The possibility of judgment aggregation for network agendas

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Abstract. Within social choice theory, the new field of judgment aggregation aims at reaching collective judgments on a set of logically interconnected propositions. I investigate decision problems, in which the agenda is a *network*, composed of atomic propositions and connection rules between them. Networks can represent various realistic decision problems, including most concrete examples given in the literature. Nevertheless, networks are unexplored so far due to problems when modelling connection rules in standard propositional logic. By extending the logic, I prove that, for any network, decision rules satisfying the common conditions always exist, in contrast to the literature's emphasis on impossibilities. I also characterise the class of such decision rules, and propose a simple way to select a decision rule.

Key words: judgment aggregation, collective inconsistency, possibility theorems, network, connection rule, formal logic, material conditional, subjunctive conditional

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1 Introduction

The newly arising model of *judgment aggregation* takes a different approach to collective decision-making from the classical model of preference aggregation. A collective decision is not represented as *one* choice between *many* alternatives, but as *many* decisions on *single* propositions (issues, questions), where the propositions are interconnected. The new model is very general, since propositions can represent *beliefs* (e.g. "pollution creates global warming"), *desires* (e.g. "global warming is undesirable") or *act preferences* (e.g. "measure X against pollution should be taken"). Moreover, the model is close to real decision situations, for at least two reasons. First, it does not require individuals to rank many complex alternatives (I am rarely able) but only to have an opinion on different issues; and, second, it accounts for the fact that real choice situations in committees, management boards, governments, societies etc. often consist indeed in reaching decisions on several interconnected issues, such as whether to believe that a rule has been abused, whether to abolish a rule, whether to hire someone, etc.

In this paper, I analyse an important type of judgment aggregation problem: deciding simultaneously whether to accept certain atomic propositions and whether to accept certain links (constraints) between these propositions. For instance, consider a company board that has to decide on its product management strategy. The board members disagree on whether the following atomic propositions are true or false.

a : the demand for products sold under an old brand is about to decline;

b : a new brand should be created now;

c : more money should be spent on marketing.

Not surprisingly, they also disagree on the *connections* between a, b and c. Some believe that $a \to b$ ("*if* the demand for products sold under an old brand is about to decline *then* a new brand should be created now"). Others believe that $c \leftrightarrow a$ ("more money should be spent on marketing *if and only if* the demand for products sold under an old brand is about to decline"). Even others think that $a \to (b \wedge c)$ ("if a then [b and c]"). Each connection rule establishes a *constraint* on the decisions on a, b and c: if $a \to b$ is accepted it becomes inconsistent to accept a and reject c; if $c \leftrightarrow a$ is accepted it becomes inconsistent to accept exactly one of a and c; and if $a \to (b \wedge c)$ is accepted it becomes inconsistent to accept a and reject b or c (or both).

Reaching a collective decision is non-trivial. The decisions on a, b, c should neither ignore some person's opinion on constraints between a, b, c, nor blindly accept these constraints. So, the group must decide both on a, b, c *and* on each connection rule. However, the natural proposal of taking a majority vote on each (atomic or non-atomic) proposition often leads to inconsistent outcomes, as illustrated in Table 1.

	a	b	c	$a \to b$	$c \leftrightarrow a$	$a \to (b \wedge c)$
1/3 of the board	Yes	Yes	No	Yes	No	No
1/3 of the board	No	No	No	Yes	Yes	Yes
1/3 of the board	Yes	No	Yes	No	Yes	No
Majority	Yes	No	No	Yes	Yes	No

Table 1: Inconsistencies under majority voting

In Table 1, two connection rules, namely $a \to b$ and $c \leftrightarrow a$, receive majority approval, but the majority decisions on a, b and c violate these two constraints: given the acceptance of a, the constraint $a \to b$ would have required the acceptance of b, and the constraint $c \leftrightarrow a$ would have required the acceptance of c.

By Table 1, propositionwise majority rule may generate logically inconsistent collective judgments for the agenda at hand. Are there any other "acceptable" decision rules that avoid this flaw? A negative answer would be no surprise, given the various impossibility theorems cited below. Many of these theorems would indeed apply to the agenda at hand if we were to model the connection rules ($a \to b, c \leftrightarrow a, a \to (b \wedge c)$) as *material* (bi)conditionals. However, as I will show, this would be a misrepresentation of connection rules. Under an adequate representation of connection rules, the agenda at hand and more generally all *networks* do allow for "acceptable" decision rules with consistent outcomes, a perhaps somewhat surprising finding.

In Section 2, I introduce the model of judgment aggregation. In Section 3, I define the logic used to adequately represent connection rules. In Section 4, I prove the main possibility result for networks. In Section 5, I come to a full characterisation of the relevant class of decision rules, by focussing on so-called *simple* networks. In Section 6, I propose easy methods to construct decision rules. Section 7 contains conclusions.

On an informal basis, judgment aggregation has been discussed already for a while (e.g. Kornhauser and Sager 1986, Pettit 2001, Brennan 2001, Chapman 2002). List and Pettit (2002) formalise judgment aggregation using standard propositional logic, and prove a first impossibility result. Stronger impossibility results are derived by Pauly and van Hees (2004), Dietrich (2004a/b), Gärdenfors (2004), Nehring and Puppe (2004) and van Hees (2004). To escape impossibilities, one may for instance restrict the domain of the aggregation rule (List 2003), restrict the independence condition to *premises* (Dietrich 2004a), use *fusion operators* (Pigozzi 2004), or use *sequential* decision rules (List 2004b and Dietrich and List 2005). The models of Pauly and van Hees (2004) and van Hees (2004) allow for *degrees* of acceptance. The probability of "correct" collective judgments is analysed in by Bovens and Rabinowicz (2004) and List (2004a). Characterisations of the class of aggregation rules satisfying various collective rationality conditions (and other conditions) are provided in Nehring and Puppe (2004) and Dietrich and List (2005).

2 Basic notions of the model

Consider a group of persons $1, ..., n$ ($n \geq 2$), facing a collective decision problem.

The language. Following the generalised model of judgment aggregation presented in Dietrich (2004b), propositions need not be stated in standard propositional logic. Rather, one can use any logic satisfying mild conditions, such as (L1)-(L5) in Dietrich (2004b).[1] This permits several more expressive logics, including the *conditional logic* used in this paper to define networks. The language $\mathbf{L}$ of our conditional logic contains *atomic propositions* without logical connectives and *non-atomic propositions* with the logical operators $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (if-then), $\leftrightarrow$ (if and only if). As defined later, $\rightarrow$ and $\leftrightarrow$ are *subjunctive*, not *material* (bi)conditionals. Formally, $\mathbf{L}$ is the (smallest) set such that (i) $\mathbf{L}$ contains the given *atomic propositions* a, b, c, ..., and (ii) whenever $\mathbf{L}$ contains two propositions p and q, then $\mathbf{L}$ also contains $\neg p$, $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$ and $(p \leftrightarrow q)$. For convenience, I drop brackets when there is no ambiguity, e.g. I write $c \rightarrow (a \wedge b \wedge c)$ for $(c \rightarrow ((a \wedge b) \wedge c))$.

The semantics of the logic, especially of $\rightarrow$ and $\leftrightarrow$, is defined in the next section.

The agenda. The *agenda* is the set of propositions under consideration; it is a non-empty subset $X \subseteq \mathbf{L}$, where (i) X contains no doubly-negated propositions ($\neg\neg p$), and (ii) X is a union of proposition-negation pairs $\{p, \neg p\}$. Thus

$$X = \{p, \neg p : p \in X^+\}, \text{ where } X^+ := \{p \in X : p \text{ is not a negated proposition}\}.$$

The agenda of the example in the introduction is given by $X^+ = \{a, b, c, a \rightarrow b, c \leftrightarrow a, a \rightarrow (b \wedge c)\}$.

Judgment sets. A *judgment set* (held by a person or the collective) is a subset $A \subseteq X$, where $p \in A$ stands for "the person/the collective accepts proposition p". A judgment set A can be more or less rational. It is *fully rational* if it is both *complete* (i.e. $p \in A$ or $\neg p \in A$ for each pair $p, \neg p \in X$) and *(logically) consistent* (as defined in the next section).

[1] Formally, a *logic* (*with negation operator*) consists of (i) a non-empty set $\mathbf{L}$ (of "propositions") such that if $p \in \mathbf{L}$ then $\neg p \in \mathbf{L}$ (i.e. each proposition can be negated), and (ii) an ("entailment") relation $\models (\subseteq \mathcal{P}(\mathbf{L}) \times \mathbf{L})$, relating sets $A \subseteq \mathbf{L}$ to propositions $p \in \mathbf{L}$; "$A \vDash p$" is read "A (logically) entails p". $\mathbf{L}$ defines the syntax and $\models$ the semantics of the logic. $\mathbf{L}$ is the set of all formable sentences; $\mathbf{L}$ might contain the propositions $a, b, \neg c, c \wedge d, a \rightarrow c$. $\vDash$ represents the rules according to which certain sets of propositions entail other propositions; one might have $\{a, a \rightarrow b\} \models b$. A set $A \subseteq \mathbf{L}$ is "*inconsistent*" if there is a $p \in \mathbf{L}$ such that $A \vDash p$ and $A \vDash \neg p$.

