Judgment aggregation by quota rules

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It is known that majority voting among several individuals on logically interconnected propositions may generate irrational collective judgments. We generalize majority voting by considering quota rules, which accept each proposition if and only if the number of individuals accepting it exceeds some (proposition-specific) threshold. After characterizing quota rules, we prove necessary and sufficient conditions under which their outcomes satisfy various rationality conditions. We also consider sequential quota rules, which adjudicate propositions sequentially, letting earlier judgments constrain later ones. While ensuring rationality, sequential rules may be path-dependent. We characterize path-independence and prove its equivalence to strategy-proofness under mild conditions. Our results generalize earlier (im)possibility theorems. JEL Classification Number: D71.

1 Introduction

How can a group of individuals make collective judgments on multiple logically connected propositions based on the individuals' judgments on these propositions? It is natural for the group to take a majority vote on each proposition. But propositionwise majority voting does not guarantee 'rational' collective judgments, as a simple example illustrates. Suppose a three-member government has to make judgments on the following propositions:

a : Country X has weapons of mass destruction.

b : Action Y should be taken against country X.

$b \leftrightarrow a$: Action Y should be taken against country X if and only if country X has weapons of mass destruction.

The judgments of the three government members are as shown in table 1, each individually consistent.

	a	$b \leftrightarrow a$	b
Individual 1	True	True	True
Individual 2	False	False	True
Individual 3	False	True	False
Majority	False	True	True

Table 1

Then a majority rejects a (i.e. holds that country X does not have any weapons of mass destruction); a majority accepts $b \leftrightarrow a$ (i.e. holds that action Y should be taken if and only if country X has such weapons); and yet a majority accepts b (i.e. holds that action Y should be taken), an inconsistent set of collective judgments. Problems of this kind are sometimes called 'discursive dilemmas' (Pettit 2001). Can we modify propositionwise majority voting to avoid such problems?

This paper addresses a general class of judgment aggregation rules: quota rules. Under a *quota rule*, a proposition is collectively accepted if and only if the number

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of individuals accepting it is greater than or equal to some threshold, which may depend on the proposition in question. Propositionwise majority voting is a special quota rule with a simple majority threshold for every proposition. Generally, as propositions may differ in status and importance, the threshold may vary from proposition to proposition. In many real-world decision-making bodies, a higher acceptance threshold is required for more important propositions (e.g. constitutional amendments or taking action Y against country X) than for less important ones (e.g. ordinary legislation).

After characterizing the class of quota rules, we prove necessary and sufficient conditions under which a quota rule guarantees 'collective rationality'. We address each of the rationality conditions of (weak and strong) consistency, deductive closure and completeness. We show that the agenda of propositions under consideration determines whether these conditions can be met. If the interconnections between the propositions are above a certain complexity, no quota rule guarantees full 'collective rationality'.

So how can rational collective judgments be achieved? In the real world, groups often consider different propositions not simultaneously, but sequentially, letting earlier judgments constrain later ones. Under a *sequential quota rule*, a group considers different propositions in a sequence and takes a vote (applying the relevant acceptance threshold) only on those propositions on which the judgments are not yet constrained by earlier judgments.

Sequential quota rules guarantee collective consistency by design (and sometimes completeness and deductive closure), but may be *path-dependent*: the order in which the propositions are considered may affect the outcome. We show that a sequential quota rule is path-*in*dependent if and only if its corresponding ordinary quota rule is collectively rational in an appropriate sense, which implies that path-*in*dependence is a demanding condition.

Path-dependence matters for two reasons. First, path-dependent sequential rules are obviously vulnerable to manipulation by changes of the decision-path. Second, and less obviously, path-dependent sequential rules are also vulnerable to strategic voting. We show that, under mild conditions, strategy-proofness of a sequential quota rule is logically equivalent to its path-*in*dependence.

Our findings show that groups forming collective judgments on multiple propositions may face a trade-off between democratic responsiveness, collective rationality and strategy-proofness.

The problem of judgment aggregation was first formalized by List and Pettit (2002, 2004), who also proved a first impossibility theorem. Stronger impossibility results were proved by Pauly and van Hees (2004), Dietrich (2004a/b), Gärdenfors (2004), Nehring and Puppe (2004a) and van Hees (2004). List (2003), Dietrich (2004a) and Pigozzi (2004) proved possibility results. Nehring and Puppe (2004b) investigated the related framework of property spaces and proved a characterization of collective consistency similar to the one given here; we extend their contribution by considering also the other rationality conditions discussed in the literature, such as deductive closure, and sequential aggregation rules. Path-dependence and strategy-proofness were first discussed in List (2004) and Dietrich and List (2004), but not with respect to quota rules; we extend the latter contributions by fully characterizing path-(in)dependence and strategy-proofness under sequential quota rules. All proofs are given in an appendix.

2 Judgment aggregation: the basic model

We consider a group of individuals $N = \{1, 2, \ldots, n\}$ $(n \geq 2)$, which seeks to make collective judgments on logically connected propositions.

2.1 The propositions under consideration

Propositions are represented in formal logic. For simplicity, we use standard propositional logic, but our results are also true in more general logics.[2] Here our propositional language $\mathbf{L}$ contains *atomic propositions* without logical connectives and *non-atomic propositions* with the logical connectives $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), $\leftrightarrow$ (if and only if). Formally, $\mathbf{L}$ is the (smallest) set such that (i) $\mathbf{L}$ contains the given *atomic propositions* a, b, c, ..., and (ii) whenever $\mathbf{L}$ contains two propositions p and q, then $\mathbf{L}$ also contains $\neg p$, $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$ and $(p \leftrightarrow q)$. As a notational convention, we drop external brackets around propositions, e.g. instead of $((a \wedge b) \rightarrow c)$ we write $(a \wedge b) \rightarrow c$.

A *truth-value assignment* is a function assigning the value 'true' or 'false' to each proposition in $\mathbf{L}$, with standard properties.[3] A set of propositions $S \subseteq \mathbf{L}$ is *consistent* if there exists some truth-value assignment for which all propositions in S are true, and *inconsistent* otherwise; S is *minimal inconsistent* if S is inconsistent, but every proper subset of S is consistent. For example, $\{a \wedge b, \neg a, \neg b\}$ is inconsistent and $\{a \wedge b, \neg a\}$ is minimal inconsistent. A set $S \subseteq \mathbf{L}$ *entails* a proposition $p \in \mathbf{L}$ if $S \cup \{\neg p\}$ is inconsistent.

The *agenda* is the set of propositions under consideration; it is a finite non-empty subset $X \subseteq \mathbf{L}$, where (i) X contains no double-negated propositions $(\neg\neg p)$, (ii) X is a union of proposition-negation pairs $\{p, \neg p\}$, (iii) X does not contain any tautologies (propositions true for every truth-value assignment) or contradictions (propositions false for every truth-value assignment).

For simplicity, we introduce a modified negation operator $\sim$, where

$$\sim p := \begin{cases} \neg p & \text{if } p \text{ is not the negation of some other proposition,} \\ q & \text{if } p = \neg q \text{ for some other proposition } q. \end{cases}$$

In the example above, the agenda is $X := \{a, b, b \leftrightarrow a, \sim a, \sim b, \sim (b \leftrightarrow a)\}$.

2.2 Individual and collective judgment sets

Each individual i's *judgment set* is a subset $A_i \subseteq X$, where $p \in A_i$ means 'individual i accepts proposition p'. A *profile (of individual judgment sets)* is an n-tuple $(A_1, \ldots, A_n)$.

A *(judgment) aggregation rule* is a function F that maps each profile $(A_1, \ldots, A_n)$ in a given domain to a *(collective) judgment set* $F(A_1, \ldots, A_n) = A \subseteq X$, where $p \in A$ means 'the group accepts proposition p'.

We introduce several rationality conditions on a judgment set A (individual or collective): (i) A is *complete* if it contains at least one member of each pair $p, \sim$

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[2] We can use any formal language $\mathbf{L}$ satisfying the minimal conditions (L1)-(L5) in Dietrich (2004b). In addition to standard propositional logic as defined above, this permits several more expressive logics, including predicate calculi, modal logics and conditional logics.

[3] A truth-value assignment satisfies the following. For any p,$q \in \mathbf{L}$, $\neg p$ is true if and only if p is false; $p \wedge q$ is true if and only if both p and q are true; $p \vee q$ is true if and only if at least one of p or q is true; $p \rightarrow q$ is true if and only if it is not the case that [p is true and q is false]; $p \leftrightarrow q$ is true if and only if p and q are both true or both false.

$p \in X$; (ii) A is *weakly consistent* if it contains at most one member of each pair $p, \sim p \in X$; (iii) A is *consistent* if there exists some truth-value assignment for which all propositions in A are true (i.e. A is a consistent set); (iv) A is *deductively closed* if, whenever a consistent subset $B \subseteq A$ entails a proposition $p \in X$, then $p \in X$. These rationality conditions are logically interrelated as follows.

Lemma 1 *For any judgment set A,*
(a) *consistency implies weak consistency;*
(b) *given deductive closure, consistency is equivalent to weak consistency;*
(c) *given completeness, consistency is equivalent to the conjunction of weak consistency and deductive closure.*

A judgment set is *fully rational* if it is complete and consistent (hence also weakly consistent and deductively closed, by lemma 1). We call the set of all possible profiles of fully rational judgment sets the *universal domain.*

3 Quota rules and collective rationality

After defining and characterizing the class of quota rules for judgment aggregation, we prove necessary and sufficient conditions under which a quota rule satisfies various collective rationality conditions and generalize earlier (im)possibility results.

3.1 Quota rules

Consider any *family of thresholds* $(m_p)_{p\in X}$ such that, for each proposition $p \in X$, $m_p \in \{1, ..., n\}$. A *quota rule* is the aggregation rule $F_{(m_p)_{p\in X}}$ with universal domain given by

$$F_{(m_p)_{p\in X}}(A_1, ..., A_n) \quad := \{p \in X : |\{i \in N : p \in A_i\}| \geq m_p\} \qquad \text{for each profile } (A_1, \ldots, A_n).$$

Informally, each proposition $p \in X$ is collectively accepted if and only if it is accepted by at least m_p individuals.

