Equilibrium and the Core in Alonso's Discrete Population Model of Land Use*

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Abstract

Conventional wisdom tells us that with no market failure and local non-satiation of preferences, the core is at least as large as the collection of competitive equilibrium allocations. We confirm this for a standard model featuring private ownership of land. Next we consider the public land ownership version of the model. If the role of land ownership and rent distribution is assumed by a government that ploughs back rent (at least in excess of its agricultural value) to its citizens, the equilibrium remains efficient, but no longer need be in the core.

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1 Introduction

Consider a multi-commodity generalization of the Alonso (1964) model, the workhorse of urban economic theory. The economy has land, the interval $[0, 1)$, where the origin is the central business district or CBD. Each consumer must commute to the CBD to work or to pick up their endowment of consumption commodity. Only one consumer can be adjacent to the CBD. If his parcel is $[0, s)$, then the next consumer incurs transport cost ts, where $t \in \mathbb{R}_+$ is the commuting input per unit distance from the CBD in terms of consumption good, as measured from the front of a person's parcel. The other consumers incur even greater commuting costs. Traders must use intervals of land. As Berliant and Fujita (1992) have shown, any equilibrium allocation is efficient.[1] But what about the possibility of improving utility by forming a coalition? The more land consumed by the agent closest to the CBD, the less land and the less standard commodities are available for consumption (due to the increased commuting cost of the consumers farther from the CBD). This observation raises the question if there is an incentive to exclude one agent. In the next section we demonstrate that the answer is negative for exchange economies with privately owned land; a theorem shows that an equilibrium allocation cannot be improved upon by any coalition. The subsequent section will reverse the answer in the public land ownership model, where a public administration owns land and distributes the rent.[2] An equilibrium allocation is still efficient in the weak sense of Pareto, but can be improved upon by a subcoalition; it might not be efficient in the strong sense of belonging to the core.[3]

General equilibrium models of public ownership are rare. This is not without reason. An important rationale for public ownership is the presence of increasing returns to scale. Now with increasing returns, profits are maximized when the quantity supplied is infinite or zero and neither case generally equilibrates with demand. Guesnerie (1975) has analyzed what happens if public management of firms with non-convex production technologies follows the rule of marginal cost pricing (which is just the first order condition for profit maximization). A general equilibrium exists (under his conditions), but need not be Pareto optimal.[4] Basically, Guesnerie shows that the global

[1] See Berliant and LaFountain (forthcoming) for a graphical treatment.

[2] The public land ownership model is described in detail in Fujita (1986, section 1.2; 1989, pp. 60-63) for the model of the New Urban Economics with a continuum of consumers. He attributes its origins to Solow (1973).

[3] In fact, the core is empty for the example we provide.

[4] In fact, in Guesnerie's model, all equilibria may be Pareto inefficient.

condition of profit maximization is essential to the first welfare theorem. In our analysis, however, this global condition is fulfilled and the equilibrium is Pareto optimal indeed. We have stumbled upon a more subtle complication of public management: it is weakly efficient (in the sense of Pareto), but not strongly efficient (in the sense of belonging to the core).

Our result is more subtle than the emptiness of the core of the three-person voting game (Aumann, 1987). In such a game, the excluded player gets nothing. Hence this player is willing to accept an arbitrarily small offer, in particular less than any of the two incumbent players receives. Consequently the incumbents prefer to share with the outsider, rather than with each other. In our model, however, an excluded player still gets her share of rent, so it is harder to engage her in a coalition.

There are two points to be made relative to the recent literature on city formation, for example Lucas and Rossi-Hansberg (2002). First, we have postulated in our work an exogenously given CBD. In most models of city formation, the CBD or location of firms is endogenous, and there is an explicit agglomeration externality used to determine these locations. Thus, equilibria are usually not efficient, and thus not in the core. However, these models all have embedded in them a model of consumer location and commuting, making our analysis relevant. For example, conditional on the spatial distribution of firms, one might want to consider the consumer location problem. Second, and more importantly, the recipient and distribution of land rent in these models is ill-defined. The models are not closed in the sense of material balance, and it is not specified who receives land rent. Our point here is that predictions can be quite sensitive to how this part of the model is specified.