Aggregation rules. A *profile* is an n-tuple $(A_1, \ldots, A_n)$ of (individual) judgment sets $A_i \subseteq X$. A *(judgment) aggregation rule* is a function F that maps each profile $(A_1, \ldots, A_n)$ in a given domain to a *(collective) judgment set* $F(A_1, \ldots, A_n) = A \subseteq X$. Often, the domain of F is the *universal domain*, i.e. the set of all profiles of fully rational judgment sets. F is *complete/consistent/fully rational* if F generates a complete/consistent/fully rational collective judgment set for each profile in its domain. For instance:

• F is *(propositionwise) majority rule* if $F(A_1, ..., A_n) = \{p \in X$: more persons i have $p \in A_i$ than $p \notin A_i\}$ for all $(A_1, ..., A_n)$ in the domain of F. Defined on the universal domain, this rule is not consistent, as seen in the introduction.

• F is a *dictatorial* rule if there is a person j (the "dictator") such that $F(A_1, ..., A_n) = A_j$ for all $(A_1, ..., A_n)$ in the domain of F. Defined on the universal domain, this rule is fully rational, but of course far from democratic.

3 A logic that can express connection rules

Requirements on the representation of connection rules. It is crucial to use a logic that ascribes the intended meaning to connection rules such as $a \rightarrow b, c \leftrightarrow a, a \rightarrow (b \wedge c)$. Specifically, the logic should satisfy the following two conditions.

(a) The *acceptance* of a connection rule p establishes exactly the intended logical constraints on judgments on atomic propositions, i.e. p is consistent with exactly the "right" sets atomic propositions and negated atomic propositions. For instance, $a \rightarrow b$ is inconsistent with $\{a, \neg b\}$ but consistent with each of $\{a, b\}$, $\{\neg a, b\}$, $\{\neg a, \neg b\}$.

(b) The *rejection* of a connection rule p does *not* constrain the judgments on atomic propositions, i.e. $\neg p$ is consistent with *each* (consistent) set of atomic propositions and negated atomic propositions. For instance, $\neg(a \rightarrow b)$ is consistent with each of $\{a, b\}$, $\{a, \neg b\}$, $\{\neg a, b\}$, $\{\neg a, \neg b\}$.

To illustrate (b), consider again our starting example, in which a is "the demand for products sold under an old brand is about to decline", and b is "a new brand should be created now". Consider an agent who believes that $\neg(a \rightarrow b)$, i.e. that a forthcoming decline in demand does *not* imply the need for a new brand. This belief is intuitively perfectly consistent with any opinions on a and b: the agent might or might not believe that demand will decline, and might or might not believe that a new brand is needed. The logic should respect this intuition.

The failure of the material conditional. Material (bi)conditionals (used in standard propositional logic) satisfy (a) but not (b). Consider $a \rightarrow b$. Interpreted as a material conditional, $a \rightarrow b$ is equivalent to $\neg a \vee b$ (a is false or b is true), and $\neg(a \rightarrow b)$ is equivalent to $a \wedge \neg b$ (a is true and b is false). So, under a material interpretation,

• (a) holds, because $a \rightarrow b$ is inconsistent with $\{a, \neg b\}$ (as desired) and consistent with each of $\{a, b\}, \{\neg a, b\}, \{\neg a, \neg b\}$ (as desired);

• (b) is violated, because $\neg(a \rightarrow b)$, far from imposing no constraints, is inconsistent with all sets containing $\neg a$ or containing b.

It is well-known that material conditionals misrepresent the intended meaning of most conditional statements in common language. The (in common language clearly false) conditional "if the earth falls on the sun then we freeze" is *true* under a material interpretation because the earth does *not* fall on the sun. This clash between our intuition and the material conditional is due to the fact that, in common language, "if a then b" is not a statement about whether a and b are true in the *actual* world, but about whether b is true in world(s) in which a is true, for instance in worlds in which the earth falls on the sun; in other words, "if a then b" means "if a *were* true ceteris paribus, then b *would be* true", not "a *is* false or b *is* true".

A conditional logic. By the last remarks, the truth value of $a \to b$ is not simply a function of the actual truth values of a and b, but of their truth values in *possible worlds*. This leads to the notion of a *subjunctive conditional*, which was formalised by D. Lewis (1973) using a *conditional logic* and is now well-established in non-classical logic. I will use a standard version of conditional logic, often denoted C^+ (other related versions could also be used). For further reference, e.g. Priest (2001).

The language $\mathbf{L}$ was already defined in Section 2. I now endow $\mathbf{L}$ with semantics. For comparison, recall that in *standard* propositional logic (not in our logic!) an *interpretation* is given by a ("truth") function $v : \mathbf{L} \to \{T, F\}$, assigning to each proposition a truth value, such that, for any propositions $p, q \in \mathbf{L}$,

($\neg$) $v(\neg p) = T$ if and only if $v(p) = F$,

($\wedge$) $v(p \wedge q) = T$ if and only if $v(p) = T$ and $v(q) = T$,

($\vee$) $v(p \vee q) = T$ if and only if $v(p) = T$ or $v(q) = T$,

($\to_{\text{material}}$) $v(p \to q) = T$ if and only if $v(p) = F$ or $v(q) = T$,

($\leftrightarrow$) $v(p \leftrightarrow q) = T$ if and only if $v(p \to q) = T$ and $v(q \to p) = T$.

By contrast, I define an *interpretation* as a triple $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$, where:

- W is a non-empty set, whose members are called *(possible) worlds*;
- $(R_p)_{p \in \mathbf{L}}$ is a family of binary relation on W (wR_pw' is interpreted as "world w' is similar to world w, and p is true in w'"), such that, for any $w, w' \in W$ and $p \in \mathbf{L}$, (i) if wR_pw' then $v_{w'}(p) = T$ (an obvious requirement given the interpretation of wR_pw') and (ii) if $v_w(p) = T$ then wR_pw (since w is similar to itself);
- $(v_w)_{w \in W}$ is a family of ("truth") functions $v_w : \mathbf{L} \to \{T, F\}$, assigning to each proposition $p \in \mathbf{L}$ its truth value $v_w(p)$ in world $w \in W$, such that, for each world $w \in W$ and any propositions $p, q \in \mathbf{L}$, the truth function $v = v_w$ satisfies ($\neg$), ($\wedge$), ($\vee$), ($\leftrightarrow$) and

($\to$) $v_w(p \to q) = T$ if and only if $v_{w'}(q) = T$ for each world $w' \in W$ with wR_pw'.

By ($\to$), $p \to q$ is true in world w just in case q is true in every world w' similar to w and with true p. This captures the intuition of conditional statements. For instance, "if the earth falls on the sun then we freeze" is plausibly false, because we do *not* freeze in those worlds similar to the actual world except that the earth falls on the sun. The requirements (a) and (b) on the representation of connection rules are now both satisfied.[2]

By definition, a set of propositions $A \subseteq \mathbf{L}$ is *(logically) consistent* if some world of some interpretation makes each $p \in A$ true, i.e. if there exists an interpretation $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$ and a world $w \in W$ such that $v_w(p) = T$ for all $p \in A$. Finally, a set $A \subseteq \mathbf{L}$ *(logically) entails* a proposition $p \in \mathbf{L}$ ("$A \vDash p$") if, for every interpretation $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$ and every world $w \in W$ such that $v_w(q) = T$

[2] Regarding (b) (where the material conditional fails), this follows from Lemma 1 applied to sets A containing no (non-negated) connection rules (note that (1) and (2) are then vacuously true).

for all $q \in A$, we have $v_w(p) = T$. Intuitively, A is consistent in case all members of A *can be* simultaneously true, and A entails p if *whenever* each $q \in A$ is true p is true.

4 A general possibility theorem

After having defined our logic, including our subjunctive interpretation of $\rightarrow$ and $\leftrightarrow$, I come to the possibility theorem for networks.

Definition 1 *(a) A* connection rule *is a (subjunctive) conditional $p \rightarrow q$ or biconditional $p \leftrightarrow q$, where p and q are atomic propositions or conjunctions of many atomic propositions.*

(b) The agenda X is a network *if each $p \in X^+$ is either an atomic proposition or a connection rule.*

In a network, atomic propositions represent particular issues, and connection rules represent potential links between these issues. If a connection rule is accepted, it establishes a constraint on how to decide issues; if it is rejected, there is no constraint.