A quota rule $F_{(m_p)_{p\in X}}$ $(= F_m)$ is *uniform* if the acceptance threshold takes the same value $m_p = m$ for all $p \in X$. Examples of uniform quota rules are *propositionwise majority rule* (where $m = \lceil (n+1)/2 \rceil$, with $\lceil x \rceil$ defined as the smallest integer greater than or equal to x), *propositionwise special majority rule* (where $\lceil (n+1)/2 \rceil < m < n$) and *propositionwise unanimity rule* (where $m = n$).

It is easy to characterize the class of quota rules using the following conditions.

Universal domain. The domain of F is the universal domain, i.e. the set of all possible profiles of fully rational individual judgment sets.

Anonymity. For every two profiles $(A_1, ..., A_n)$, $(A_{\pi(1)}, ..., A_{\pi(n)})$ in the domain of F, where $\pi : N \mapsto N$ is any permutation of the individuals, $F(A_1, ..., A_n) = F(A_{\pi(1)}, ..., A_{\pi(n)})$.

Responsiveness. For every proposition $p \in X$, there exist at least two profiles $(A_1, ..., A_n)$, $(A^*_1, ..., A^*_n)$ in the domain of F such that $p \in F(A_1, ..., A_n)$ and $p \notin F(A^*_1, ..., A^*_n)$.

Independence. For every proposition $p \in X$ and profiles $(A_1, ..., A_n)$, $(A_1^*, ..., A_n^*)$ in the domain of F, if [for all individuals i, $p \in A_i$ if and only if $p \in A_i^*$], then [$p \in F(A_1, ..., A_n)$ if and only if $p \in F(A_1^*, ..., A_n^*)$].

Two profiles are *i-variants* of each other if they coincide for all individuals except possibly i.

Monotonicity. For every proposition $p \in X$, individual i, and pair of i-variants $(A_1, ..., A_n)$, $(A_1, ..., A_i', ..., A_n)$ in the domain of F with $p \notin A_i$ and $p \in A_i'$, if $p \in F(A_1, ..., A_n)$ then $p \in F(A_1, ..., A_i', ..., A_n)$.

Universal domain states that every possible profile of fully rational individual judgment sets is admissible. Anonymity requires giving equal consideration to all individuals' judgment sets. Responsiveness rules out that some proposition in the agenda is never accepted or never rejected. Independence requires propositionwise aggregation, i.e. the collective judgment on a proposition depends only on the individuals' judgments on that proposition and not on their judgments on other propositions. Monotonicity requires that an additional individual's support for an accepted proposition does not lead to the rejection of that proposition.

Proposition 1 *An aggregation rule has universal domain, is anonymous, responsive, independent and monotonic if and only if it is a quota rule* $F_{(m_p)_{p \in X}}$ *for some family of thresholds* $(m_p)_{p \in X}$.[4]

3.2 Necessary and sufficient conditions for collective rationality

We began with the observation that propositionwise majority voting does not guarantee rational collective judgments. Quota rules generalize propositionwise majority voting by allowing any family of thresholds $(m_p)_{p \in X}$ instead of the same threshold $m = \lceil (n+1)/2 \rceil$ for all propositions. Can we specify the thresholds such that the corresponding quota rule guarantees collective rationality?

We call an aggregation rule *complete* (respectively: *weakly consistent*, *consistent*, *deductively closed* and *fully rational*) if it generates, for every profile in its domain, a complete (respectively: weakly consistent, consistent, deductively closed and fully rational) collective judgment set.

Theorem 1 *A quota rule* $F_{(m_p)_{p \in X}}$ *is*

(a) *complete if and only if*

$$m_p + m_{\sim p} \leq n + 1 \text{ for every pair } p, \sim p \in X; \tag{1}$$

(b) *weakly consistent if and only if*

$$m_p + m_{\sim p} > n \text{ for every pair } p, \sim p \in X; \tag{2}$$

(c) *consistent if and only if*

$$\sum_{p \in Z} m_p > n(|Z| - 1) \text{ for every minimal inconsistent set } Z \subseteq X; \tag{3}$$

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[4] The result remains true if responsiveness is dropped and each $m_p \in \{0, ..., n+1\}$ (rather than $\{1, ..., n\}$), permitting degenerate quota rules.

(d) *deductively closed if and only if*

$$\sum_{p\in Z\backslash\{q\}} m_p - m_{\sim q} \geq n(|Z|-2) \quad \begin{array}{l}\textit{for every member } q \textit{ of every}\\ \textit{minimal inconsistent set } Z\subseteq X.\end{array} \tag{4}$$

The inequalities in (2) are just the inequalities in (3) restricted to minimal inconsistent sets of the form $Z=\{p,\sim p\}\subseteq X$. Nehring and Puppe (2004a) have proved a result similar to part (c) of theorem 1 using the 'intersection property', which we briefly discuss in our concluding remarks.

To see how strongly the consistency requirement restricts the thresholds $(m_p)_{p\in X}$, note that the inequality in (3) is equivalent to $\frac{1}{|Z|}\sum_{p\in Z} m_p > n(1-1/|Z|)$. So the average threshold m_p for the acceptance of p (averaging over $p\in Z$) must exceed $n(1-1/|Z|)$, a value approaching n as the size of a minimal inconsistent set Z increases.

By combining the inequalities in theorem 1, we obtain conditions under which a quota rule satisfies more than one rationality condition.

Corollary 1 *A quota rule* $F_{(m_p)_{p\in X}}$ *is*

(a) *complete and weakly consistent if and only if*

$$m_p + m_{\sim p} = n+1 \textit{ for every pair } p,\sim p\in X;$$

(b) *consistent and deductively closed if and only if*

$$\sum_{p\in Z\backslash\{q\}} m_p + \min\{m_q, n+1-m_{\sim q}\} > n(|Z|-1) \quad \begin{array}{l}\textit{for every member } q \textit{ of every}\\ \textit{minimal inconsistent set } Z\subseteq X;\end{array} \tag{5}$$

(c) *fully rational if and only if*

$$\begin{array}{ll} m_p + m_{\sim p} = n+1 & \textit{for every pair } p,\sim p\in X, \textit{ and}\\ \sum_{p\in Z} m_p > n(|Z|-1) & \textit{for every minimal inconsistent set } Z\subseteq X.\end{array} \tag{6}$$

3.3 The special case of uniform quota rules

As noted above, an important special class of quota rules are the uniform ones, where the acceptance threshold is the same for all propositions. Here the inequalities characterizing consistency and deductive closure reduce to some simple conditions.

Corollary 2 *Let z be the size of the largest minimal inconsistent set $Z\subseteq X$.*

(a) *A uniform quota rule F_m is consistent if and only if $m > n-n/z$. In particular, for $n\neq 2,4$, propositionwise majority rule (where $m=\lceil (n+1)/2\rceil$) is consistent if and only if $z\leq 2$; if $n=2$, it is always consistent; if $n=4$, it is consistent if and only if $z\leq 3$.*

(b) *A uniform quota rule F_m is deductively closed if and only if $m=n$ (i.e. F_m is propositionwise unanimity rule) or $z\leq 2$. In particular, if $n\geq 3$, propositionwise majority rule (where $m=\lceil (n+1)/2\rceil$) is deductively closed if and only if $z\leq 2$; if $n=2$, it is always deductively closed.*

(c) *A uniform quota rule F_m is consistent and deductively closed if and only if $m=n$ (i.e. it is propositionwise unanimity rule) or [$z\leq 2$ and $m>n/2$].*

Note that z indicates how complex the logical interconnections in the agenda are; $z \leq 2$ corresponds to an agenda without any non-trivial interconnections, i.e. without any minimal inconsistent sets of more than two propositions. Propositionwise unanimity rule is always consistent and deductively closed, at the expense of significant incompleteness. By contrast, propositionwise special majority rule is consistent if and only if the acceptance threshold for every proposition exceeds $n(1-1/z)$, which approaches 1 as z increases, and it is deductively closed only in the special case $z \leq 2$. Propositionwise majority rule (when $n \geq 3$) is consistent and deductively closed only in the special case $z \leq 2$. These results generalize the 'discursive dilemma' with which we began.

3.4 A general (im)possibility result

By combining theorem 1 and proposition 1, we can characterize the types of agendas X for which there exist fully rational aggregation rules that satisfy the conditions introduced in the previous section.

Corollary 3 *An aggregation rule with universal domain is anonymous, responsive, independent, monotonic and fully rational if and only if it is a quota rule* $F_{(m_p)_{p\in X}}$ *satisfying* (6) *above. In particular, there exists an aggregation rule with these properties if and only if the system* (6) *admits a solution* $(m_p)_{p\in X}$ *in* $\{1,...,n\}^X$.

This corollary can be seen as an impossibility result: the (in)equalities in (6) have solutions only for special agendas with few logical connections between propositions.

3.5 An example

For the agenda $X := \{a, b, b \leftrightarrow a, \sim a, \sim b, \sim (b \leftrightarrow a)\}$ from our initial example, the minimal inconsistent subsets $Z \subseteq X$ are $\{a, \sim a\}$, $\{b, \sim b\}$, $\{b \leftrightarrow a, \sim (b \leftrightarrow a)\}$, $\{a, \sim b, b \leftrightarrow a\}$, $\{\sim a, b, b \leftrightarrow a\}$, $\{a, b, \sim (b \leftrightarrow a)\}$ and $\{\sim a, \sim b, \sim (b \leftrightarrow a)\}$. We show that there exists no fully rational quota rule for this agenda. Assume, for a contradiction, that $F_{(m_p)_{p\in X}}$ is fully rational. Then, by part (c) of corollary 1,

$$m_a + m_{\sim a} = m_b + m_{\sim b} = m_{b\leftrightarrow a} + m_{\sim(b\leftrightarrow a)} = n+1, \tag{7}$$

$$m_a + m_{\sim b} + m_{b\leftrightarrow a} > 2n \text{ and } m_{\sim a} + m_b + m_{b\leftrightarrow a} > 2n, \tag{8}$$

$$m_a + m_b + m_{\sim(b\leftrightarrow a)} > 2n \text{ and } m_{\sim a} + m_{\sim b} + m_{\sim(b\leftrightarrow a)} > 2n. \tag{9}$$

By adding the two inequalities in (8), we obtain $m_a + m_{\sim a} + m_b + m_{\sim b} + 2m_{b\leftrightarrow a} > 4n$. By (7), $n+1+n+1+2m_{b\leftrightarrow a} > 4n$, hence $2m_{b\leftrightarrow a} > 2n-2$, i.e. $m_{b\leftrightarrow a} = n$. An analogous argument for the two inequalities in (9) yields $m_{\sim(b\leftrightarrow a)} = n$. So $m_{b\leftrightarrow a} + m_{\sim(b\leftrightarrow a)} = 2n > n+1$, which violates (7).