It is known that models with a continuum of agents, such as variants of the standard monocentric city model of the New Urban Economics, can have the property that equilibrium allocations are not efficient and thus are not in the core; see Berliant, Papageorgiou and Wang (1990). This phenomenon is entirely due to the fact that there is a continuum of agents in the model. To avoid this problem, we employ Alonso's model. It features a finite number of discrete agents.

2 Exchange economies

Consider an exchange economy with $l+1$ commodities and I consumers indexed by i with initial endowments comprised of land $[\zeta_i, \zeta_i + \sigma_i)$ and standard commodities net of transport costs $\omega_i - \zeta_i t \in \mathbb{R}^l_+$, where $[\zeta_1, \zeta_1 + \sigma_1), ..., [\zeta_I, \zeta_I + \sigma_I)$ partition[5] the world $[0, 1)$ and $t \in \mathbb{R}^l_+$ is the unit commuting input. The consumers have preference relationships $\succsim_i$ that are complete preorders on $\mathbb{R}^{l+1}_+$; only the quantity of land, e.g. σ_i, is assumed to matter, not the location, e.g. ζ_i. The quantity of land is taken to be the first commodity. A preference relationship is called *locally nonsatiated* if every neighborhood of any commodity bundle contains a strictly preferred commodity bundle. Formally, $\succsim_i$ is *locally nonsatiated* if for every commodity bundle $(\sigma_i, x_i) \in \mathbb{R}^{l+1}_+$ and for every $\epsilon > 0$, there exists $(\sigma'_i, x'_i) \in \mathbb{R}^{l+1}_+$ with $(\sigma'_i, x'_i) \succ_i (\sigma_i, x_i)$ and $\| (\sigma'_i, x'_i) - (\sigma_i, x_i) \| < \epsilon$. For example, the assumption that preferences are strictly monotonic is stronger. In general, we use the notation z_i to denote the front or driveway location of the parcel used by consumer i. An *allocation* is a vector of intervals and of consumption bundles $([z_i, z_i + s_i), x_i - z_i t)_{i=1}^I$, where for all i, $x_i \geq z_i t$. An allocation $([z_i, z_i + s_i), x_i - z_i t)_{i=1}^I$ is called *feasible* if $[z_i, z_i + s_i)_{i=1}^I$ partition $[0, 1)$ (formally $\cup_{i=1}^I [z_i, z_i + s_i) = [0, 1)$ and for all $i \neq j$, $1 \leq i, j \leq I$, $[z_i, z_i + s_i) \cap [z_j, z_j + s_j) = \varnothing$) and $\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i$. A feasible allocation $([z_i, z_i + s_i), x_i - z_i t)_{i=1}^I$, an integrable price density $p : [0, 1) \to \mathbb{R}_+$ and a price vector $q \in \mathbb{R}^l_+$ constitute an *equilibrium* if for each trader i, $\int_{z_i}^{z_i+s_i} p(z) dm(z) + q \cdot x_i \leq \int_{\zeta_i}^{\zeta_i+\sigma_i} p(z) dm(z) + q \cdot \omega_i$, $(s'_i, x'_i - z'_i t) \succ_i (s_i, x_i - z_i t) \Rightarrow \int_{z'_i}^{z'_i+s'_i} p(z) dm(z) + q \cdot x'_i > \int_{\zeta_i}^{\zeta_i+\sigma_i} p(z) dm(z) + q \cdot \omega_i$. An *equilibrium allocation* is the allocation component of an equilibrium.