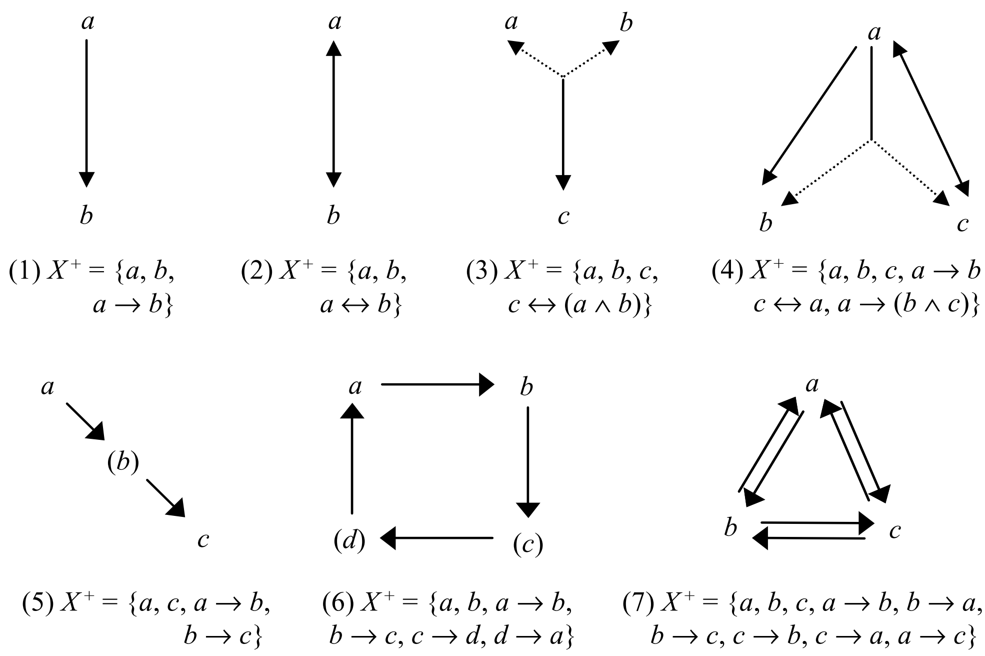

Figure 1: Examples of networks

Figure 1 shows seven networks, where the network (4) represents the marketing strategy example of the introduction. In general, each network X can be represented by a graph as follows.

• Nodes contain atomic propositions, where an atomic proposition is put in brackets if it is not contained in X but only occurs in a connection rule contained in X;

• Each arrow represents a connection rule $p \rightarrow q$ or $p \leftrightarrow q$ in X, where the arrow is bidirectional in the case of a biconditional, and where the arrow contains bifurcations in the case that p and/or q is a conjunction of more than one atomic proposition.

Let me give some concrete examples illustrating the broad applications of networks.

• The agenda is either of the networks (3), (4), (5), (7), where a, b, c are as in the introductory example of a company board deciding its marketing strategy. In each of these networks, a different set of connection rules (constraints) is under consideration.

• An expert commission has to reach collective judgments on environmental issues. Their agenda is the network (1), where

a : carbon dioxide emissions cause global warming;

b : taxes on kerosene should be introduced.

• The government of a state has to decide on its anti-terrorism policy. The agenda is the network (2), where (as in Dietrich and List 2005)

a : country X has weapons of mass destruction;

b : action Y should be taken against country X.

• As in the notorious *doctrinal paradox* (e.g. Kornhauser and Sager 1986), a court has to decide a trial against a company. The agenda is the network (3), where

a : the company has broken the contract (with another company);

b : the contract was legally valid;

c : the company is liable (to pay damages to the other company).

If the agenda X is a network, does there exist any "acceptable" procedure to reach collective judgments? A procedure is standardly considered as "acceptable" if it generates rational collective judgments *and* is suitably democratic. The rationality requirement means (for networks) essentially that the judgments on atomic propositions should respect all accepted connection rules. The democraticity requirement can be interpreted to mean that the aggregation rule should satisfy *independence*, *anonymity*, *responsiveness*, *monotonicity*, and *universal domain*; for definitions of these standard conditions, see for instance Dietrich and List (2005) (the conditions also imply *strategy-proofness*; see Dietrich and List 2004). This is a demanding interpretation, illustrating the strength of the below possibility result. The most notable condition is *independence*, which requires propositionwise aggregation: the collective judgment on each proposition should be determined solely by the individual judgments on this proposition, regardless of any individual judgments on other propositions – an analogue of Arrow's *independence of irrelevant alternatives* for preference aggregation.

As is easily shown, for any agenda X, the class of aggregation rules satisfying the above democraticity conditions is the class of *quota rules*, defined in Dietrich and List (2005) and closely related to Nehring and Puppe's (2004) *voting by quota*. Under a quota rule, each proposition $p \in X$ is accepted if and only if the number of persons accepting it exceeds some (proposition-specific) threshold. Specifically, consider any *family of thresholds* $(m_p)_{p \in X}$, where $m_p \in \{1, ..., n\}$ is the threshold for proposition $p \in X$. To each such family corresponds a quota rule, $F_{(m_p)_{p \in X}}$, defined as the aggregation rule with universal domain given by

$$F_{(m_p)_{p \in X}}(A_1, ..., A_n) := \{p \in X : \text{at least } m_p \text{ persons } i \text{ have } p \in A_i\}.$$

A natural requirement is that $m_{\neg p} = n + 1 - m_p$ for any pair $p, \neg p \in X$, which ensures that exactly one member of each pair $p, \neg p \in X$ is accepted under each profile[3] (implying *completeness*). A quota rule with this property can be written as $F_{(m_p)_{p\in X^+}}$, since the threshold for any $\neg p \in X$ follows from that for $p \in X^+$. More precisely, to each family $(m_p)_{p\in X^+}$ of numbers in $\{1, ..., n\}$ corresponds a quota rule, $F_{(m_p)_{p\in X^+}}$, defined as

$$F_{(m_p)_{p\in X^+}} := F_{(m_p)_{p\in X}}, \text{ where } m_{\neg p} := n + 1 - m_p \text{ for each } \neg p \in X.$$

A quota rule $F_{(m_p)_{p\in X^+}}$ is an attractive solution to the aggregation problem *provided* that it is consistent, i.e. generate consistent judgment sets. But does there exist any consistent quota rules $F_{(m_p)_{p\in X^+}}$?

Theorem 1 *If the agenda is a network, there exists a consistent quota rule* $F_{(m_p)_{p\in X^+}}$.

Corollary 1 *If the agenda is a network, there exists a fully rational (i.e. consistent and complete), independent, anonymous, monotonic and responsive aggregation rule with universal domain.*

This possibility result may appear surprising. Previous results suggest, in short, that agendas with sufficiently interconnected propositions lead to impossibilities. By contrast, the above result holds for all networks, however highly interconnected they might be. The possibility relies crucially on the use of *subjunctive* conditionals. Indeed, for most networks – for instance the networks (2)-(7) in Figure 1 – the possibility would disappear if the connection rules were replaced by *material* (bi)conditionals. So, some impossibilities in the literature (such as Proposition 1 in Dietrich and List 2004) are artefacts of the use of material (bi)conditionals.

To prove Theorem 1, I start with a lemma. Two rationality conditions on judgment sets $A \subseteq X$ are:

$$\text{for any connection rule } p \in A, \text{ we have } \neg p \notin A; \tag{1}$$

$$\begin{array}{l}\text{for any set of connection rules } S \subseteq A \text{ and} \\ \text{any connection rule } p \in X, \text{ if } S \models p \text{ then } p \in A.\end{array} \tag{2}$$

Lemma 1 *Let X be a network. Any set $A \subseteq X$ satisfying (1) and (2) is consistent if and only if its subset $\{p \in A : p$ is not a negated connection rule$\}$ is consistent.*

Proof. Suppose X is a network and $A \subseteq X$ satisfies (1) and (2). Put $A_0 := \{p \in A$: p is not a negated connection rule$\}$. Obviously, if A is consistent then so is A_0. Now assume A_0 is consistent.

Claim 1: For any $\neg(p \to q) \in A$, there exists a conjunct of q, denoted $a_{p\to q}$, which is not a conjunct of p nor of any q' such that $p \to q' \in A$ or $p \leftrightarrow q' \in A$ or $q' \leftrightarrow p \in A$.

Assume for contradiction that the claim is false for $\neg(p \to q) \in A$. I show that $p \to q \in A$, which contradicts (1). Let S be the set of all propositions in A of the form $p \to q'$ or $p \leftrightarrow q'$ or $q' \leftrightarrow p$. By (2), it is sufficient to show that $S \models p \to q$. So, consider any world w of any interpretation $(W, (R_r)_{r\in\mathbf{L}}, (v_w)_{w\in W})$ such that each member of S is true in w. To show that $p \to q$ is true in w, consider any world $w^* \in W$ such that wR_pw^*. I have to show that q is true in w^*, i.e. that each conjunct

[3] This follows because the number of persons accepting p is n minus the number accepting $\neg p$.

of q is true in w^*. Let a be any conjunct of q. If a is a conjunct of p, a is true in w^* since p is true in w^*. Now let a not be a conjunct of p. Then, by assumption, there exists an $r \in S$, where r is either $p \to q'$ or $p \leftrightarrow q'$ or $q' \leftrightarrow p$, such that a is a conjunct of q'. As r is true in w and $wR_p w^*$, q' is true in w^*. Hence a is true in w^*, q.e.d.

Claim 2: For any $\neg(p \leftrightarrow q) \in A$,

(α) *either* there exists a conjunct of q, denoted $a_{\hat{p}\leftrightarrow q}$, which is not a conjunct of p nor of any q' such that $p \to q' \in A$ or $p \leftrightarrow q' \in A$ or $q' \leftrightarrow p \in A$,

(β) *or* there exists a conjunct of p, denoted $a_{p\leftrightarrow\hat{q}}$, which is not a conjunct of q nor of any p' such that $q \to p' \in A$ or $q \leftrightarrow p' \in A$ or $p' \leftrightarrow q \in A$.