But, for a slightly modified agenda, there is a fully rational quota rule. Replace the biconditional $b \leftrightarrow a$ (action should be taken if and only if country X has weapons) by the simple conditional $a \rightarrow b$ (if country X has weapons, then action Y should be taken). The new agenda is thus $X := \{a, b, a \rightarrow b, \sim a, \sim b, \sim (a \rightarrow b)\}$. The minimal inconsistent sets $Z \subseteq X$ are now $\{a, \sim a\}$, $\{b, \sim b\}$, $\{a \rightarrow b, \sim (a \rightarrow b)\}$, $\{\sim a, \sim (a \rightarrow b)\}$, $\{b, \sim (a \rightarrow b)\}$ and $\{a, \sim b, a \rightarrow b\}$. By part (c) of corollary 1, a quota rule $F_{(m_p)_{p\in X}}$ is fully rational if and only if

$$\begin{aligned} &m_a + m_{\sim a} = m_b + m_{\sim b} = m_{a\rightarrow b} + m_{\sim(a\rightarrow b)} = n+1, \\ &m_{\sim a} + m_{\sim(a\rightarrow b)} > n \text{ and } m_b + m_{\sim(a\rightarrow b)} > n \text{ and } m_a + m_{\sim b} + m_{a\rightarrow b} > 2n. \end{aligned}$$

By expressing each $m_{\sim p}$ as $n+1-m_p$, the three inequalities become

$$-m_a+m_{\sim(a\to b)} > -1 \text{ and } -m_{\sim b}+m_{\sim(a\to b)} > -1 \text{ and } m_a+m_{\sim b}-m_{\sim(a\to b)} > n-1;$$

equivalently,

$$m_{\sim(a\to b)} \geq m_a \text{ and } m_{\sim(a\to b)} \geq m_{\sim b} \text{ and } m_{\sim(a\to b)} \leq m_a+m_{\sim b}+1.$$

The only solution to these inequalities in $\{1,...,n\}^X$ is $m_a = m_{\sim b} = m_{\sim(a\to b)} = n$, i.e. a unanimity threshold for each of a, $\sim b$ and $\sim(a \to b)$ and a threshold of 1 for each of $\sim a$, b and $a \to b$. So, in our example, the proposition that country X has weapons of mass destruction is accepted only if all individuals accept that proposition, whereas the proposition that action Y should be taken and the proposition that weapons require action are each accepted as soon as they are accepted by just one individual, a questionable aggregation rule.

Further, for the original agenda and also the modified one, the size of the largest minimal inconsistent set is $z = 3$, so by corollary 2 a uniform quota rule F_m is consistent if and only if $m > \frac{2}{3}n$ and deductively closed if and only if $m = n$. By implication, for both agendas, there exists no fully rational uniform quota rule.

3.6 The computational usefulness of the inequalities

In addition to providing new theoretical insights, the inequalities in theorem 1 and its corollaries are computationally useful. First, suppose we wish to verify whether a *given* quota rule $F_{(m_p)_{p\in X}}$ satisfies some rationality condition. Without theoretical results, we would have to consider every profile in the universal domain and determine whether the collective judgment set for that profile satisfies the required condition. The number of such profiles grows exponentially in the group size n (of course, it also depends on the structure of the agenda). By contrast, the number of inequalities in each part of theorem 1 does not depend on n; it is determined only by the structure of the agenda. So, by using our inequalities, verifying the rationality of a given quota rule is computationally feasible even for large group sizes.

Second, suppose we wish to verify, for a given agenda and a given number of individuals, whether there *exists* a fully rational quota rule. Even for a small n, this task is computationally hard. There are n^k possible quota rules for n individuals and k propositions, and, for each of these n^k rules, we would have to consider every possible profile and check the rationality of the outcome under that profile, where the number of such profiles grows exponentially in n. But, if we use corollary 3, the problem reduces to verifying whether the system of linear (in)qualities (6) admits a solution $(m_p)_{p\in X}$ in $\{1,...,n\}^X$, a computationally feasible task; the simplex procedure can be used.

4 Sequential quota rules and path-dependence

We have seen that, for agendas above a certain complexity, there exists no fully rational quota rule. A group can solve this problem by making judgments on multiple propositions sequentially, letting earlier judgments constrain later ones. We now consider the class of sequential quota rules, which are always consistent, but may be path-dependent. After formally defining path-dependence, we prove necessary and sufficient conditions for its avoidance. In the subsequent section, we address the relation between path-dependence of a sequential quota rule and its manipulability by strategic voting.

4.1 Sequential quota rules

A *decision-path* is a one-to-one function $\Omega : \{1, 2, ..., k\} \to X$, with $k = |X|$, where $p_1 := \Omega(1)$, $p_2 := \Omega(2)$, ..., $p_k := \Omega(k)$ are the propositions considered first, second, ..., last. The decision-path may reflect either the temporal order in which the propositions come up or an order of priority among the propositions. When the propositions are considered along a given decision-path, the group's judgment on a new proposition may be logically constrained by its judgments on earlier propositions. If so, the group derives its judgment on the new proposition from those earlier judgments. If not, the group takes a vote on the new proposition, applying a certain acceptance threshold. This generalizes the approach in List (2004) by allowing different acceptance thresholds for different propositions.

For any decision-path Ω and any family of thresholds $(m_p)_{p\in X}$, a *sequential quota rule* $F_{\Omega,(m_p)_{p\in X}}$ is the aggregation rule with universal domain given as follows. For each profile $(A_1, \ldots, A_n)$,

$$F_{\Omega,(m_p)_{p\in X}}(A_1, ..., A_n) := \Phi_k,$$

where the set Φ_k is obtained recursively in k steps: for $t = 1, ..., k$,

$$\Phi_t := \begin{cases} \Phi_{t-1} \cup \{p_t\} & \text{if } \Phi_{t-1} \text{ entails } p_t \textit{ or } \left[\begin{array}{l} \Phi_{t-1} \cup \{p_t\} \text{ is consistent } \textit{and} \\ |\{i \in N : p_t \in A_i\}| \geq m_{p_t} \end{array}\right], \\ \Phi_{t-1} & \text{otherwise,} \end{cases}$$

with $\Phi_0 := \varnothing$.

Informally, for each t, Φ_t is the set of propositions accepted up to step t; p_t is accepted at step t if *either* past judgments require the acceptance of p_t *or* [past judgments are consistent with p_t *and* the group votes the accept p_t].

As before, a sequential quota rule $F_{\Omega,(m_p)_{p\in X}}$ $(= F_{\Omega,m})$ is *uniform* if the acceptance threshold takes the same value $m_p = m$ for all $p \in X$. A sequential quota rule is always consistent by design (hence also weakly consistent). Whether it is also complete and deductively closed depends on the decision-path Ω and the family of thresholds $(m_p)_{p\in X}$.[5]

4.2 An example

To illustrate that the outcome of a sequential quota rule may depend on the decision-path, consider our first example, where the agenda is $X := \{a, b, b \leftrightarrow a, \sim a, \sim b, \sim (b \leftrightarrow a)\}$ and there are three individuals with judgment sets $A_1 = \{a, b \leftrightarrow a, b\}$, $A_2 = \{\sim a, \sim (b \leftrightarrow a), b\}$ and $A_3 = \{\sim a, b \leftrightarrow a, \sim b\}$, as shown in table 1. Suppose the group uses a sequential quota rule $F_{\Omega,m}$, with a simple majority threshold $m = 2$ for every proposition $p \in X$. Consider two different decision-paths, Ω_1 and Ω_2, as shown in table 2.

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[5] Consider a (natural) decision-path in which each proposition $p \in X$ and its negation $\sim p$ are adjacent, i.e. $\sim p$ comes immediately before or after p, and suppose that the thresholds m_p and $m_{\sim p}$ satisfy $m_p + m_{\sim p} \leq n + 1$, meaning that the corresponding quota rule $F_{(m_p)_{p\in X}}$ is complete. Then the sequential quota rule $F_{\Omega,(m_p)_{p\in X}}$ is complete (and hence deductively closed by consistency). Informally, the reason is that, when the sequential decision process reaches a pair of adjacent propositions $p, \sim p$, either the past judgments entail p or $\sim p$, in which case p or $\sim p$ is accepted, or the past judgments entail neither p nor $\sim p$, in which case again p or $\sim p$ is accepted since the relation $m_p + m_{\sim p} \leq n + 1$ ensures that the support for p or $\sim p$ exceeds the appropriate threshold.

t	1	2	3	4	5	6
$\Omega_1(t)$	a	$\sim a$	$b \leftrightarrow a$	$\sim (b \leftrightarrow a)$	b	$\sim b$
$\Omega_2(t)$	b	$\sim b$	$b \leftrightarrow a$	$\sim (b \leftrightarrow a)$	a	$\sim a$

Table 2

It is easy to see that the decision-paths Ω_1 and Ω_2 lead to different outcomes. Under Ω_1, $\sim a$ and $b \leftrightarrow a$ are each accepted by a vote and $\sim b$ is accepted by inference, resulting in the judgment set $\{\sim a, b \leftrightarrow a, \sim b\}$. In our example, the government first forms the view that country X has no weapons and that weapons are the required justification for action before deriving the view that no action should be taken. Under Ω_2, b and $b \leftrightarrow a$ are each accepted by a vote and a is accepted by inference, resulting in the judgment set $\{a, b \leftrightarrow a, b\}$. Here the government first forms the view that action should be taken and that weapons are the required justification for action before deriving the view that country X has weapons.