A *coalition* is a subset S of $\{1, ..., I\}$. For a coalition S, a *coalition reallocation* is a vector of intervals and of consumption bundles $([z'_i, z'_i + s'_i), x'_i - z'_i t)_{i \in S}$ with $x'_i \geq z'_i t$ for all $i \in S$, with $[z'_i, z'_i + s'_i)_{i \in S}$ partitioning $\cup_{i \in S} [\zeta_i, \zeta_i + \sigma_i)$ (formally $\cup_{i \in S} [z'_i, z'_i + s_i) = \cup_{i \in S} [\zeta_i, \zeta_i + \sigma_i)$ and for all $i \neq j$, $i, j \in S$, $[z'_i, z'_i + s_i) \cap [z'_j, z'_j + s_j) = \varnothing$) and $\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i$. A feasible allocation $([z_i, z_i + s_i), x_i - z_i t)_{i=1}^I$ is in the *core* if for all coalitions S there is no coalition reallocation $([z'_i, z'_i + s'_i), x'_i - z'_i t)_{i \in S}$ which is superior in the sense that $(s'_i, x'_i - z'_i t) \succsim_i (s_i, x_i - z_i t)$ (for all $i \in S$) and $(s'_i, x'_i - z'_i t) \succ_i (s_i, x_i - z_i t)$ (for some $i \in S$).

If this "no coalition reallocation" condition holds for the grand coalition, $S = \{1, ..., I\}$, the feasible allocation is *efficient*. A core allocation is clearly efficient, but an efficient allocation need not be in the core.

We now adapt Theorem 1 of Debreu and Scarf (1963, attributed to Shapley) to the generalized Alonso model.

[5] The formal definition of a partition is given below in this paragraph.

Generalized First Welfare Theorem: If preferences are locally nonsatiated, then any equilibrium allocation is in the core.

Proof: Suppose not. Then for some coalition S there is a coalition reallocation $([z_i', z_i' + s_i'), x_i' - z_i't)_{i\in S}$ with $[z_i' + s_i')_{i\in S}$ partitioning $\cup_{i\in S}[\zeta_i, \zeta_i + \sigma_i)$, $\sum_{i\in S} x_i' = \sum_{i\in S} \omega_i$, $(s_i', x_i' - z_i't) \succsim_i (s_i, x_i - z_i t)$ (for all $i \in S$) and $(s_i', x_i' - z_i't) \succ_i (s_i, x_i - z_i t)$ (for some $i \in S$). By the equilibrium condition, $\int_{z_i'}^{z_i'+s_i'} p(z)dm(z) + q \cdot x_i' > \int_{\zeta_i}^{\zeta_i+\sigma_i} p(z)dm(z) + q \cdot \omega_i$ (for some $i \in S$). By local nonsatiation, for all $\varepsilon > 0$ there are s_i^ε and x_i^ε within distance ε from s_i' and x_i' such that $(s_i^\varepsilon, x_i^\varepsilon - z_i't) \succ_i (s_i, x_i - z_i t)$. By the equilibrium condition, $\int_{z_i'}^{z_i'+s_i^\varepsilon} p(z)dm(z) + q \cdot x_i^\varepsilon > \int_{\zeta_i}^{\zeta_i+\sigma_i} p(z)dm(z) + q \cdot \omega_i$ (for all $i \in S$). By Lebesgue's dominated convergence theorem and continuity of the (linear) value function on the left hand side, $\int_{z_i'}^{z_i'+s_i'} p(z)dm(z) + q \cdot x_i' \geq \int_{\zeta_i}^{\zeta_i+\sigma_i} p(z)dm(z) + q \cdot \omega_i$ (for all $i \in S$). Summing, $\sum_{i\in S}[\int_{z_i'}^{z_i'+s_i'} p(z)dm(z) + q \cdot x_i'] > \sum_{i\in S}[\int_{\zeta_i}^{\zeta_i+\sigma_i} p(z)dm(z) + q \cdot \omega_i]$. But since $([z_i' + s_i'))_{i\in S}$ partition $\cup_{i\in S}[\zeta_i, \zeta_i + \sigma_i)$ and $\sum_{i\in S} x_i' = \sum_{i\in S} \omega_i$, we have $\sum_{i\in S}[\int_{z_i'}^{z_i'+s_i'} p(z)dm(z) + q \cdot x_i'] = \sum_{i\in S}[\int_{\zeta_i}^{\zeta_i+\sigma_i} p(z)dm(z) + q \cdot \omega_i]$; that is a contradiction. Q.E.D.