Assume for contradiction that neither (α) nor (β) holds for $\neg(p \leftrightarrow q) \in A$. I show that $p \leftrightarrow q \in A$, which contradicts (1). Let S_p be the set of all propositions in A of the form $p \to q'$ or $p \leftrightarrow q'$ or $q' \leftrightarrow p$, and S_q the set of all propositions in A of the form $q \to p'$ or $q \leftrightarrow p'$ or $p' \leftrightarrow q$. By (2), it is sufficient to show that $S_p \cup S_q \models p \leftrightarrow q$. So it is sufficient to show that $S_p \models p \to q$ and that $S_q \models q \to p$. These entailments can be shown by the same procedure as in the proof of claim 1, q.e.d.

I write "$p \supset q$" for "$\neg p \vee q$", and "$p \supset\subset q$" for "$(p \supset q) \wedge (q \supset p)$"; so, $p \supset q$ and $p \supset\subset q$ are equivalent to material (bi)conditionals. Let $\overline{A_0}$ be the set of propositions arising from A_0 by replacing each connection rule $r \in A_0$ by its material counterpart $\overline{r}$, obtained from r by replacing "$\to$" by "$\supset$", or "$\leftrightarrow$" by "$\supset\subset$". Since A_0 is consistent and $r \models \overline{r}$ for each connection rule r, $\overline{A_0}$ is also consistent. Now consider an interpretation $(W, (R_p)_{p\in X}, (v_w)_{w\in W})$ and a world $w \in W$ subject to the following conditions:

(w1) In w, the truth values of atomic propositions are such that each member of $\overline{A_0}$ is true (which is possible since $\overline{A_0}$ is consistent).

(w2) For any $\neg(p \to q) \in A$ there is a world, denoted $w_{p\to q} \in W\backslash\{w\}$, in which all atomic proposition except $a_{p\to q}$ are true; moreover, we have $wR_p w_{p\to q}$, but not $wR_s w_{p\leftrightarrow q}$ for any $s \in \mathbf{L}\backslash\{p\}$.

(w3) For any $\neg(p \leftrightarrow q) \in A$ such that (α) holds, there is a world, denoted $w_{\hat{p}\leftrightarrow q} \in W\backslash\{w\}$, in which all atomic propositions except $a_{\hat{p}\leftrightarrow q}$ are true; moreover, we have $wR_p w_{\hat{p}\leftrightarrow q}$, but not $wR_s w_{\hat{p}\leftrightarrow q}$ for any $s \in \mathbf{L}\backslash\{p\}$.

(w4) For any $\neg(p \leftrightarrow q) \in A$ such that (β) holds, there is a world, denoted $w_{p\leftrightarrow\hat{q}} \in W\backslash\{w\}$, in which all atomic propositions except $a_{p\leftrightarrow\hat{q}}$ are true; moreover, we have $wR_q w_{p\leftrightarrow\hat{q}}$, but not $wR_s w_{p\leftrightarrow\hat{q}}$ for any $s \in \mathbf{L}\backslash\{q\}$.

(w5) Each world $w' \in W$ that is neither w nor any of the worlds defined in (w2)-(w4) is not reachable from w, i.e. there is no $r \in \mathbf{L}$ with $wR_r w'$.

To complete the proof, consider any $r \in A$. I show that r is true in w.

Case 1: r is a possibly negated atomic proposition. Then $r \in \overline{A_0}$. Thus r is true in w by (w1).

Case 2: r is a conditional $s \to t$. Let $w' \in W$ be any world with $wR_s w'$. I have to show that t is true in w'. If $w' = w$, then s is true in w by $wR_s w$. As $s \supset t = \overline{r} \in \overline{A_0}$, $s \supset t$ is true in w by (w1). Since s and $s \supset t$ are true in w, so is t. Now suppose $w' \neq w$. Then by (w5), w' is one of the worlds defined in (w2)-(w4). Assume w' is a world defined in (w2) (the proofs for (w3) and (w4) are analogous). In other words, $w' = w_{p\to q}$ for some $p, q \in \mathbf{L}$. By $wR_s w_{p\to q}$ and (w2), $p = s$. By (w2), all atomic propositions except $a_{p\to q}$ are true in $w_{p\to q}$. By $p \to t = s \to t \in A$ and claims 1, $a_{p\to q}$ is not a conjunct of t. So t is true in $w_{p\to q} = w'$.

Case 3: r is a biconditional $s \leftrightarrow t$. $s \leftrightarrow t$ is true in w in case $s \rightarrow t$ and $t \rightarrow s$ are true in w. Both of these claims can be shown by a procedure analogous to that for case 2.

Case 4: r is a negated conditional $\neg(p \rightarrow q)$. To show that r is true in w, I show that $p \rightarrow q$ is false in w. This is so because the world $w_{p\rightarrow q}$ satisfies $wR_p w_{p\rightarrow q}$ by (w2), and q is false in $w_{p\rightarrow q}$ as q contains the conjunct $a_{p\rightarrow q}$ which is false in $w_{p\rightarrow q}$.

Case 5: r is a negated biconditional $\neg(p \leftrightarrow q)$. To show that r is true in w, I show that $p \leftrightarrow q$ is false in w, i.e. that $p \rightarrow q$ or $q \rightarrow p$ is false in w. By claim 2, there are two cases (α) and (β). In case (α), $p \rightarrow q$ is false in w because the world $w_{\hat{p}\leftrightarrow q}$ satisfies $wR_p w_{\hat{p}\leftrightarrow q}$ by (w3), and q is false in $w_{\hat{p}\leftrightarrow q}$ as q contains the conjunct $a_{\hat{p}\leftrightarrow q}$ which is false in $w_{\hat{p}\leftrightarrow q}$. In case (β), $q \rightarrow p$ is false in w because the world $w_{p\leftrightarrow\hat{q}}$ satisfies $wR_q w_{p\leftrightarrow\hat{q}}$ by (w4), and p is false in $w_{p\leftrightarrow\hat{q}}$ as p contains the conjunct $a_{p\leftrightarrow\hat{q}}$ which is false in $w_{p\leftrightarrow\hat{q}}$. ■

Proof of Theorem 1. Let X be a network. In part A, I prove the claim in a special case; the general case is proven in part B. Let $\mathcal{B}$ be the set of all atomic propositions in X, and $\mathcal{C}$ the set of all connection rules in X.

A. In this part, I assume that X contains each atomic proposition a that occurs in some connection rule in X. I show that the quota rule $F := F_{(m_p)_{p\in X^+}}$ with $m_p = n$ for all $p \in X^+$ is consistent. Consider any profile $(A_1, ..., A_n)$ in the universal domain, and let me show that $A := F(A_1, ..., A_n)$ is consistent. A satisfies the assumptions (1) and (2) of Lemma 1. This follows easily from the definition of F and the fact that each A_i satisfies (1) and (2). So, by Lemma 1, it is sufficient to show that $A_0 := \{p \in A :$ p is not a negated connection rule$\}$ is consistent. To do so, I define an interpretation and a world in which each member of A_0 is true (which is much easier than for A). Let $(W, (R_p)_{p\in\mathbf{L}}, (v_w)_{w\in W})$ be an interpretation and $\overline{w} \in W$ a world such that

(1) for any atomic proposition a, $v_{\overline{w}}(a) = T$ if and only if $a \in A_0$.

(2) for any world $w \in W\backslash\{\overline{w}\}$, $v_w(a) = T$ for *every* atomic proposition a.

Now let $p \in A_0$. To show that $v_{\overline{w}}(p) = T$, we distinguish for cases (note that the case in which p is a negated connection rule does not exist).

Case 1: p is an atomic proposition a. So $v_{\overline{w}}(a) = T$ by (1), q.e.d.

Case 2: p is a negated atomic proposition $\neg a$. By $\neg a \in A_0$ and the definition of F, there is a person who accepts $\neg a$. This person does not accept a. So $a \notin A_0$, again by the definition of F. Hence $v_{\overline{w}}(a) = F$ by (1). So $w_{\overline{w}}(\neg a) = T$, q.e.d.

Case 3: p is a conditional $p \rightarrow q$. To show that $v_{\overline{w}}(p \rightarrow q) = T$, consider any world $w \in W$ such that $\overline{w}R_p w$, and let us show that $v_w(q) = T$. If $w \neq \overline{w}$, then $v_w(q) = T$ by (2). Now suppose $w = \overline{w}$. Then $\overline{w}R_p\overline{w}$. Hence p is true in $\overline{w}$. So every conjunct of p is true in $\overline{w}$. Hence, by (1), every conjunct of p belongs to A_0. Since $p \rightarrow q$ and each conjunct of p are contained in A_0, they are contained in each of $A_1, ..., A_n$ by the definition of F. Since they together entail each conjunct of q, each conjunct of q is contained in each of $A_1, ..., A_n$. So each conjunct of q belongs to A_0, again by the definition of F. So, by (1), each conjunct of q is true in $\overline{w}$. Hence q is true in $\overline{w}$, q.e.d.