4.3 Necessary and sufficient conditions for path-independence

Formally, a sequential quota rule $F_{\Omega,(m_p)_{p\in X}}$ is *path-dependent* if there exist two decision-paths Ω_1 and Ω_2, a profile $(A_1, \ldots, A_n)$ and a proposition $p \in X$ such that

$$p \in F_{\Omega_1,(m_p)_{p\in X}}(A_1, \ldots, A_n) \text{ and } p \notin F_{\Omega_2,(m_p)_{p\in X}}(A_1, \ldots, A_n),$$

and *path-independent* otherwise; $F_{\Omega,(m_p)_{p\in X}}$ is *strongly path-dependent* if there exist two decision-paths Ω_1 and Ω_2, a profile $(A_1, \ldots, A_n)$ and a proposition $p \in X$ such that

$$p \in F_{\Omega_1,(m_p)_{p\in X}}(A_1, \ldots, A_n) \text{ and } \sim p \in F_{\Omega_2,(m_p)_{p\in X}}(A_1, \ldots, A_n),$$

and *weakly path*-in*dependent* otherwise. Strong path-dependence implies path-dependence; path-independence implies weak path-independence.

When is a sequential quota rule path-dependent, when not? By combining a result in List (2004) with theorem 1 above, we can answer this question.

Theorem 2 *A sequential quota rule* $F_{\Omega,(m_p)_{p\in X}}$ *is*

(a) *weakly path-independent if and only if the corresponding ordinary quota rule* $F_{(m_p)_{p\in X}}$ *is consistent, i.e. if and only if* (3) *above holds;*

(b) *path-independent if and only if the corresponding ordinary quota rule* $F_{(m_p)_{p\in X}}$ *is consistent and deductively closed, i.e. if and only if* (5) *above holds.*

We can also address the special case of a uniform sequential quota rule, combining theorem 2 and corollary 2.

Corollary 4 *Let z be the size of the largest minimal inconsistent set $Z \subseteq X$.*

(a) *A uniform sequential quota rule $F_{\Omega,m}$ is weakly path-independent if and only if $m > n - n/z$. In particular, for $n \neq 2, 4$, a sequential majority rule (where $m = \lceil (n+1)/2 \rceil$) is weakly path-independent if and only if $z \leq 2$; if $n = 2$, it is always weakly path-independent; if $n = 4$, it is weakly path-independent if and only if $z \leq 3$.*

(b) *A uniform sequential quota rule $F_{\Omega,m}$ is path-independent if and only if $m = n$ (i.e. it is a sequential unanimity rule) or [$z \leq 2$ and $m > n/2$].*

Our example above illustrates this result: sequential majority voting is path-dependent because we have $z = 3$ (with $n = 3$), which violates $z \leq 2$.

5 Path-independence and strategy-proofness

Path-dependent sequential quota rules are obviously vulnerable to manipulation by agenda setters who can influence the order in which the propositions are considered. In our example, an agenda setter who cares about taking action Y will set the decision-path Ω_2, whereas one who cares about avoiding action Y will set the decision-path Ω_1. But path-dependent rules are also vulnerable to strategic voting, i.e. to the misrepresentation of judgments by the individuals; specifically, under mild conditions, we show that strategy-proofness is equivalent to path-*in*dependence. We also note that ordinary quota rules are always strategy-proof, although their use is limited given their rationality violations.

5.1 Strategy-proofness

We now assume that each individual has not only a judgment set, but also an underlying preference relation – possibly only partial – over all possible judgment sets. This assumption captures the idea that, in comparing different collective judgment sets as potential outcomes, individuals will prefer some judgment sets to others.

Formally, each individual i has a preference relation $\succsim_i$ over all possible judgment sets of the form $A \subseteq X$. We assume that preference relations are reflexive and transitive (but not necessarily complete). We also require that $\succsim_i$ is *compatible* with individual i's judgment set A_i as follows. We say that one judgment set, A, *agrees* with another, A^*, on a proposition $p \in X$ if either both or none of A and A^* contains p. Now $\succsim_i$ is *compatible* with A_i if the following holds: whenever two judgment sets A and A^* are such that [for all propositions $p \in X$, if A^* agrees with A_i on p, then so does A], then $A \succsim_i A^*$. Informally, compatibility of $\succsim_i$ with A_i requires that, if one judgment set is at least as close as another to an individual's own judgments on the propositions, then the individual weakly prefers the first judgment set to the second. In particular, an individual most prefers his or her own judgment set.

Now we can define strategy-proofness of an aggregation rule F.

Strategy-proofness. For every profile $(A_1, ..., A_n)$ in the domain of F, every individual i and any preference relation $\succsim_i$ compatible with A_i, $F(A_1, ..., A_n) \succsim_i F(A_1, ..., A_i^*, ..., A_n)$ for every i-variant $(A_1, ..., A_i^*, ..., A_n)$ in the domain of F.

Informally, strategy-proofness requires that, for every profile, each individual weakly prefers the collective judgment set that is obtained from expressing his or her own judgment set truthfully to any collective judgment set that would be obtained from misrepresenting his or her judgment set (where other individuals' judgment sets are held fixed).

Proposition 2 *(Dietrich and List 2004) An aggregation rule with universal domain is strategy-proof if and only if it is independent and monotonic.*

This proposition immediately implies that ordinary quota rules are strategy-proof, as they are independent and monotonic by proposition 1. But, as we have seen, such rules often generate rationality violations. Are sequential quota rules ever strategy-proof?

5.2 An example

Consider again our example of the three-member government with judgments as shown in table 1. Suppose the government uses a sequential majority rule with decision-path Ω_1 as shown in table 2. Assuming that all three government members express their judgments truthfully, the decision-path Ω_1 leads to the collective judgment set $\{\sim a, b \leftrightarrow a, \sim b\}$, i.e. a decision not to take action Y against country X, as shown above. But suppose individual 2 cares strongly about taking action Y, i.e. the acceptance of proposition b. Specifically, the following preference relation is compatible with individual 2's judgment set A_2:

$$\{\sim a, \sim (b \leftrightarrow a), b\} \succ_2 \{a, (b \leftrightarrow a), b\} \succ_2 \{a, \sim (b \leftrightarrow a), \sim b\} \succ_2 \{\sim a, b \leftrightarrow a, \sim b\},$$

where $\succ_2$ is the strong component of $\succsim_2$.

If individual 2 strategically expresses the judgment set $A_2^* = \{a,\ b \leftrightarrow a, b\}$ instead of his or her truthful judgment set $A_2 = \{\sim a,\ \sim (b \leftrightarrow a), b\}$, then sequential majority voting leads to the collective judgment set $\{a, b \leftrightarrow a, b\}$ instead of $\{\sim a, b \leftrightarrow a, \sim b\}$, where $\{a, b \leftrightarrow a, b\} \succ_2 \{\sim a, b \leftrightarrow a, \sim b\}$. So, by pretending to believe that country X has weapons and that weapons justify action, individual 2 can bring about the preferred decision to take action Y against country X. Hence sequential majority rule on the given agenda with decision-path Ω_1 is not strategy-proof.

5.3 Necessary and sufficient conditions for strategy-proofness

To state necessary and sufficient conditions for strategy-proofness of a sequential quota rule, we first introduce a simple condition on the representation of such a rule.

Note that, for a fixed decision-path Ω, two different families of thresholds $(m_p)_{p \in X}$ and $(m_p^*)_{p \in X}$ may yield the same aggregation rule (i.e. mapping from profiles to collective judgments). Let $X = \{a, \sim a\}$, $\Omega(1) = a$, $\Omega(2) = \sim a$, $m_a = m_{\sim a} = m_a^* = (n+1)/2$ (with n odd) and $m_{\sim a}^* = 1$. The rules $F_{\Omega,(m_p)_{p \in X}}$ and $F_{\Omega,(m_p^*)_{p \in X}}$ both accept a whenever a majority supports a, and $\sim a$ whenever a majority supports $\sim a$. This is obvious for $F_{\Omega,(m_p)_{p \in X}}$ and holds for $F_{\Omega,(m_p^*)_{p \in X}}$ because any submajority acceptance of $\sim a$ at step 2 in the recursive decision process is overruled by the majority acceptance of a at step 1. So $F_{\Omega,(m_p)_{p \in X}}$ and $F_{\Omega,(m_p^*)_{p \in X}}$ represent the same aggregation rule, though $F_{\Omega,(m_p)_{p \in X}}$ does so more transparently.

We say that m_p is the *effective threshold* for proposition $p \in X$ under the aggregation rule F if, for all profiles $(A_1, ..., A_n)$ in the domain of F, $p \in F(A_1, ..., A_n)$ if and only if $|\{i \in N : p \in A_i\}| \geq m_p$. A sequential quota rule $F_{\Omega,(m_p)_{p \in X}}$ is *transparent* if, for any proposition $p \in X$, [if there exists an effective threshold for p, then m_p is this threshold]. Transparency is a weak requirement: every sequential quota rule $F_{\Omega,(m_p)_{p \in X}}$ can – if not yet transparent – be made transparent by adjusting some of the thresholds.

Proposition 3 *For every sequential quota rule* $F_{\Omega,(m_p)_{p \in X}}$*, there exists a transparent sequential quota rule* $F_{\Omega,(m_p^*)_{p \in X}}$ *with the same decision-path* Ω *such that, for every profile* $(A_1, ..., A_n)$ *in the universal domain,* $F_{\Omega,(m_p)_{p \in X}}(A_1, ..., A_n) = F_{\Omega,(m_p^*)_{p \in X}}(A_1, ..., A_n)$.

To obtain $(m_p^*)_{p \in X}$, define, for each $p \in X$, m_p^* to be the effective threshold for p if there exists such an effective threshold and $m_p^* = m_p$ otherwise.

Now we can state the logical relation between strategy-proofness and path-independence.

Theorem 3 *A complete or deductively closed transparent sequential quota rule* $F_{\Omega,(m_p)_{p\in X}}$ *is strategy-proof if and only if it is path-independent.*[6]

By combining theorem 3 with theorem 2 above, we can characterize strategy-proofness in terms of our inequalities on the family of thresholds.

Corollary 5 *A complete or deductively closed transparent sequential quota rule* $F_{\Omega,(m_p)_{p\in X}}$ *is strategy-proof if and only if* (3) *above holds.*

Together with corollary 4 above, theorem 3 finally implies a result on sequential majority and unanimity rules.