Remark: One may replace the equality in the material balance conditions by a strict inequality, both in the definition of equilibrium and of the core, but then one must assume free disposal to obtain the Generalized First Welfare Theorem.

Corollaries:

1. An equilibrium allocation is efficient. (Take $S = \{1, ..., I\}$.) This is the First Welfare Theorem. It motivates the name of the Theorem above.
2. An equilibrium allocation is individually rational. (Take $S = \{i\}$.)

3 Economies with public land ownership

In many papers[6] land is not owned by the consumers, but by an absentee landlord or a government. In this literature the absentee landlord or the government is a broker between the farmers and the urban consumers, buying land at the rent that prevails in agriculture and reselling it at a higher rate to the consumers. Strictly speaking, this modeling approach is inconsistent with the premises of neoclassical economics. Why would *only* the absentee

[6] See the surveys of Fujita (1986, 1989).

landlord or the government be able to arbitrage between the farmers and the urban consumers? Are farmers irrational? We circumvent this problem by focusing on the so-called closed city model, where land is not purchased from farmers but instead is owned by the absentee landlord or the government from the outset.

Our model given in the previous section encompasses the situation with an absentee landlord without modification. Simply endow one agent, who obtains utility from consumption good but not from land, with all the land. The generalized first welfare theorem applies. There is no incentive to exclude a consumer by forming a coalition. True, the central consumer inflicts an enormous opportunity cost on the other consumers, who all incur transport cost in crossing his parcel. In equilibrium, however, this opportunity cost is reflected in the rent he pays. By excluding this consumer, the others can no longer tap his initial wealth endowment. The gain of commuting cost reduction is offset by the loss of rent he contributes to the other agents, including the landlord.

The situation with a government is different. Index the government agent by $i = 0$. It owns all the land, has no preferences (or equivalently is indifferent among all allocations), and redistributes rent to the consumers.[7] What the latter can achieve in terms of land and standard commodities, individually or in a coalition, depends not only on the commodity endowment of the agents involved, but also on rent and titles to rent, hence prices. Whereas in the preceding section the question of whether an equilibrium is in the core depended only on the equilibrium allocation, it now also depends on prices and rent titles. We may minimize this complication of the core concept by following the urban economic postulate that there is only one non-land or "numeraire" commodity ($l = 1$). The price of this commodity is normalized to 1 ($q = 1$). Indeed, since we merely want to show that an equilibrium need not be in the core, a simple example is good enough. The definition of equilibrium is modified by simple inclusion of $\theta_i \int_0^1 p(z)dm(z)$ in the budget of consumer i, where $(\theta_i)_{i=1}^I$ are the exogenously given rent shares. A coalition without the government has no land and, therefore, no potential to generate a superior assignment to its members if land is an essential commodity. For a coalition with the government, $\{0\} \cup S$, where S is a subset of $\{1, ..., I\}$, a *coalition reallocation* is a vector of intervals and of consumption bundles $([z_i', z_i' + s_i'), x_i' - z_i't)_{i \in S}$ with $[z_i', z_i' + s_i')_{i \in S}$ parti-

[7] We assume directly that the government exhausts its budget, so there is no role for local non-satiation.