Case 4: p is a biconditional $p \leftrightarrow q$. Then p is true in $\overline{w}$ just in case $p \rightarrow q$ and $q \rightarrow p$ are true in $\overline{w}$. The truth of $p \rightarrow q$ and of $q \rightarrow p$ in $\overline{w}$ can be shown by the method used in case 3, q.e.d.

B. I now drop the previous assumption that each atomic proposition occurring in a connection rule in X is contained in X. I will reduce this general case to the previous special case. Consider the extended agenda $\overline{X} := X \cup \{a, \neg a : a$ is an atomic proposition occurring in some proposition in $X\}$. By part A, there exists a consistent quota rule $F_{(m_p)_{p \in \overline{X}^+}}$ for the agenda $\overline{X}$. Now consider the quota rule $F_{(m_p) p \in X^+}$ for the agenda X. As one easily checks, for each $(A_1, ..., A_n)$ in the domain of $F_{(m_p)_{p \in X^+}}$,

$$F_{(m_p)_{p \in X^+}}(A_1, ..., A_n) = F_{(m_p)_{p \in \overline{X}^+}}(\overline{A_1}, ..., \overline{A_n}) \cap X,$$

where, for each person i, $\overline{A_i}$ is any complete and consistent judgment set for the agenda $\overline{X}$ such that $A_i \subseteq \overline{A_i}$. So the consistency of $F_{(m_p)_{p \in \overline{X}^+}}$ implies that of $F_{(m_p)_{p \in X^+}}$. ■

5 A characterisation theorem

The above possibility result leaves open the question *how large* the possibility space is. For a given network X, what type of thresholds m_p, $p \in X^+$, is allowed if one wants the quota rule $F_{(m_p)_{p \in X^+}}$ to be consistent? For so-called *simple* networks, a compact answer to this question can be given.

Definition 2 *A network X is* simple *if (i) each connection rule in X is a conditional $a \to b$, in which a and b are distinct atomic propositions, and (ii) for each $a \to b \in X$, we have $a \in X$ and $b \in X$.*

For instance, the networks (1) and (7) in Figure 1 are simple. In graphic terms, a network is simple if and only if its graph contsins (i) no arrow that is bidirectional or bifurcating or pointing from a proposition to itself, and (ii) no atomic proposition in brackets.

Theorem 2 *If X is a simple network, a quota rule $F_{(m_p)_{p \in X^+}}$ is consistent if and only if*

$$m_b \leq m_a + m_{a \to b} - n \text{ for each connection rule } a \to b \in X.$$

This characterises consistency in terms of a system of simple linear inequalities. The more connection rules X contains, the more inequalities have to be satisfied to achieve consistency. For the network X with $X^+ = \{a, b, a \to b\}$, a quota rule $F_{(m_p)_{p \in X^+}}$ is consistent just in case $m_b \leq m_a + m_{a \to b} - n$ holds, hence for instance if $n = 10$, $m_a = m_{a \to b} = 8$ and $m_b = 6$. For the network X with $X^+ = \{a, b, c, a \to b, b \to c\}$, a quota rule $F_{(m_p)_{p \in X^+}}$ is consistent just in case $m_b \leq m_a + m_{a \to b} - n$ and $m_c \leq m_b + m_{b \to c} - n$, hence for instance in case $n = 10$, $m_a = m_{a \to b} = 8$, $m_b = m_{b \to c} = 6$ and $m_c = 2$.

Before coming to the proof, let me note some corollaries. If each $p \in X^+$ has the same ("uniform") threshold $m = m_p$ then all inequalities in Theorem 2 reduce to the same inequality $m \leq m + m - n$, hence to $m \geq n$, i.e. $m = n$. So:

Corollary 2 *If X is a simple network (containing at least one connection rule), there exists exactly one consistent quota rule $F_{(m)_{p \in X^+}}$ with a uniform threshold m for all $p \in X^+$; it is given by the unanimity threshold $m = n$.*

Theorem 2 also yields two general properties of consistent quota rules. First, since the inequality in Theorem 2 implies that $m_b \leq m_a$, the acceptance threshold must decrease (weakly) along each path in X. Formally:

Definition 3 *Let X be a simple network.*

(i) $a \in X$ is a parent *of $b \in X$ (and b a* child *of a) if $a \to b \in X$;*

(ii) A path *(in X) is a sequence $(a_1, ..., a_k)$ $(k \geq 2)$ of atomic propositions such that $a_1 \to a_2 \in X$, $a_2 \to a_3 \in X$, ..., $a_{k-1} \to a_k \in X$;*

(iii) $a \in X$ is an ancestor *of $b \in X$ (and b a* descendant *of a) if there exists a path $(a_1, ..., a_k)$ with $a_1 = a$ and $a_k = b$.*

Corollary 3 *If X is a simple network, any consistent quota rule $F_{(m_p)_{p \in X^+}}$ satisfies*

$$m_b \leq m_a \text{ for each } a, b \in X \text{ such that } b \text{ is a descendant of } a.$$

Second, if the networks contains *cycles* the inequalities of Theorem 2 impose a rather severe restriction on the choice of thresholds:

Definition 4 *A* cycle *in a simple network X is a path $(a_1, ..., a_k)$ in X with $a_1 = a_k$.*

Corollary 4 *If X is a simple network, any consistent quota rule $F_{(m_p)_{p \in X^+}}$ satisfies*

$$m_{a_1} = ... = m_{a_k} \text{ and } m_{a_1 \to a_2} = ... = m_{a_{k-1} \to a_k} = n \text{ for each cycle } (a_1, ..., a_k).$$

Proof. Let X be a simple network and $F_{(m_p)_{p \in X^+}}$ consistent. Any two distinct members a_j and a_l of a cycle $(a_1, ..., a_k)$ are descendants of each other. So, by Corollary 3, $m_{a_j} \leq m_{a_l}$ and $m_{a_l} \leq m_{a_j}$, hence $m_{a_j} = m_{a_l}$. Moreover, for any $1 \leq j < k$, by Theorem 2 $m_{a_{j+1}} \leq m_{a_j} + m_{a_j \to a_{j+1}} - n$. Hence, by $m_{a_{j+1}} = m_{a_j}$, we have $0 \leq m_{a_j \to a_{j+1}} - n$, so $m_{a_j \to a_{j+1}} = n$. ■

For instance, suppose $F_{(m_p)_{p \in X^+}}$ is a consistent quota rule for the simple network (7) in Figure 1. Since (a, b, c, a) is a cycle, $m_a = m_b = m_c$ and $m_{a \to b} = m_{b \to c} = m_{c \to a} = n$. Moreover, since (c, b, a, c) is another cycle, $m_{c \to b} = m_{b \to a} = m_{a \to c} = n$. So, in summary, one must assign the same threshold to *all* atomic propositions in X, and the unanimity threshold n to *all* connection rules in X – a strong restriction.

Now let me prove Theorem 2. The proof uses the following lemma.

Lemma 2 *Let X be a simple network. Any set $A \subseteq X$ satisfying (1) is consistent if and only if its subset $\{p \in A : p \text{ is not a negated connection rule}\}$ is consistent.*

Proof. Suppose X is a simple network and $A \subseteq X$ satisfies (1). By Lemma 1, it is sufficient to show that (2) holds. So, consider any set of connection rules $S \subseteq A$ and any connection rule $a \to b \in X$ such that $S \models a \to b$. To show $a \to b \in A$, I prove $a \to b \in S$. For contradiction, suppose $a \to b \notin S$. I show that $S \cup \{\neg(a \to b)\}$ is consistent, violating $S \models a \to b$. I define an interpretation $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$ containing a world in which each $p \in S \cup \{\neg(a \to b)\}$ is true. Let W be a binary set $W = \{w, w'\}$, where

(i) In world w, each atomic proposition is true.

(ii) In world w', b is false and all other atomic propositions are true.

(iii) From world w, R_a reaches both worlds w and w'.

(iv) From world w, each $R_{a'}$, where a' is any atomic proposition different from a, reaches only world w.

I show that each $p \in S \cup \{\neg(a \to b)\}$ is true in world w. Let $p \in S \cup \{\neg(a \to b)\}$.

Case 1: $p = \neg(a \to b)$. Then p is true in w because, by (iii), from w R_a reaches the world w', in which b is false by (ii).

Case 2: $p = a' \to b'$ with $a' \neq a$. Then p is true in w because, by (iv), from w $R_{a'}$ reaches only world w, in which b' is true by (i).