Corollary 6 *Let* z *be the size of the largest minimal inconsistent set* $Z \subseteq X$*.*

(a) *If* n *is odd, a sequential majority rule* $F_{\Omega,m}$ *(where* $m = \lceil (n+1)/2 \rceil$*) is strategy-proof if and only if* $z \leq 2$*.*

(b) *A sequential unanimity rule* $F_{\Omega,m}$ *(where* $m = n$*) is always strategy-proof.*

Our results show that strategy-proofness of sequential quota rule is a demanding condition. Moreover, among the class of uniform sequential quota rules, a sequential majority rule is strategy-proof only in the special case $z \leq 2$; a sequential unanimity rule is always strategy-proof, but again only at the expense of significant incompleteness.

6 Concluding remarks

Before summarizing our findings, let us indicate an avenue for further generalization. Quota rules are by definition anonymous. The non-anonymous generalization of a quota rule is a *committee rule*. Here each proposition $p \in X$ is endowed not with a threshold m_p but with a set $\mathcal{C}_p$ of winning coalitions $C \subseteq N$, where $N \in \mathcal{C}_p$, $\emptyset \notin \mathcal{C}_p$, and [if $C \in \mathcal{C}_p$ and $C \subseteq C^* \subseteq N$, then $C^* \in \mathcal{C}_p$]. For each family $(\mathcal{C}_p)_{p\in X}$ of sets of winning coalitions, a *committee rule* $F_{(\mathcal{C}_p)_{p\in X}}$ is the aggregation rule with universal domain given by

$$F_{(\mathcal{C}_p)_{p\in X}}(A_1, ..., A_n) = \{p \in X : \{i \in N : p \in A_i\} \in \mathcal{C}_p\} \text{ for each profile } (A_1, ..., A_n).$$

Nehring and Puppe's 'voting by committees' (2004a,b) is a committee rule with the additional property that [$C \in \mathcal{C}_p$ if and only if $N \backslash C \notin \mathcal{C}_{\sim p}$] for each $p \in X$ and each $C \subseteq N$.

Can our results on collective rationality under quota rules be generalized to committee rules? Nehring and Puppe (2004a,b) have proved that 'voting by committees' is consistent if and only if the family $(\mathcal{C}_p)_{p\in X}$ satisfies the 'intersection property'. Generally, the following can be shown in analogy to theorem 1 above, where part (c) corresponds to Nehring and Puppe's result. A committee rule $F_{(\mathcal{C}_p)_{p\in X}}$ is

(a) complete if and only if

$$C \in \mathcal{C}_p \text{ or } N \backslash C \in \mathcal{C}_{\sim p} \text{ for every pair } p, \sim p \in X \text{ and coalition } C;$$

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[6] This result and corollary 5 also holds if, instead of requiring the sequential quota rule $F_{\Omega,(m_p)_{p\in X}}$ to be complete or deductively closed, we require the corresponding ordinary quota rule $F_{(m_p)_{p\in X}}$ to be complete or deductively closed.

(b) weakly consistent if and only if

$$C \notin \mathcal{C}_p \text{ or } N \backslash C \notin \mathcal{C}_{\sim p} \text{ for every pair } p, \sim p \in X \text{ and coalition } C;$$

(c) consistent if and only if

$$\bigcap_{p \in Z} C_p \neq \emptyset \quad \begin{array}{l} \text{for every minimal inconsistent set } Z \subseteq X \\ \text{and coalitions } C_p \in \mathcal{C}_p,\ p \in Z; \end{array}$$

(d) deductively closed if and only if

$$\bigcap_{p \in Z \backslash \{q\}} C_p \in \mathcal{C}_{\sim q} \quad \begin{array}{l} \text{for every minimal inconsistent set } Z \subseteq X, \\ \text{proposition } q \in Z, \text{ and coalitions } C_p \in \mathcal{C}_p,\ p \in Z \backslash \{q\}. \end{array}$$

Using this generalization of theorem 1, our subsequent results can be generalized to the non-anonymous case too.

In conclusion, our findings have clarified the scope of rational judgment aggregation under ordinary and sequential quota rules. For each of the rationality conditions of completeness, (weak and strong) consistency and deductive closure, we have shown that a quota rule satisfies the given condition if and only if its family of acceptance thresholds satisfies an appropriate system of inequalities. As full rationality is often impossible to achieve under a quota rule, we have also considered sequential quota rules. Such rules guarantee consistency (and sometimes also completeness and deductive closure) but are path-dependent whenever the corresponding ordinary quota rule exhibits certain rationality violations. Path-dependent sequential quota rules are, in turn, vulnerable to various forms of strategic manipulation. So a group making judgments on interconnected propositions – such as whether to take a certain action, what counts as a justification for that action and whether the justification holds – may have a hard time doing so in a way that is simultaneously democratic, rational and strategy-proof.

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A Appendix

In the following we write $S \vDash p$ as an abbreviation for 'S entails p'. For each $S \subseteq \mathbf{L}$, we define $\overline{S} := \{p \in \mathbf{L} : S \vDash p\}$. For each judgment set A and each proposition $p \in X$, we define $A(p) := 1$ if $p \in A$ and $A(p) := 0$ if $p \notin A$.

The proof of lemma 1 uses the following result.

Lemma 2 *(List 2004) A set $S \subseteq X$ is inconsistent if and only if there exist two consistent subsets $S_1, S_2 \subseteq S$ and a proposition $p \in X$ such that S_1 entails p and S_2 entails $\sim p$.*

Proof of lemma 1. Part (a) is trivial.

(b) Assume $A \subseteq X$ is deductively closed. By part (a), consistency implies weak consistency. Now assume A is not consistent. By lemma 2, there exist consistent sets $S_1, S_2 \subseteq A$ and a proposition $p \in X$ such that $S_1 \vDash p$ and $S_2 \vDash \sim p$. By deductive closure, $p, \sim p \in A$. Hence A is not weakly consistent.

(c) Assume $A \subseteq X$ is complete. First, let A be consistent. By (a), A is weakly consistent. To prove deductive closure, consider any $p \in X$ such that $A \vDash p$. Then $A \cup \{\sim p\}$ is inconsistent. So, since A is consistent, $A \neq A \cup \{\sim p\}$. Hence $\sim p \notin A$. By completeness, $p \in A$. Conversely, suppose A is not consistent. We must show that A is not weakly consistent or not deductively closed. By lemma 2, there exist consistent sets $S_1, S_2 \subseteq A$ and a proposition $p \in X$ such that $S_1 \vDash p$ and $S_2 \vDash \sim p$. If A is deductively closed, then $p, \sim p \in X$, hence X violates deductive closure. ■

Proof of proposition 1. It is easy to see that a quota rule $F_{(m_p)_{p \in X}}$ satisfies the specified conditions. Conversely, assume that F satisfies the conditions. We show that, for any $p \in X$, there exists a threshold $m_p \in \{1, ..., n\}$ such that p is accepted if and only if at least m_p individuals accept p. Consider any $p \in X$. By responsiveness, there exists at least one profile $(A_1, ..., A_n)$ such that p is accepted; among all such profiles, choose one for which the number of individuals accepting p is minimal, and call this number m_p. By independence and anonymity, p is accepted for *every* profile with exactly m_p individuals accepting p. Using monotonicity, it follows that p is accepted in every profile with at least m_p individuals accepting p. On the other hand, p is rejected in every profile with less than m_p individuals accepting p, by definition of m_p. Since by responsiveness p is not always accepted, $m_p \neq 0$. Hence $m_p \in \{1, ..., n\}$. ■

Proof of theorem 1. We denote $F_{(m_p)_{p\in X}}$ simply by F. Also, for each $p \in X$, let n_p be the number of individuals i such that $p \in A_i$ for a given profile $(A_1, ..., A_n)$. Note that, as the profile ranges over the universal domain, for each pair $p, \sim p \in X$, the pair of numbers $(n_p, n_{\sim p})$ ranges over the set $\{(k, n-k) : k = 0, 1, ..., n\}$.

(a) F is complete if and only if, for each pair $p, \sim p \in X$, we have

$$\begin{array}{ll} & \text{for each profile, if } p \text{ is rejected then } \sim p \text{ is accepted,} \\ \text{equivalently,} & \text{for each profile, if } n_p < m_p \text{ then } n_{\sim p} \geq m_{\sim p}, \\ \text{equivalently,} & \text{for each } 0 \leq k \leq n, \text{ if } k < m_p \text{ then } n-k \geq m_{\sim p}, \\ \text{equivalently,} & \text{if } k = m_p - 1 \text{ then } n-k \geq m_{\sim p}, \\ \text{equivalently,} & m_p + m_{\sim p} \leq n+1. \end{array}$$

(b) F is weakly consistent if and only if, for each pair $p, \sim p \in X$, we have

$$\begin{array}{ll} & \text{for each profile, if } p \text{ is accepted then } \sim p \text{ is rejected,} \\ \text{equivalently,} & \text{for each profile, if } n_p \geq m_p \text{ then } n_{\sim p} < m_{\sim p}, \\ \text{equivalently,} & \text{for each } 0 \leq k \leq n, \text{ if } k \geq m_p \text{ then } n-k < m_{\sim p}, \\ \text{equivalently,} & \text{if } k = m_p \text{ then } n-k < m_{\sim p}, \\ \text{equivalently,} & m_p + m_{\sim p} > n. \end{array}$$

(c) First, assume that F is not consistent. We show that at least one of the inequalities is violated. By assumption, there exists a profile $(A_1, ..., A_n)$ for which $F(A_1, ..., A_n)$ is inconsistent. Let $Z \subseteq F(A_1, ..., A_n)$ be a minimal inconsistent set. Since in the profile $(A_1, ..., A_n)$ exactly $n - n_p$ individuals reject each given $p \in Z$, a rejection of some proposition in Z by some individual i occurs exactly $\sum_{p\in Z}(n - n_p)$ times in $(A_1, ..., A_n)$. On the other hand, since Z is inconsistent, each of the n individuals rejects at least one proposition in Z. So, a rejection of some proposition in Z by some individual i occurs at least n times in $(A_1, ..., A_n)$. Hence $\sum_{p\in Z}(n - n_p) \geq n$. So, since for all $p \in Z$ we have $n_p \geq m_p$ (by $p \in F(A_1, ..., A_n)$), it follows that

$$\begin{array}{ll} & \sum_{p\in Z}(n - m_p) \geq n, \\ \text{equivalently,} & n|Z| - \sum_{p\in Z} m_p \geq n \\ \text{equivalently,} & \sum_{p\in Z} m_p \leq n(|Z| - 1). \end{array} \tag{10}$$

This violates the inequality for Z.