tioning $[0,1)$ and $\sum_{i\in S} x_i' = \sum_{i\in S}\omega_i - Rentleak$. Here $Rentleak$ is the rent that leaks to nonmembers of the coalition. It is well-defined only if we limit coalition reallocations to equilibria for the economy consisting only of coalition members; call an equilibrium price density for this sub-economy p^S. This limitation only makes our result in this section stronger, in the sense we explain at the end of this paragraph. Now $Rentleak = \sum_{i\notin S}\theta_i \int_0^1 p^S(z)dm(z)$. An equilibrium allocation is in the *core* if there is no coalition reallocation $([z_i', z_i' + s_i'), x_i' - z_i't)_{i\in S}$ with $(s_i', x_i' - z_i't) \succsim_i (s_i, x_i - z_it)$ (for all $i \in S$) and with $(s_i', x_i' - z_i't) \succ_i (s_i, x_i - z_it)$ (for some $i \in S$). Following Fujita (1989, p. 60), we presume that rent is evenly divided among consumers, namely that $\theta_i = 1/I$, $i = 1, ..., I$. The purpose of this section is to provide a simple example where an equilibrium allocation is not in the core, and in fact we will show that the core is empty. This result will be quite robust, in the following sense. Alternatively one might model rent shares as coalition dependent, by assuming that consumers who are excluded from a coalition with the government have no title to the government rent proceeds. In this case $Rentleak$ is zero, so that the superior coalition reallocation we will construct remains applicable (for our non-decreasing utility function). Our model and results can accommodate any version of these property rights, from complete enforcement of rent payments to consumers that are not members of a coalition to the regime where consumers that are not members of a coalition have no right to rent proceeds (or any intermediate regime with partial rights to land rent shares).

With a government, the equilibrium is still efficient. The proof is as follows. For the grand coalition $Rentleak$ is zero. Begin with an Alonso economy with a government and public land ownership. Take the equilibrium allocation we wish to test for efficiency. Use this equilibrium allocation as the initial endowments (including redistributed rent) for a new exchange economy with the same I consumers but without the government. The equilibrium allocation remains an equilibrium allocation in this new exchange economy without the government but with altered initial endowments. The equilibrium of the exchange economy is efficient by Corollary 1 to the Generalized First Welfare Theorem.

Surprisingly, an equilibrium allocation is efficient but need not be in the core. We will show this in the simplest case, $I = \{1, 2\}$, with equal endowments $\frac{1}{8} < \omega < 1.319$ and equal preferences induced by the good-old utility function $\ln(s) + x - zt$, where $t \leq 0.9231$. As is well-known, quasi-linearity of the utility function renders the demand for land independent of the consumption of numeraire for allocations with positive levels of numeraire consumption. We will suppose without loss of generality for the remainder of the paper that 1 lives closer to the CBD than 2. The contract curve in this model is defined to be the set of Pareto optima such that 1's marginal rate of substitution is equal to 2's marginal rate of substitution plus t. This is the analog of the equality of marginal rates of substitution in the standard general equilibrium model, and it is also the Muth (1969)-Mills (1972) condition for the Alonso model.[8] The familiar intuition is that at an optimum, if this equality does not hold, then a Pareto dominating feasible allocation can be found as follows. If 1's marginal rate of substitution is greater than 2's marginal rate of substitution plus t, then 1's land parcel can be made slightly larger and 2's land parcel made slightly smaller, covering the increased commuting cost for 2 and generating a surplus of numeraire. Of course, an analogous argument can be made if the inequality is reversed. Given the functional form of utility, the contract curve features land consumption independent of composite good consumption for each of the two consumers. It is determined by the equation: $\frac{1}{s_1} = \frac{1}{1-s_1} + t$,[9] where we use the assumption that total endowment of land is 1. By the quadratic formula, the solution is $s^* = \frac{2+t-\sqrt{4+t^2}}{2t} \leq \frac{1}{2}$.

