) given L4, $A \models p$ *if and only if* $A \cup \{\neg p\}$ *is inconsistent.*

(f) given L4, $A \models p$ *if and only if* $A^* \models p^*$*, for any p-variant* p^* *and any set* A^* *arising from* A *by replacing elements* $q \in A$ *by one or more q-variants.*[12]

Under L1-L4 the entailment relation $\models$ can be retrieved from the inconsistency notion using (e); see also footnote 11. By part (f), under L1-L4 double-negations "$\neg\neg$" have no effect on entailments.

Proof. (a) follows from L1 and L2.

(b) If $A \models p$, then $A \cup \{\neg p\}$ is inconsistent since $A \cup \{\neg p\} \models p$ by L2 and $A \cup \{\neg p\} \models \neg p$ by L1 and L2.

(c) Assume $A \models p$ where A is consistent. By L3, $A \cup \{p\}$ or $A \cup \{\neg p\}$ is consistent. As $A \cup \{\neg p\}$ is inconsistent by (b), $A \cup \{p\}$ is consistent.

(d) For any inconsistent set $A \subseteq \mathbf{L}$ and any $p \in \mathbf{L}$, by L2 $A \cup \{\neg p\}$ is inconsistent, and hence by L4 $A \models p$.

(e) One direction follows from (b), the other one from L4.

(f) Let A^* and p^* be as specified. By (e) and L4, $A \models p$ if and only if $A \cup \{\neg p\}$ is inconsistent, and $A^* \models p^*$ if and only if $A^* \cup \{\neg p^*\}$ is inconsistent. So the claim follows from Proposition 2(f). ■

With Propositions 2 and 3 in place, I can now prove Proposition 1 on the relations between the various rationality conditions on judgment sets.

Proof of Proposition 1. Assume L1-L3 and let $A \subseteq X$ be any judgment set.

(a) If A is consistent then it is weakly consistent by Proposition 2(c). Now assume A is deductively closed. Of course, consistency still implies weak consistency. Conversely, suppose A is not consistent. So A entails each $p \in X$ by Proposition 3(d). Hence, by deductive closure, $A = X$. So A is not weakly consistent.

(b) First, let $A \subseteq X$ be fully rational. Then A is complete, and Proposition 2(c) weakly consistent. To prove deductive closure, consider any $p \in X$ such that $A \models p$. By $A \models p$, the set $A \cup \{\neg p\}$ is inconsistent by Proposition 3(b). First assume p is not a negated proposition; then $p, \neg p \in X$ by the definition of agendas. Since A is consistent, $A \neq A \cup \{\neg p\}$, and hence $\neg p \notin A$, which implies $p \in A$ by A's completeness. Now assume p is a negated proposition, say $p = \neg q$; then $q, \neg q \in X$ by the definition of agendas. Since $A \cup \{\neg p\} = A \cup \{\neg\neg q\}$ is inconsistent, so is $A \cup \{q\}$ by Proposition 2(f). So, as A is consistent, $A \neq A \cup \{q\}$, and hence $q \notin A$, which implies $\neg q = p \in A$ by A's completeness.

Now assume also L4 and let A be complete, weakly consistent and deductively closed. Assume for contradiction that A is not fully rational. Then, as A is complete, it is not consistent. So, by (a) and A's deductive closure, A is not weakly consistent, contradiction the assumption. ■

6 Judgment aggregation in general logics vs. in a specific logic

Not all results on judgment aggregation can or should be stated in general logics: some results require a specific logic, but this logic need not be standard propositional logic.

The general logics model is appropriate for results about agendas (decision problems) that are characterised by certain *logical relations* (of inconsistency or entailment) *between* the propositions, regardless of the specific logic generating these relations or the specific syntactic form of the propositions. Examples are the *weakly connected* agenda of Theorem 1, and, if suitably redefined in the general logics model, Gärdenfors' *Boolean algebra* agenda, Dietrich's (2004) *atomic* agenda, and Nehring and Puppe's (2004) *totally blocked* agenda.

By contrast, the restriction to a specific logic is required for results about agendas (decision problems) that are characterised by propositions *of a particular syntactic form*, available in a particular logic. I give two examples. Dietrich's (2005) possibility theorem holds for *network* agendas X, which belong the conditional logic C^+ and contain only propositions of the following syntactic forms (and their negations): atomic propositions $a, b, c, ...$, and *connection rules* $p \rightarrow q$ or $p \leftrightarrow q$, where p and q are conjunctions $a_1 \wedge ... \wedge a_k$ of atomic propositions $a_1, ..., a_k$, $k \geq 1$. Pauly and van Hees' (2004) Theorem 3 holds for *atomically closed* agendas X, which belong to standard propositional logic and contain (i) each atomic proposition a that occurs in some proposition in X, and (ii) the propositions $a \wedge b, \neg a \wedge b, a \wedge \neg b, \neg a \wedge \neg b$ for any atomic propositions $a, b \in X$. Network agendas and atomically closed agendas are not even definable in a general logic $(\mathbf{L}, \models)$.

7 Conclusion

I have argued that a large variety of collective decision problems can be studied within judgment aggregation, where each decision problem requires a particular logic that can express the propositions under consideration. Given the multitude of logics and their complexity, one might have feared that a separate approach is needed for each type of decision problem (each logic), and that judgment aggregation looses its unity and simplicity as a field. But, fortunately, it is possible to work and prove results in a simple and general model not restricted to any particular logic. I have presented a model of judgment aggregation in general logics, open to any logic satisfying the minimal conditions L1-L3 (and perhaps L4,L5); this includes standard propositional logic as well as many modal, conditional, and predicate logics.

Despite of the generality of the conditions, they allow one to derive interesting results. In order to demonstrate this, I have first generalised existing impossibility theorems about systematic aggregation rules to general logics; and then I have shown that many tools underlying the typical proofs in judgment aggregation are also available in general logics. The framework L1-L3 (plus perhaps L5 if the agenda is infinite) is often appropriate when consistency and completeness are the only rationality conditions under consideration. Condition L4 may become additionally relevant if one studies the rationality condition of deductive closure.

I hope to have convinced the reader that working with general logics does not make judgment aggregation more difficult, but far more general, and more transparent by removing unnecessary special assumptions.

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