Conversely, assume that there is some minimal inconsistent set $Z \subseteq X$ with $\sum_{p\in Z} m_p \leq n(|Z| - 1)$, hence by (10) $\sum_{p\in Z}(n - m_p) \geq n$. We construct a profile $(A_1, ..., A_n)$ for which the group accepts each $p \in Z$, and hence generates an inconsistent judgment set. Since Z is minimal inconsistent, for each $p \in Z$ the set $Z\backslash\{p\}$ is consistent, and so $Z\backslash\{p\}$ may be extended to a (complete and consistent) judgment set, denoted $A_{\sim p}$. By $\sum_{p\in Z}(n - m_p) \geq n$, it is possible to assign to every individual i exactly one proposition $p_i \in Z$ in such a way that each $p \in Z$ is assigned to at most $n - m_p$ individuals. Define A_i as $A_{\sim p_i}$. For each $p \in Z$, at most $n - m_p$ individuals do not accept p, hence at least m_p individuals accept p. So $p \in F(A_1, ..., A_n)$ for each $p \in Z$.

(d) First, assume that F is not deductively closed. We show that at least one of the inequalities is violated. By assumption, there exists a profile $(A_1, ..., A_n)$, a consistent subset $R \subseteq F(A_1, ..., A_n)$ and a $p^* \in X$ such that $R \vDash p^*$ but $p^* \notin F(A_1, ..., A_n)$. Let $S \subseteq R$ be minimal such that $S \vDash p^*$. Writing q for $\sim p^*$, the set $Z := S \cup \{q\}$ is minimal inconsistent. Since in the profile $(A_1, ..., A_n)$ exactly $n - n_p$ individuals reject each given $p \in Z$, a rejection of some proposition in Z by some individual i

occurs exactly $\sum_{p\in Z}(n-n_p)$ times in $(A_1,...,A_n)$. On the other hand, since Z is inconsistent, each of the n individuals rejects at least one proposition in Z, so that a rejection of some proposition in Z by some individual i occurs at least n times in $(A_1,...,A_n)$. Hence $\sum_{p\in Z}(n-n_p)\geq n$, or $\sum_{p\in Z\backslash\{q\}}(n-n_p)+(n-n_q)\geq n$, or $\sum_{p\in Z\backslash\{q\}}(n-n_p)+n_{\sim q}\geq n$. Using that $n_{\sim q}<m_{\sim q}$ (by $\sim q=p^*\notin F(A_1,...,A_n)$) and that, for all $p\in Z\backslash\{q\}$, $n_p\geq m_p$ (by $p\in F(A_1,...,A_n)$), it follows that

$$\begin{array}{rl} & \sum_{p\in Z\backslash\{q\}}(n-m_p)+m_{\sim q}>n, \\ \text{equivalently,} & n|Z\backslash\{q\}|-\sum_{p\in Z\backslash\{q\}}m_p+m_{\sim q}>n, \\ \text{equivalently,} & n(|Z|-1)-\sum_{p\in Z\backslash\{q\}}m_p+m_{\sim q}>n \\ \text{equivalently,} & \sum_{p\in Z\backslash\{q\}}m_p-m_{\sim q}<n(|Z|-2). \end{array} \tag{11}$$

This violates the inequality for Z.

Conversely, assume there is a minimal inconsistent set $Z\subseteq X$ and an element $q\in Z$ such that $\sum_{p\in Z\backslash\{q\}}m_p-m_{\sim q}<n(|Z|-2)$, i.e. by (11) $\sum_{p\in Z\backslash\{q\}}(n-m_p)+m_{\sim q}>n$. We construct a profile $(A_1,...,A_n)$ for which each $p\in Z\backslash\{q\}$ but not $\sim q$ is accepted. This is a violation of deductive closure because $Z\backslash\{q\}$ is consistent and entails $\sim q$. For each $p\in Z$, let $A_{\sim p}$ be some extension of $Z\backslash\{p\}$ to a (complete and consistent) judgment set. By $\sum_{p\in Z\backslash\{q\}}(n-m_p)+m_{\sim q}>n$ we have $\sum_{p\in Z\backslash\{q\}}(n-m_p)+(m_{\sim q}-1)\geq n$. So it is possible to assign to every individual i exactly one proposition $p_i\in Z$ in such a way that each $p\in Z\backslash\{q\}$ is assigned to at most $n-m_p$ individuals and q is assigned to at most $m_{\sim q}-1$ individuals. Let A_i be $A_{\sim p_i}$. Then, for each $p\in Z\backslash\{q\}$, at most $n-m_p$ individuals do not accept p, hence at least m_p individuals accept p. So $p\in F(A_1,...,A_n)$ for each $p\in Z\backslash\{p\}$. Moreover, at most $m_{\sim q}-1$ individuals do not accept q, i.e. accept $\sim q$. So $\sim q\notin F(A_1,...,A_n)$. ■

Proof of corollary 1. Part (a) is trivial.

(b) Let $\mathcal{Z}$ be the set of minimal inconsistent sets $Z\subseteq X$. $F_{(m_p)_{p\in X}}$ is consistent if and only if $\sum_{p\in Z}m_p>n(|Z|-1)$ for every $Z\in\mathcal{Z}$, or equivalently

$$\sum_{p\in Z\backslash\{q\}}m_p+m_q>n(|Z|-1)\quad\begin{array}{l}\text{for every member}\\ q\text{ of every }Z\in\mathcal{Z}.\end{array} \tag{12}$$

Further, $F_{(m_p)_{p\in X}}$ is deductively closed if and only if $\sum_{p\in Z\backslash\{q\}}m_p-m_{\sim q}\geq n(|Z|-2)$ for every member q of every $Z\in\mathcal{Z}$, or equivalently

$$\sum_{p\in Z\backslash\{q\}}m_p+n+1-m_{\sim q}>n(|Z|-1)\quad\begin{array}{l}\text{for every member}\\ q\text{ of every }Z\in\mathcal{Z}.\end{array} \tag{13}$$

The claim follows from the fact that the conjunction of (12) and (13) is equivalent to (5).

(c) $F_{(m_p)_{p\in X}}$ is fully rational if and only if it is (i) complete and weakly consistent, and (ii) consistent; by part (a), (i) is equivalent to the equations in (6), and (ii) is equivalent to the inequalities in (6). ■

Proof of corollary 2. (a) By theorem 3, F_m is consistent if and only if, for all minimal inconsistent $Z\subseteq X$, $n(|Z|-1)<\sum_{p\in Z}m$, i.e. $n|Z|-n<|Z|m$, i.e. $m>n-n/|Z|$. The latter inequality holds for all minimal inconsistent $Z\subseteq X$ just in case $m>n-n/z$. Let $m=\lceil(n+1)/2\rceil$. First, assume n is odd, hence $m=(n+1)/2$. Then F_m is consistent if and only if $(n+1)/2>n-n/z$, which is easily seen to be equivalent to $z \leq 2$. Now let n be even, hence $m = n/2 + 1$. Then F_m is consistent if and only if $n/2 + 1 > n - n/z$, i.e. $n/z > n/2 - 1$. This inequality always holds if $n = 2$; if $n = 4$, it holds just in case $z \leq 3$; if $n \geq 6$, it holds just in case $z \leq 2$.

(b) By theorem 3, F_m is deductively closed if and only if, for all minimal inconsistent $Z \subseteq X$ and any $q \in Z$, $\sum_{p \in Z\setminus\{q\}} m - m \geq n(|Z|-2)$, i.e. $m(|Z|-2) \geq n(|Z|-2)$, i.e. $m = n$ or $|Z| \leq 2$. The latter inequality holds for all minimal inconsistent $Z \subseteq X$ just in case $z \leq 2$. Now let $m = \lceil (n+1)/2 \rceil$. First, let $n \geq 3$. Hence $m \neq n$. So F_m is deductively closed if and only if $z \leq 3$. Second, let $n = 2$. Then $m = n$. So F_m is deductively closed.

(c) By parts (a) and (b), F_m is consistent and deductively closed if and only if $m > n - n/z$ and $[z \leq 2$ or $m = n]$, i.e. (i) $[m > n - n/z$ and $m = n]$ or (ii) $[m > n - n/z$ and $z \leq 2]$. Note that (i) is equivalent to $m = n$. Further, (ii) is equivalent to $[m > n/2$ and $z \leq 2]$: if $z \leq 2$, then we have $z = 2$ (because $z \neq 1$, as X contains no contradictions), and hence $n - n/z = n/2$. ■

Proof of corollary 3. The result follows immediately from theorem 1 and proposition 1. ■

Given theorem 1 and corollary 2, only the following equivalence remains to be shown in order to prove theorem 2.

Proposition 4 *A sequential quota rule* $F_{\Omega,(m_p)_{p\in X}}$ *is*

(a) *weakly path-independent if and only if the corresponding quota rule* $F_{(m_p)_{p\in X}}$ *is consistent;*

(b) *path-independent if and only if the corresponding quota rule* $F_{(m_p)_{p\in X}}$ *is consistent and deductively closed.*

The proof of proposition 4 relies on two lemmas.

Lemma 3 *For every sequential rule* $F_{\Omega,(m_p)_{p\in X}}$*, profile* $(A_1, ..., A_n)$*, and step* $t \in \{0, ..., k\}$*, we have* $\Phi_t \subseteq \overline{\Phi_t \cap F_{(m_p)_{p\in X}}(A_1, \dots, A_n)}$ *(where* Φ_t *is as in the definition of* $F_{\Omega,(m_p)_{p\in X}}(A_1, ..., A_n)$*).*

Proof of lemma 3. Consider any family $(m_p)_{p\in X}$ and profile $(A_1, ..., A_n)$. We prove $\Phi_t \subseteq \overline{\Phi_t \cap F_{(m_p)_{p\in X}}(A_1, \dots, A_n)}$ by induction on $t \in \{0, ..., k\}$.

If $t = 0$, the claim follows from $\Phi_0 = \emptyset$.

Now let $t > 0$ and assume $\Phi_{t-1} \subseteq \overline{\Phi_{t-1} \cap F_{(m_p)_{p\in X}}(A_1, \dots, A_n)}$. If $p_t \notin \Phi_t$, then $\Phi_t = \Phi_{t-1}$; hence the claim holds by induction hypothesis.