As already discussed, a first welfare theorem holds in this model, so we can use the contract curve and s^* to solve for an equilibrium. A candidate equilibrium price is given by $p(z) = \frac{1}{s^*}$ for $0 \leq z \leq s^*$, $p(z) = \frac{1}{s^*} - \rho^*(z - s^*)$ for $s^* \leq z \leq 1$, where ρ^* will be determined by the equal treatment condition, namely that the two consumers (who have the identical endowments and utility functions) are at the same utility level in equilibrium.[10] The appendix contains

[8] See Berliant and Fujita (1992) and Berliant and LaFountain (forthcoming).

[9] As discussed in detail in Berliant and Fujita (1992) and Berliant and LaFountain (forthcoming), the contract curve in the Alonso model can be described in a modified Edgeworth box.

[10] Mirrlees (1972) and Wildasin (1986) discuss how consumers with identical endowments and utility functions can have different utility levels at a utilitarian optimum. However, at an equilibrium allocation, consumers with the same endowments and utility functions must be at the same utility level, though they don't necessarily consume the same bundles. Even though equilibrium allocations in our model are efficient, they are not necessarily utilitarian optimal. Notice also that core generally does not require that identical consumers are at the same utility level, and that core and utilitarian optima are not necessarily related. Finally, the utility possibilities frontier is symmetric in our model when consumers are identical. The proximate cause of the empty core in the public ownership model is the relationship between the utility possibility frontiers of the grand coalition on the one hand and the coalitions consisting of one consumer and the government on the other.

the tedious algebraic details of calculations sufficient to show that this is in fact an equilibrium rent density. The intuition is that for the innermost consumer, the price is equal to their marginal willingness to pay for land given an allocation of land on the contract curve. The rent density on the outer consumer's parcel must exceed the inner consumer's marginal willingness to pay for more land. It must also decline more slowly than commuting cost, so the inner consumer has no incentive to shift its parcel outward. Finally, the outer consumer's marginal willingness to pay for land must exceed the rent density at each point, so they buy the entire parcel. The functional form for price density we have chosen is the simplest that has all of these features.

Next, we show that this equilibrium is not in the core. To see this, consider the coalition of the government and one consumer, say $\{0, 1\}$. This coalition has at its disposal the endowments of both the government and consumer 1. Thus, consumer 1 gets all of the land and its own endowment of composite good; it pays no commuting cost, but must pay half the total rent to consumer 2. Then the utility level of consumer 1 becomes $\ln(1) + \omega - \frac{1}{2}Rent \geq \omega/2$. So the equilibrium does not belong to the core if $\omega/2 > \ln(s^*) + \omega + \frac{1}{2s^*} - 1$ or $\omega < 2 - 2\ln(s^*) - \frac{1}{s^*}$. For $t = 0.9231$, $s^* = 0.39018$ and the upper bound reads $\omega < 1.319$. Now $\frac{ds^*(t)}{dt} = \frac{8-\sqrt{64+16t^2}}{(2t)^2\sqrt{4+t^2}} < 0$. Since the upper bound is increasing in s^*, hence decreasing in t, it follows that $\frac{1}{8} \leq \omega < 1.319$ guarantees that for $t \leq 0.9231$, the equilibrium allocation does not belong to the core.

The intuition behind this result is that for a range of parameters, the equilibrium is characterized by high rent collections from the innermost consumer and thus large transfers to the outermost consumer. Notice that the utility that can be achieved for a consumer in a coalition of the consumer and the government is independent of commuting cost, since the consumer enjoys all of the land and pays no commuting cost. Hence, the coalition of the government and the innermost consumer could block by using an equilibrium rent density that is relatively low so that rent collections and transfers to the other consumer are also low. In essence, the additional utility provided to the innermost consumer by adding at least half the total land endowment to his bundle combined with a lower rent transfer to consumer 2 yield higher utility.