Case 3: $p = a \to b'$. Then $b' \neq b$ since $a \to b \notin S$. So b' is true in both worlds w and w', by (i) and (ii). Hence $a \to b'$ is true in w. ■

Proof of Theorem 2. Let X be a simple network. A set of propositions $Y \subseteq X$ is *minimal inconsistent* if it is inconsistent and every proper subset of Y is consistent; $Y \subseteq X$ is a *non-trivial* minimal inconsistent set if Y is minimal inconsistent and not of the form $\{p, \neg p\}$. For any set $Z \subseteq X$, a *path in* Z is a finite sequence $(a_1, ..., a_k)$ $(k \geq 2)$ such that $a_1 \to a_2 \in Z, a_2 \to a_3 \in Z, ..., a_{k-1} \to a_k \in Z$; it is called *acyclic* if $a_1, ..., a_k$ are pairwise distinct. I will show that the non-trivial minimal inconsistent subsets of X are precisely the sets of the form

(*) $Y = \{a_1, a_1 \to a_2, a_2 \to a_3, ..., a_{k-1} \to a_k, \neg a_k\}$, where $(a_1, ..., a_k)$ is an acyclic path in X.

A. In this part, I show that if Y takes the form (*) then Y is a non-trivial minimal inconsistent subset of X. Suppose Y has the form (*). To see that Y is inconsistent, suppose for contradiction that there is an interpretation $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$ and a world $w \in W$ in which each $p \in Y$ is true. Since a_1 and $a_1 \to a_2$ are true in w, so is a_2. Since a_2 and $a_2 \to a_3$ are true in w, so is a_3. Repeating this argument, it follows that a_k is true in w. But $\neg a_k$ is also true in w, a contradiction.

To see that Y is *minimal* inconsistent, I prove that any subset $Z \subseteq Y$ with $|Z| = |Y| - 1$ is consistent. This is easily done by defining an interpretation $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$ such that some world in W – in fact *every* world in W – makes each $p \in Z$ true:

- if $Z = Y \backslash \{\neg a_k\}$, let $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$ be any interpretation such that $a_1, ..., a_k$ are true in every world $w \in W$;
- if $Z = Y \backslash \{a_1\}$, let $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$ be any interpretation such that $a_1, ..., a_k$ are false in every world $w \in W$;
- if $Z = Y \backslash \{a_j \to a_{j+1}\}$ for some $1 \leq j < k$, let $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$ be any interpretation such that $a_1, ..., a_j$ are true and $a_{j+1}, ..., a_k$ false in every world $w \in W$.

Finally, Y is a *non-trivial* minimal inconsistent set since $k \geq 2$, q.e.d.

B. I now show the converse of part A: if Y is a non-trivial minimal inconsistent subset of X then Y takes the form (*). Let Y be a non-trivial minimal inconsistent subset of X. I may write $Y = B^+ \cup B^- \cup C^+ \cup C^-$, where B^+, B^-, C^+, C^- are, respectively, sets of atomic propositions, negated atomic propositions, connection rules, negated connection rules.

Claim 1: $C^- = \emptyset$, i.e. $Y = B^+ \cup B^- \cup C^+$.

Since Y is a *non-trivial* minimal inconsistent set, Y satisfies the condition (1) of Lemma 2. So, by Lemma 2, Y is inconsistent if and only if its subset $B^+ \cup B^- \cup C^+$ is inconsistent. Since Y is inconsistent, so is $B^+ \cup B^- \cup C^+$. However, since Y is *minimal* inconsistent, no proper subset of Y is inconsistent. So $B^+ \cup B^- \cup C^+$ is not a proper subset of Y. Hence $C^- = \emptyset$, q.e.d.

Claim 2: $C^+ \neq \emptyset$.

For contradiction let $C^+ = \emptyset$. So $Y = B^+ \cup B^-$ by claim 1. For any $a \in B^+$ and $\neg b \in B^-$ we have $a \neq b$. For otherwise $\{a, \neg b\} = \{a, \neg a\}$, which is inconsistent, implying that $Y = \{a, \neg a\}$; this contradicts that Y is a *non-trivial* minimal inconsistent set. So Y is consistent, contradicting that Y is minimal inconsistent, q.e.d.

A sequence $(a_1, ..., a_k)$ is a *subsequence* of another sequence $(b_1, ..., b_l)$ if $(a_1, ..., a_k) = (b_{j_1}, ..., b_{j_m})$ for some indices $1 \leq j_1 < j_2 < ... < j_m \leq l$. A path in Y is called a *maximal path in* Y if it is not a subsequence ("subpath") of another path in Y.

Claim 3: There exists a maximal path in Y, denoted $(a_1, ..., a_k)$. (I will later show that $Y = \{a_1, a_1 \to a_2, ..., a_{k-1} \to a_k, \neg a_k\}$)

Assume for contradiction the claim is false. I show that Y is consistent, a contradiction. Let $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$ be an interpretation and $\overline{w} \in W$ a world such that:

(i) any atomic proposition a is true in $\overline{w}$ if and only if $a \in Y$ or there exists a path in Y, $(b_1, ..., b_m)$, with $b_1 \in Y$ and $b_m = a$;

(ii) from the world $\overline{w}$, for each atomic proposition a, R_a does not access any world $w \in W \backslash \{\overline{w}\}$.

Consider any $p \in Y$, and let me show that p is true in $\overline{w}$.

Case 1: $p = a \in B^+$. Then by (i) a is true in $\overline{w}$.

Case 2: $p = \neg a \in B^-$. Assume for contradiction that $\neg a$ is false in $\overline{w}$, i.e. that a is true in $\overline{w}$. Then, by (i), either $a \in Y$ or there exists a path in Y, $(b_1, ..., b_m)$, such that $b_1 \in Y$ and $b_m = a$. First assume $a \in Y$. As Y contains a and $\neg a$ and is minimal inconsistent, we have $Y = \{a, \neg a\}$, contradicting that Y is a *non-trivial* minimal inconsistent set. Now assume there exists a path in Y, $(b_1, ..., b_m)$, such that $b_1 \in Y$ and $b_m = a$. The set $Y^* := \{b_1, b_1 \to b_2, ..., b_{m-1} \to b_m, \neg b_m\}$ is inconsistent, by an argument analogous to one made in part A. So $Y = Y^*$, since $Y^* \subseteq Y$ and Y is minimal inconsistent. But then $(b_1, ..., b_m)$ is a maximal path in Y, contradicting the starting assumption.

Case 3: $p = a \to b \in C^+$. By (ii), from $\overline{w}$ R_a reaches no world except perhaps $\overline{w}$. If a is false in $\overline{w}$, then from $\overline{w}$ R_a does not reach $\overline{w}$, hence reaches no world, so that $a \to b$ is true in $\overline{w}$. Now assume a is true in $\overline{w}$, and let me show that b is true in $\overline{w}$. By (i), either $a \in Y$ or there is a path in Y, $(b_1, ..., b_m)$, with $b_1 \in Y$ and $b_m = a$. In the first case, (a, b) is a path in Y; in the second case, $(b_1, ..., b_m, b)$ is a path in Y since $b_m \to b = a \to b \in Y$. So, by (i), b is true in $\overline{w}$, q.e.d.

Claim 4: $a_1 \in Y$.

Assume for contradiction that $a_1 \notin Y$. Since Y is minimal inconsistent and by claim 3 contains $a_1 \to a_2$, $Y \backslash \{a_1 \to a_2\}$ is consistent. So there is an interpretation $(W, (R_p)_{p \in \mathbf{L}}, (v_w)_{w \in W})$ and a world $\overline{w} \in W$ such that

$$v_{\overline{w}}(p) = T \text{ for all } p \in Y \backslash \{a_1 \to a_2\}. \tag{3}$$

Now let $(W, (R^*_p)_{p \in \mathbf{L}}, (v^*_w)_{w \in W})$ be a new interpretation with the same set of worlds W, satisfying the following conditions.

(1*) In each world $w \in W \backslash \{\overline{w}\}$, any atomic proposition has the same truth value as before.

(2*) In the world $\overline{w}$, a_1 is false and all other atomic propositions have the same truth value as before.

(3*) From the world $\overline{w}$, $R^*_{a_1}$ accesses no world.

(4*) $R^*_a = R_a$ for each atomic proposition $a \neq a_1$.

I show that each $p \in Y$ is true in the world $\overline{w}$ of the new interpretation $(W, (R^*_p)_{p\in\mathbf{L}}, (v^*_w)_{w\in W})$. Then Y is consistent, a contradiction. There are four cases.

Case 1: $p = a \in B^+$. By $a_1 \notin Y$, we have $a \neq a_1$. So $v^*_{\overline{w}}(a) = v_{\overline{w}}(a)$ by (2*), hence $v^*_{\overline{w}}(a) = T$ by (3).

Case 2: $p = \neg a \in B^-$. If $a = a_1$ then $v^*_{\overline{w}}(\neg a) = T$ by (2*). If $a \neq a_1$ then $v^*_{\overline{w}}(\neg a) = v_{\overline{w}}(\neg a)$ by (2*), hence $v^*_{\overline{w}}(\neg a) = T$ by (3).

Case 3: $p = a_1 \to b \in C^+$. By (3*), $R^*_{a_1}$ accesses no world from $\overline{w}$. So, vacuously, b is true in every world $w \in W$ with $\overline{w}R^*_{a_1}w$. Hence $a_1 \to b$ is true in $\overline{w}$.