Now suppose $p_t \in \Phi_t$. Then $\Phi_t = \Phi_{t-1} \cup \{p_t\}$. So $\Phi_t \subseteq \overline{\Phi_{t-1} \cap F_{(m_p)_{p\in X}}(A_1, \dots, A_n)} \cup \{p_t\}$ by induction hypothesis. Hence it is sufficient to prove that

$$\overline{\Phi_{t-1} \cap F_{(m_p)_{p\in X}}(A_1, \dots, A_n)} \cup \{p_t\} \subseteq \overline{\Phi_t \cap F_{(m_p)_{p\in X}}(A_1, \dots, A_n)}.$$

Since $\overline{\Phi_{t-1} \cap F_{(m_p)_{p\in X}}(A_1, \dots, A_n)} \subseteq \overline{\Phi_t \cap F_{(m_p)_{p\in X}}(A_1, \dots, A_n)}$ (by $\Phi_{t-1} \subseteq \Phi_t$), it is sufficient to show that $p_t \in \overline{\Phi_t \cap F_{(m_p)_{p\in X}}(A_1, \dots, A_n)}$. By $p_t \in \Phi_t$, there are two cases: (i) $\Phi_{t-1} \vDash p_t$, or (ii) $p_t \in F_{(m_p)_{p\in X}}(A_1, \dots, A_n)$. Under case (ii), the claim is trivial. Under case (i), the induction hypothesis implies $\overline{\Phi_{t-1} \cap F_{(m_p)_{p\in X}}(A_1, \dots, A_n)} \vDash p_t$; hence $p_t \in \overline{\Phi_{t-1} \cap F_{(m_p)_{p\in X}}(A_1, \dots, A_n)}$, as required. ■

Lemma 4 *Let* $(m_p)_{p\in X}$ *be given. For any profile* $(A_1, \dots, A_n)$ *and any proposition* $p \in X$*, [some consistent subset* $S \subseteq F_{(m_p)_{p\in X}}(A_1, \dots, A_n)$ *entails* p*] if and only if [*$p \in F_{\Omega,(m_p)_{p\in X}}(A_1, \dots, A_n)$ *for some decision-path* Ω*].*

Proof of lemma 4. Consider any profile $(A_1, \ldots, A_n)$ and proposition $p \in X$.

First, let there exist a decision-path Ω with $p \in F_{\Omega,(m_p)_{p\in X}}(A_1, \ldots, A_n)$. Let $\Phi_0, ..., \Phi_k$ $(k = |X|)$ be as in the definition of $F_{\Omega,(m_p)_{p\in X}}(A_1, \ldots, A_n)$, and $t \in \{1, 2, ..., k\}$ be such that $p = p_t$. By $p \in F_{\Omega,(m_p)_{p\in X}}(A_1, \ldots, A_n)$ we have $p \in \Phi_t$. So there are only two possible cases: (i) $\Phi_{t-1} \vDash p$ *or* (ii) $\left[\begin{array}{c}\Phi_{t-1} \cup \{p\} \text{ is consistent } \textit{and} \\ |\{i \in 1, 2, \ldots, n : p \in A_i\}| \geq m_p\end{array}\right]$. In case (i), we put $S := \Phi_{t-1} \cap F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$, which is consistent; since $\Phi_{t-1} \vDash p$ and since $\Phi_{t-1} \subseteq \overline{S}$ by lemma 3, we have $\overline{S} \vDash p$, hence $S \vDash p$. In case (ii), we put $S := \{p\}$, which is again consistent and entails p.

Conversely, assume that $S \subseteq F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$ is consistent and entails p. Let Ω be a decision-path that begins with the propositions in S, followed by proposition p, followed by all other propositions; specifically, $\Omega(m) = p_m \in S$ for all $m = 1, ..., s = |S|$, and $\Omega(s + 1) = p$. Let the sets $\Phi_0, ..., \Phi_k$ $(k = |X|)$ be as in the definition of $F_{\Omega,(m_p)_{p\in X}}(A_1, \ldots, A_n)$. We show by induction that $\Phi_m = \{p_1, ..., p_m\}$ for each $m = 1, ..., s$.

If $m = 1$, then $\Phi_1 = \{p_1\}$ since $p_1 \in F_{\Omega,(m_p)_{p\in X}}(A_1, ..., A_n)$.

Now let $1 < m \leq s$ and assume that $\Phi_{m-1} = \{p_1, ..., p_{m-1}\}$. Since $\Phi_{m-1} \cup \{p_m\} = \{p_1, ..., p_m\} \subseteq S$, $\Phi_{m-1} \cup \{p_m\}$ is consistent. So, as $p_m \in F_{\Omega,(m_p)_{p\in X}}(A_1, ..., A_n)$, we have $\Phi_m = \Phi_{m-1} \cup \{p_m\} = \{p_1, ..., p_m\}$, as desired.

In particular, $\Phi_s = \{p_1, ..., p_s\} = S$. By $S \vDash p$, we have $p \in \Phi_{s+1}$, so $p \in F_{\Omega,(m_p)_{p\in X}}(A_1, \ldots, A_n)$. ■

Proof of proposition 4. (a) First, suppose $F_{\Omega,(m_p)_{p\in X}}$ is not weakly path-independent, i.e. strongly path-dependent. Then there exist a profile $(A_1, \ldots, A_n)$, a proposition $p \in X$ and two decision-paths Ω_1 and Ω_2 such that

$$p \in F_{\Omega_1,(m_p)_{p\in X}}(A_1, \ldots, A_n) \text{ and } \sim p \in F_{\Omega_2,(m_p)_{p\in X}}(A_1, \ldots, A_n).$$

So, by lemma 4, there exists a consistent set $S_1 \subseteq F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$ that entails p, and a consistent set $S_2 \subseteq F_{(m_p)_{p\in X}}(A_1, ..., A_n)$ that entails $\sim p$. Hence, by lemma 2, $F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$ is inconsistent, i.e. $F_{(m_p)_{p\in X}}$ is not consistent.

Conversely, suppose $F_{(m_p)_{p\in X}}$ is not consistent. Consider any profile $(A_1, \ldots, A_n)$ for which $F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$ is inconsistent. By lemma 2, there exist two consistent subsets $S_1, S_2 \subseteq F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$ such that $S_1 \vDash p$ and $S_2 \vDash \sim p$. Hence, by lemma 4, there exist decision-path Ω_1 and Ω_2 such that $p \in F_{\Omega_1,(m_p)_{p\in X}}(A_1, \ldots, A_n)$ and $\sim p \in F_{\Omega_1,(m_p)_{p\in X}}(A_1, \ldots, A_n)$. So $F_{\Omega,(m_p)_{p\in X}}$ is not weakly path-independent.

(b) First, suppose $F_{(m_p)_{p\in X}}$ is consistent and deductively closed. We show that, for every decision-path Ω_1 and profile $(A_1, ..., A_n)$, $F_{\Omega_1,(m_p)_{p\in X}}(A_1, ..., A_n) = F_{(m_p)_{p\in X}}(A_1, ..., A_n)$, which implies path-independence. Consider any Ω_1 and $(A_1, ..., A_n)$. Let the sets Φ_0, ..., Φ_k and propositions p_1, ..., p_k $(k = |X|)$ be as in the definition of $F_{\Omega_1,(m_p)_{p\in X}}(A_1, ..., A_n)$. We show by induction that $\Phi_t = F_{(m_p)_{p\in X}}(A_1, ..., A_n) \cap \{p_1, ..., p_t\}$ for all $t \in \{0, ..., k\}$; the case $t = k$ then yields our claim.

For $t = 0$, the claim is trivial by $\Phi_0 = \emptyset$.

Now let $0 < t \leq k$ and assume $\Phi_{t-1} = F_{(m_p)_{p\in X}}(A_1, ..., A_n) \cap \{p_1, ..., p_{t-1}\}$. We have to show that $p_t \in \Phi_t$ is equivalent to $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)$. First, assume $p_t \in \Phi_t$. Then, by definition of Φ_t, either (i) $\Phi_{t-1} \vDash p_t$ or (ii) $[\Phi_{t-1} \cup \{p_t\}$ is consistent and $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)]$. In case (ii), we obviously have $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)$. In case (i), we have $F_{(m_p)_{p\in X}}(A_1, ..., A_n) \cap \{p_1, ..., p_{t-1}\} \vDash p_t$ by induction hypothesis; since $F_{(m_p)_{p\in X}}(A_1, ..., A_n) \cap \{p_1, ..., p_{t-1}\}$ is consistent by the consistency of $F_{(m_p)_{p\in X}}$, we have $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)$ by the deductive closure of $F_{(m_p)_{p\in X}}$. Now assume that $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)$. By induction hypothesis, $\Phi_{t-1} \subseteq F_{(m_p)_{p\in X}}(A_1, ..., A_n)$. Hence, $\Phi_{t-1} \cup \{p_t\} \subseteq F_{(m_p)_{p\in X}}(A_1, ..., A_n)$. So, by the consistency of $F_{(m_p)_{p\in X}}$, $\Phi_{t-1} \cup \{p_t\}$ is consistent. Hence, as $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)$, we have $p_t \in \Phi_t$ by definition of Φ_t.

Conversely, suppose $F_{(m_p)_{p\in X}}$ is not consistent or not deductively closed. If $F_{(m_p)_{p\in X}}$ is not consistent, then the result follows from part (a). Suppose now $F_{(m_p)_{p\in X}}$ is consistent, but not deductively closed. Then there is a profile $(A_1, \ldots, A_n)$, a consistent set $S \subseteq F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$ and a proposition $p \in X$ such that $S \vDash p$ and $p \notin F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$. So, on the one hand, by lemma 4, there exists a decision-path Ω_1 such that $p \in F_{\Omega_1,(m_p)_{p\in X}}(A_1, \ldots, A_n)$, and, on the other hand, $p \notin F_{\Omega_2,(m_p)_{p\in X}}(A_1, \ldots, A_n)$ for any decision-path Ω_2 with $\Omega_2(1) = p$. This implies path-dependence. ■

Proof of theorem 2. Given theorem 1 and corollary 2, the result follows from proposition *4*. ■

Proof of proposition 3. Consider any sequential quota rule $F_{\Omega,(m_p)_{p\in X}}$. For each $p \in X$, define the new threshold m_p^* as the effective threshold for p if p has an effective threshold, and as m_p otherwise.