Case 4: $p = a \to b \in C^+$ with $a \neq a_1$. As by claim 3 $(a_1, ..., a_k)$ is a maximal path in Y, $b \neq a_1$. From $\overline{w}$ R^*_a accesses exactly the same worlds w as R_a by (4*), where $v^*_w(b) = v_w(b)$ by (1*) and (2*). So $v^*_{\overline{w}}(a \to b) = v_{\overline{w}}(a \to b)$, and hence $v^*_{\overline{w}}(a \to b) = T$ by (3), q.e.d.

Claim 5: $\neg a_k \in Y$.

Assume for contradiction that $\neg a_k \notin Y$. Since Y is minimal inconsistent and by claim 3 contains $a_{k-1} \to a_k$, $Y \backslash \{a_{k-1} \to a_k\}$ is consistent. So there is an interpretation $(W, (R_p)_{p\in\mathbf{L}}, (v_w)_{w\in W})$ and a world $\overline{w} \in W$ such that

$$v_{\overline{w}}(p) = T \text{ for all } p \in Y\backslash\{a_{k-1} \to a_k\}. \tag{4}$$

Now let $(W, (R^*_p)_{p\in\mathbf{L}}, (v^*_w)_{w\in W})$ be an interpretation with the same set of possible worlds W, satisfying the following conditions.

(a*) In each world $w \in W\backslash\{\overline{w}\}$, any atomic proposition has the same truth value as before.

(b*) In the world $\overline{w}$, a_k is true and all other atomic propositions have the same truth value as before.

(c*) From the world $\overline{w}$, $R^*_{a_{k-1}}$ accesses no world in $W\backslash\{\overline{w}\}$.

(d*) $R^*_a = R_a$ for each atomic proposition $a \neq a_{k-1}$.

I show that each $p \in Y$ is true in the world $\overline{w}$ of the new interpretation $(W, (R^*_p)_{p\in\mathbf{L}}, (v_w)_{w\in W})$. This entails that Y is consistent, a contradiction.

Case 1: $p = a \in B^+$. If $a = a_k$ then a is true in $\overline{w}$ by (b*). If $a \neq a_k$, then $v^*_{\overline{w}}(a) = v_{\overline{w}}(a)$ by (b*), hence $v^*_{\overline{w}}(a) = T$ by (4).

Case 2: $p = \neg a \in B^-$. By $\neg a_k \notin Y$, we have $a \neq a_k$. So $v^*_{\overline{w}}(\neg a) = v_{\overline{w}}(\neg a)$ by (b*), hence $v^*_{\overline{w}}(\neg a) = T$ by (4).

Case 3: $p = a_{k-1} \to b \in C^+$. Let $w \in W$ be a world such that $\overline{w}R^*_{a_{k-1}}w$, and let me show that b is true in w. By (c*), $w = \overline{w}$. Hence I have to show that b is true in $\overline{w}$. If $b = a_k$, this holds by (b*). Now suppose $b \neq a_k$. By $\overline{w}R^*_{a_{k-1}}\overline{w}$, we have $v^*_{\overline{w}}(a_{k-1}) = T$. Hence $v_{\overline{w}}(a_{k-1}) = T$ by (b*). So $\overline{w}R_{a_{k-1}}\overline{w}$. Hence $v_{\overline{w}}(b) = T$, as $v_{\overline{w}}(a_{k-1} \to b) = T$ by (4). Thus $v^*_{\overline{w}}(b) = T$ by (b*).

Case 4: $p = a \to b \in C^+$ with $a \neq a_{k-1}$. Let $w \in W$ be any world such that $\overline{w}R^*_a w$. I show that b is true in w. If $w = \overline{w}$ and $b = a_k$, then b is true in w by (b*). If $w \neq \overline{w}$ or $b \neq a_k$, then $v^*_w(b) = v_w(b)$ by (a*) and (b*). So I have to show that $v_w(b) = T$. By (4), $v_{\overline{w}}(a \to b) = T$. By $\overline{w}R^*_a w$ and (d*), $\overline{w}R_a w$. Combining $v_{\overline{w}}(a \to b) = T$ and $\overline{w}R_a w$, we obtain $v_w(b) = T$, as desired.

Claim 6: $Y = \{a_1, a_1 \to a_2, ..., a_{k-1} \to a_k, \neg a_k\}$, and the path $(a_1, ..., a_k)$ is acyclic (which completes the proof of part B).

The set $\{a_1, a_1 \to a_2, ..., a_{k-1} \to a_k, \neg a_k\}$ is inconsistent, by an argument analogous to one in part A. As by claims 3-5 the set is a subset of the (minimal inconsistent) set Y, it equals Y. Now assume for contradiction that $(a_1, ..., a_k)$ is not acyclic. Then there exist indices $1 \leq j < l \leq k$ such that $a_j = a_l$. Since Y is minimal inconsistent, its subset $\{a_1, a_1 \to a_2, ..., a_{j-1} \to a_j, a_l \to a_{l+1}, ..., a_k, \neg a_k\}$ is consistent; but this subset is inconsistent, again by an argument analogous to one in part A, a contradiction, q.e.d.

C. By part (c) of Theorem 1 in Dietrich and List (2005), F is consistent *if and only if*

$$\sum_{p \in Y} m_p \geq (|Z| - 1)n + 1 \quad \text{for every non-trivial minimal inconsistent set } Y \subseteq X.$$

By parts A and B above, the non-trivial minimal inconsistent sets $Y \subseteq X$ are precisely the sets of type (*). So F is consistent *if and only if*

$$m_{a_1} + m_{\neg a_k} + \sum_{j=1}^{k-1} m_{a_j \to a_{j+1}} \geq kn + 1 \text{ for every acyclic path in } X,\ (a_1, ..., a_k). \quad (5)$$

Hence the proof is completed once it is shown that:

Claim: The condition (5) holds if and only if

$$m_b \leq m_a + m_{a \to b} - n \text{ for every connection rule } a \to b \in X. \quad (6)$$

First assume (5) holds. To show (6), consider any $a \to b \in X$. Then (a, b) is an acyclic path of X. Hence, by (5), $m_a + m_{\neg b} + m_{a \to b} \geq 2n+1$. So, as $m_{\neg b} = n+1-m_b$, we have $m_a - m_b + m_{a \to b} \geq n$, hence $m_b \leq m_a + m_{a \to b} - n$, as desired.

Now suppose (6) holds. To show (5), consider any acyclic path $(a_1, ..., a_k)$ in X. Then, for each $1 \leq j < k$, $a_j \to a_{j+1} \in X$; hence, by (6), $m_{a_{j+1}} \leq m_{a_j} + m_{a_j \to a_{j+1}} - n$, or equivalently $m_{a_j} - m_{a_{j+1}} + m_{a_j \to a_{j+1}} \geq n$. Since the latter inequality holds for each $j = 1, 2, ..., k-1$, we can add these $k-1$ inequalities. This yields

$$m_{a_1} - m_{a_k} + \sum_{j=1}^{k-1} m_{a_j \to a_{j+1}} \geq (k-1)n.$$

The desired inequality follows by replacing m_{a_k} by $n + 1 - a_{\neg a_k}$. ■

6 Constructing consistent quota rules

Consider a simple network X. While Theorem 2 allows one to check easily whether a *given* quota rule $F_{(m_p)_{p \in X^+}}$ is consistent, let us now focus on the question of how to *construct* a consistent quota rule. This may not be an obvious task: if the network X is large, a large number of thresholds m_p, $p \in X^+$, has to be determined by respecting a potentially large number of inequalities. If X contains k atomic propositions, it may contain up to $k(k-1)$ connection rules $a \to b$, so that up to $k(k-1)$ inequalities have to be respected, where $k(k-1)$ grows quadratically in k.

There is a simple way to structure the problem of choosing the thresholds.

Definition 5 *Let X be a simple network.*

(i) The depth *of X is $d_X := \sup\{k$: there is a path in X of length $k\}$, interpreted as 1 if there is no path (hence no connection rule) in X.*

(ii) The level *of an atomic proposition $a \in X$ is $l_a := \sup\{k$: there is a path in X of length k ending with $a\}$, interpreted as 1 if there is no path ending with a.*

For instance, atomic propositions $a \in X$ with no parents (no a' such that $a' \to a \in X$) have level 1, atomic propositions whose parents have level 1 have level 2, etc. Figure 2 shows a simple network with three levels.

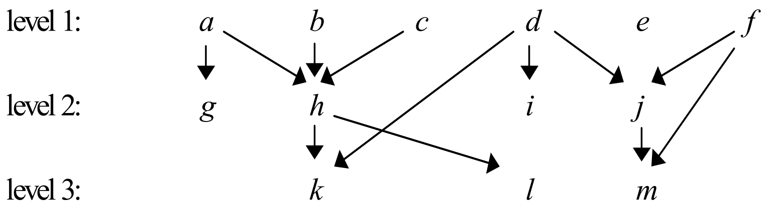

Figure 2: An acyclic simple network X of depth $d_X = 3$.

Often, propositions of high level are *act propositions*, stating that a certain collective act should be taken (a road should be built, a company should be downsized, law X should be amended, ...), whereas their ancestors describe potential *reasons* or *arguments* for act propositions (the traffic between two towns will increase, the demand for the company's products will fall, law X is inefficient, ...). One may interpret the level of a proposition as measuring how "fundamental" the proposition (issue) is, on a scale from level 1 (most fundamental) to level d_X (closest to collective action).