Claim 1: $F_{\Omega,(m_p)_{p\in X}}$ and $F_{\Omega,(m_p^*)_{p\in X}}$ generate the same judgment sets. Consider any profile $(A_1, ..., A_n) \in \mathbf{A}^n$. Let the sets Φ_t ($t = 0, ..., k$, $k = |X|$) be as given in the definition of $F_{\Omega,(m_p)_{p\in X}}(A_1, ..., A_n)$, and let Φ_t^* ($t = 0, ..., k$) be the corresponding sets for $F_{\Omega,(m_p^*)_{p\in X}}(A_1, ..., A_n)$. By a straightforward induction on t, we have $\Phi_t = \Phi_t^*$ for all t. In particular, $\Phi_k = \Phi_k^*$, i.e. $F_{\Omega,(m_p)_{p\in X}}(A_1, ..., A_n) = F_{\Omega,(m_p^*)_{p\in X}}(A_1, ..., A_n)$.

Claim 2: $F_{\Omega,(m_p^*)_{p\in X}}$ is transparent. Consider any proposition $p \in X$, and assume p has an effective threshold under $F_{\Omega,(m_p^*)_{p\in X}}$. By claim 1, p has the same effective threshold under $F_{\Omega,(m_p)_{p\in X}}$. So, by definition of m_p^*, p has effective threshold m_p^* under $F_{\Omega,(m_p)_{p\in X}}$. So, by claim 1, p has effective threshold m_p^* under $F_{\Omega,(m_p^*)_{p\in X}}$. ■

The proof of theorem 3 relies on the following lemma.

Lemma 5 *For every sequential quota rule* $F_{\Omega,(m_p)_{p\in X}}$*,*

(a) path-independence implies $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p)_{p\in X}}$*;*

(b) the converse also holds in case $F_{\Omega,(m_p)_{p\in X}}$ *or* $F_{(m_p)_{p\in X}}$ *is complete or deductively closed.*

Proof of lemma 5. (a) Let $F_{\Omega,(m_p)_{p\in X}}$ be path-independent. Consider any profile $(A_1, ..., A_n)$ and proposition $p \in X$. We have to show that $F_{\Omega,(m_p)_{p\in X}}(A_1, ..., A_n)(p) = F_{(m_p)_{p\in X}}(A_1, ..., A_n)(p)$. Let Ω_1 be some decision-path with $\Omega_1(1) = p$. Then, by path-independence, $F_{\Omega,(m_p)_{p\in X}}(A_1, ..., A_n)(p) = F_{\Omega_1,(m_p)_{p\in X}}(A_1, ..., A_n)(p)$, which equals $F_{(m_p)_{p\in X}}(A_1, ..., A_n)(p)$ by definition of Ω_1.

(b) Now let $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p)_{p\in X}}$, and assume this aggregation rule is complete or deductively closed. If it is deductively closed, then, as it is also consistent (by definition of sequential rules), it is path-independent by proposition 4 (b). If it is complete, then, as it is also consistent, it is deductively closed by lemma 1; hence it is again path-independent by proposition 4 (b). ■

Proof of theorem 3. Consider any complete or deductively closed transparent sequential quota rule $F_{\Omega,(m_p)_{p\in X}}$.

1. First, assume $F_{\Omega,(m_p)_{p\in X}}$ is path-independent. By lemma 5, $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p)_{p\in X}}$. So $F_{\Omega,(m_p)_{p\in X}}$ is independent and monotonic, hence strategy-proof by proposition 2.

2. Now assume $F_{\Omega,(m_p)_{p\in X}}$ is strategy-proof. Then $F_{\Omega,(m_p)_{p\in X}}$ is independent and monotonic by proposition 2. So, since $F_{\Omega,(m_p)_{p\in X}}$ is also anonymous, there exists a family $(m_p^*)_{p\in X} \in \{0, ..., n+1\}^X$ such that $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p^*)_{p\in X}}$, where $F_{(m_p^*)_{p\in X}}$ denotes the obvious generalisation of our definition of quota rules to the case where each m_p^* can also be 0 or $n+1$ (in which case p is always or never accepted).

Claim 1: For each $p \in X$, $m_p^* \leq n$.

Consider any $p \in X$. Let A be a (complete and consistent) judgment set such that $p \in A$. Let the propositions p_t and the sets Φ_t ($t = 0, ..., k$, $k = |X|$) be as in the definition of $F_{\Omega,(m_p)_{p\in X}}(A, ..., A)$. Note that $F_{(m_p)_{p\in X}}(A, ..., A) = A$ (since $1 \leq m_p \leq n$ for all $p \in X$). By a straightforward induction, it follows that $\Phi_t = A \cap \{p_1, ..., p_t\}$ for all $t \in \{0, ..., k\}$. In particular, $\Phi_k = A \cap \{p_1, ..., p_k\} = A$, i.e. $F_{\Omega,(m_p)_{p\in X}}(A, ..., A) = A$. So, by $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p^*)_{p\in X}}$, $F_{(m_p^*)_{p\in X}}(A, ..., A) = A$. In particular, $p \in F_{(m_p^*)_{p\in X}}(A, ..., A)$, hence $m_p^* \leq n$.

Claim 2: For each $p \in X$, $m_p^* \geq 1$.

Consider any $p \in X$. Let A be a (complete and consistent) judgment set such that $p \notin A$. By the argument used to prove claim 1, $F_{(m_p^*)_{p\in X}}(A, ..., A) = A$. In particular, $p \notin F_{(m_p^*)_{p\in X}}(A, ..., A)$, hence $m_p^* \geq 1$.

By the claims 1 and 2, each m_p^* belongs to $\{1, ..., n\}$, i.e. is a threshold in our standard sense, which will allow us to use a transparency argument. By $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p^*)_{p\in X}}$, each $p \in X$ has effective threshold m_p^*. So, by transparency, $m_p = m_p^*$ for each $p \in X$. Hence $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p)_{p\in X}}$. So, by lemma 5 (b), $F_{\Omega,(m_p)_{p\in X}}$ is path-independent. ■

Proof of corollary 6. (a) Let n be odd and consider a sequential majority rule $F_{\Omega,m}$ ($m = (n+1)/2$). To apply theorem 3 to $F_{\Omega,m}$, it is sufficient to show that $F_{\Omega,m}$ is transparent and that F_m is complete (see footnote 6). As n is odd, F_m is complete. To prove that $F_{\Omega,m}$ is transparent, consider any $p \in X$ and let there be an effective threshold m_p for p. We show that $m_p = m$ by proving first that $m_p \leq m$ and then that $m_p > m - 1$. Let A_p and $A_{\sim p}$ be (complete and consistent) judgment sets with $p \in A_p$ and $p \notin A_{\sim p}$.

$m_p \leq m$: Let $(A_1, ..., A_n)$ be a profile in which exactly m individuals i have $A_i = A_p$ and the other $n - m = m - 1$ individuals i have $A_i = A_{\sim p}$. Let the propositions $p_1, ..., p_k$ and the sets $\Phi_0, ..., \Phi_k$ ($k = |X|$) be as in the recursive definition of $F_{\Omega,m}(A_1, ..., A_n)$. By a straightforward induction that uses the fact that a majority submits the judgment set A_p, we have $\Phi_t = A_p \cap \{p_1, ..., p_t\}$ for all $t \in \{0, ..., k\}$. In particular, $\Phi_k = A_p \cap \{p_1, ..., p_k\} = A_p$, i.e. $F_{\Omega,m}(A_1, ..., A_n) = A_p$. Hence $p \in F_{\Omega,m}(A_1, ..., A_n)$. This implies $m_p \leq m$, as m_p is the effective threshold for p and m individuals accept p in $(A_1, ..., A_n)$.

$m_p > m - 1$: Now let $(A_1, ..., A_n)$ be a profile in which exactly m individuals i have $A_i = A_{\sim p}$ and the other $n - m = m - 1$ individuals i have $A_i = A_p$. By an argument analogous to the above one, we have $F_{\Omega,m}(A_1, ..., A_n) = A_{\sim p}$. Hence $p \notin F_{\Omega,m}(A_1, ..., A_n)$. This implies $m_p > m - 1$, as m_p is the effective threshold for p and $m - 1$ individuals accept p in $(A_1, ..., A_n)$.

Having shown transparency, by theorem 3 strategy-proofness is equivalent to path-independence, which is equivalent to $z \leq 2$ by corollary 4.

(b) Now consider a sequential unanimity rule $F_{\Omega,m}$ ($m = n$). By corollary 2, $F_{\Omega,m}$

is deductively closed. To apply theorem 3, we need to show that $F_{\Omega,m}$ is transparent. Consider any $p \in X$ and let there be an effective threshold m_p for p. We show that $m_p = m$, i.e. that $m_p = n$. As in part (a), let A_p and $A_{\sim p}$ be (complete and consistent) judgment sets with $p \in A_p$ and $p \notin A_{\sim p}$. Let $(A_1, ..., A_n)$ be a profile in which one individual i has $A_i = A_{\sim p}$ and $n-1$ individuals i have $A_i = A_p$. Let the propositions $p_1, ..., p_k$ and the sets $\Phi_0, ..., \Phi_k$ $(k = |X|)$ be as in the definition of $F_{\Omega,m}(A_1, ..., A_n)$. By a straightforward induction, we have $\Phi_t = A_p \cap A_{\sim p} \cap \{p_1, ..., p_t\}$ for all $t \in \{0, ..., k\}$. In particular, $\Phi_k = A_p \cap A_{\sim p} \cap \{p_1, ..., p_k\} = A_p \cap A_{\sim p}$, i.e. $F_{\Omega,m}(A_1, ..., A_n) = A_p \cap A_{\sim p}$. Hence $p \notin F_{\Omega,m}(A_1, ..., A_n)$. This implies $m_p > n-1$, as m_p is the effective threshold for p and $n-1$ individuals accept p in $(A_1, ..., A_n)$. So $m_p = n$. This proves transparency.

Now by theorem 3 strategy-proofness is equivalent to path-independence, which is satisfied by corollary 4. ■