Not all simple networks can be represented in the hierarchical way of Figure 2; for instance, if the network contains a cycle $(a_1, ..., a_k)$ then each of $a_1, ..., a_k$ has level ∞. However, each acyclic and finite simple network *can* be represented hierachically, for the following reason:

Definition 6 *A simple network X is* acyclic *if it contains no cycle, i.e. no path with the same first and last element.*

Proposition 1 *If a simple network X is acyclic and finite, its depth and set of levels satisfy $d_X < \infty$ and $\{l_a : a \text{ is an atomic proposition in } X\} = \{1, 2, ..., d_X\}$.*

Proof. Let X be as specified. As X is acyclic, each path in X, $(a_1, ..., a_k)$, consists of *pairwise distinct* propositions, hence has length $k \leq |X|$. So $d_X \leq |X|$, and hence $d_X < \infty$. Now I show that $M = \{1, ..., d_X\}$, where $M := \{l_a : a$ is an atomic proposition in $X\}$. Obviously, $M \subseteq \{1, ..., d_X\}$. To see that $\{1, ..., d_X\} \subseteq M$, note that by $d_X < \infty$ there exists a path in X of length d_X, say $(a_1, ..., a_{d_X})$; each $j \in \{1, ..., d_X\}$ belongs to M because a_j has level j. ■

Combining Theorem 2 and Proposition 1, a consistent quota rule $F_{(m_p)_{p \in X^+}}$ for an acyclic and finite simple network X can be constructed recursively in d_X steps (recall that a $a \in X$ is a parent of $b \in X$ if and only if $a \to b \in X$):

Step l $(= 1, 2, ..., d_X)$: for each atomic proposition $b \in X$ of level l, choose a threshold $m_b \in \{1, ..., n\}$ and a threshold $m_{a \to b} \in \{1, ..., n\}$ for each parent $a \in X$ of b, such that

$$m_b \leq m_a + m_{a \to b} - n \text{ for each parent } a \in X \text{ of } b. \quad (7)$$

This procedure is well-defined because, in each step l, (i) the thresholds m_a of parents of b have already been chosen (a has lower level!) and (ii) the system (7) always admits a solution, for instance the trivial solution given by $m_b = 1$ and $m_{a \to b} = n$ for all parents a of b.

However, the above procedure involves the determination of a possibly very large number of thresholds. The network of Figure 2 contains 13 atomic propositions and 11 conditionals! To reduce the number of parameters, the group might use

• the same threshold m_l for all propositions in X with the same level l ($\in \{1, ..., d_X\}$), where m_l represents how the group treats propositions (issues) of level l,

• the same threshold $m = m_{a \to b}$ for each connection rule $a \to b \in X$, where m represents how easily the group imposes constraints between propositions (issues).

Quota rules of this simple type are given by only d_X+1 parameters $(m, m_1, ..., m_{d_X})$, for instance in Figure 2 by $3 + 1 = 4$ parameters instead of $13 + 11 = 24$ parameters. To define a procedure for choosing these d_X+1 parameters, I first need to characterise such quota rules:

Corollary 5 *Let X be an acyclic and finite simple network. A quota rule $F_{(m_p)_{p \in X^+}}$ with the same threshold m for each connection rule and, for each level $l \in \{1, ..., d_X\}$, the same threshold m_l for all atomic propositions of level l is consistent if and only if*

$$m_l \leq m_{l-1} + m - n \textit{ for each level } l = 2, 3, ..., d_X. \quad (8)$$

Proof. Let X, $F_{(m_p)_{p \in X^+}}$, m and m_l ($l = 1, 2., , , .d_X$) be as specified. By Theorem 2, F is consistent if and only if

$$m_b \leq m_a + m_{a \to b} - n \text{ for each connection rule } a \to b \in X. \quad (9)$$

So I have to show that (9) is equivalent to (8).

First assume (9), and consider any level $1 < l \leq d_X$. By $l > 1$, there exists atomic propositions $a \in X$ of level $l - 1$ and $b \in X$ of level l such that $a \to b \in X$. By (9), $m_b \leq m_a + m_{a \to b} - n$. Hence $m_l \leq m_{l-1} + m - n$, as desired.

Now assume (8), and consider any $a \to b \in X$. Let l be the level of a, and k the level of b. As one easily checks, $k > l$. By (8), we have

$$m_{l+1} \leq m_l + m - n. \quad (10)$$

By $k \geq l + 1$ and (8), we also have $m_k \leq m_{l+1}$. By this and (10), we have $m_k \leq m_l + m - n$. Hence $m_b \leq m_a + m - n$, as desired. ■

Corollary 5 yields a simple recursive procedure to construct consistent quota rules $F_{(m_p)_{p \in X^+}}$ of the above type, valid again for any acyclic and finite simple network X, with depth $d_X > 1$ to avoid triviality.

Step 0: choose a threshold $m \in \{1, ..., n\}$ (to be used for all connection rules) satisfying (i) $m \geq n - (n - 1)/(d_X - 1)$.

Step l ($= 1, 2, ..., d_X$): choose a threshold $m_l \in \{1, ..., n\}$ (to be used for all propositions of level l) satisfying (ii) $m_l \geq 1 + (d_X - l)(n - m)$ and (iii) $m_l \leq m_{l-1} + m - n$ if $l > 1$.

The conditions (i)-(iii) follow from Corollary 5. Condition (iii) is obvious, and the conditions (i) and (ii) are necessary and sufficient to ensure that the choices in following steps are possible; for instance, if m violated (i) there would be *no* choices of $m_1, ..., m_{d_X}$ satisfying the system (8).

For example, consider the network of Figure 2 with depth $d_X = 3$, and assume there are $n = 10$ persons. Then one might make the following choices.

Step 0: $m = 8$ (note that $8 \geq n - (n-1)/(d_X - 1) = 10 - 9/2 = 5.5$).

Step 1: $m_1 = 8$ (note that $8 \geq 1 + (d_X - 1)(n - m) = 1 + 2 \times 2 = 5$).

Step 2: $m_2 = 6$ (note that $6 \geq 1 + (d_X - 2)(n - m) = 1 + 2 = 3$ and $6 \leq m_1 + m - n = 8 + 8 - 10 = 6$).

Step 3: $m_3 = 4$ (note that $4 \geq 1 + (d_X - 3)(n - m) = 1$ and $4 \leq m_2 + m - n = 6 + 8 - 10 = 4$).

7 Conclusion

A large class of judgment aggregation problems, including most classical examples, consist in deciding many issues (whether to take an action, adopt a belief, or adopt a desire) *and* deciding how these issues are interconnected (e.g. how certain beliefs constrain certain acts). Such decision problems can be modelled using networks: agendas X that contain atomic propositions $(a, b, c, ...)$ representing issues and connection rules $(a \rightarrow b, a \leftrightarrow (b \wedge c), ...)$ representing constraints between issues. Previous results suggest that "democratic and rational" aggregation rules are inexistent even for moderately interconnected networks. I showed that this impossibility is an artefact of representing connection rules by *material* (bi)conditionals. While the material representation of connection rules leads into impossibilities for six out of the seven networks in Figure 1, an adequate representation of connection rules leads to a possibility result, valid for any network, however complex and interconnected it might be. In a second theorem, I provided (for *simple* networks) a characterisation of the class of all aggregation rules with the desired properties. These aggregation rules are *quota rules* $F_{(m_p)_{p \in X^+}}$ for which the acceptance thresholds m_p for propositions $p \in X^+$ satisfy simple linear inequalities. These inequalities yield a handy recursive method to construct aggregation rules of the desired kind.

The inequalities on acceptance thresholds have a further implication. Suppose a group, say a state's population, has to reach ("democratic and rational") collective judgments on various propositions forming a (simple, acyclic and finite) network X. The atomic propositions can be put into a hierarchical order by assigning to each of them a *level*, reflecting how "fundamental" the proposition (issue) is. Propositions $a \in X$ of level 1 have no parents in the network, i.e. there is no $a' \in X$ with $a' \rightarrow a \in X$; propositions of level 2 are propositions whose parents have level 1; and so on (see Definition 5). Level 1 propositions are most fundamental in the sense of not receiving any *reasons* (*arguments*, *justifications*) by parental propositions in the network; typical level 1 propositions are "the state should preserve its autonomy" and "a multi-cultural society is desirable". While level 1 propositions are not given external reasons, they serve as reasons for higher-level propositions, which are typically propositions more closely linked to collective action, such as "the state should refuse to introduce the Euro" or "immigration law X should be amended". Now, by Corollary 3, in order to ensure “democratic and rational” decisions, low level propositions $a \in X$ have to be given higher acceptance thresholds m_a than their less fundamental descendants in the network. If “the state should preserve its autonomy” is an ancestor of “the state should refuse to introduce the Euro”, the former might be endowed with an acceptance threshold of 3/4 of the population, and the latter with one of 1/2 of the population; similarly, if “a multi-cultural society is desirable” is an ancestor of “immigration law X should be amended”, the former needs a higher acceptance threshold than the latter. In short, the collective must accept more fundamental propositions less easily than less fundamental ones.

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