In fact, the core is empty for this example. To see this, suppose that the core is nonempty. We proved toward the beginning of this section that any equilibrium allocation is efficient, so this applies to the equilibrium allocation we have found for our example. Thus, some consumer is as well off or worse off in the core allocation compared with the equilibrium allocation. The coalition of this mistreated consumer and the public authority can block the core allocation using the argument in the preceding paragraphs. So we have a contradiction, and the core is empty. This argument applies generally to the model with two identical consumers, so if any equilibrium allocation is not in the core, then the core is empty.

4 Conclusion

Although land is an indivisible commodity and its use inflicts extra commuting costs on more remotely located consumers, the market does not fail. Moreover, there is no incentive for a subgroup of consumers to form a coalition. This result holds for private ownership economies with land, possibly featuring an absentee landlord. For an economy with public land ownership where a government returns rent (at least in excess of its agricultural value) to its citizens, the equilibrium remains basically the same and, in particular, efficient, but becomes vulnerable to a coalition of the government and a subgroup of the citizens, even if the rent titles of the excluded citizens are honored. There is an incentive to keep the population small. This idea goes beyond the familiar notion in the literature on local public goods that wealthy communities use exclusionary zoning to bar poor residents in order to preserve their tax base. In our model we have no taxes, and thus a government has no tax base to preserve. Our argument employs the government only as a redistributive mechanism, similar to its use in some versions of the second welfare theorem. With multiple governments, we are likely to simply have replicas of the our basic example and arguments, as the governments are indifferent about allocation, and thus have no reason to compete. The idea even goes beyond Hadar, Hochman and Pines (2004) who show that a laisser-faire city may have suboptimal size. In their model, the equilibrium is not Pareto efficient.

The bottom line is that competitive equilibrium might not be the proper solution concept in the sense of predicting outcomes under public land ownership.

It is important to note that our example and arguments all apply when $t = 0$, that is when there is no commuting cost and the model is aspatial. Thus, it applies to models with public ownership in general. What is crucial to our argument is that there is an agent endowed with all of one commodity that pays out rent proceeds from the use of this commodity to other agents in terms of other goods. But well-known problems arise in such models with the definition of the core. For example, in a standard general equilibrium model with production and exogenous profit shares, if the firms can make positive profits in equilibrium (for example, if they are not endowed with constant returns to scale technologies), then what a firm does in a coalition is not internalized within the coalition. Coalition production economies can help address this problem, though these models are not quite analogous to ours, where there is no production or profit.

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5 Appendix

Consumer 1 pays rent $\frac{1}{s^*}s^* = 1$, while consumer 2 pays rent $[\frac{1}{s^*} - \frac{\rho}{2}(1 - s^*)](1 - s^*)$ and commuting cost ts^*. Half the total rent is $\frac{1}{2}[\frac{1}{s^*} - \frac{\rho}{2}(1 - s^*)^2]$, which is decreasing in ρ. The utility levels of 1 and 2 are, respectively, $u_1 = \ln(s^*) + \omega + \frac{1}{2}[\frac{1}{s^*} - \frac{\rho}{2}(1 - s^*)^2] - 1$ and $u_2 = \ln(1 - s^*) + \omega + \frac{1}{2}[\frac{1}{s^*} - \frac{\rho}{2}(1 - s^*)^2] - [\frac{1}{s^*} - \frac{\rho}{2}(1 - s^*)](1 - s^*) - ts^*$. Subtracting,

$$u_1 - u_2 = [\frac{1}{s^*} - \frac{\rho}{2}(1 - s^*)](1 - s^*) + ts^* - 1 - \ln(\frac{1}{s^*} - 1) \tag{1}$$

To begin the proof, we first show that there is a $\rho^* \in (\frac{1-2s^*}{s^*(1-s^*)^2}, \frac{1}{s^*})$ such that $u_1 - u_2 = 0$ by using the intermediate value theorem. For $\rho = \frac{1-2s^*}{s^*(1-s^*)^2}$, substituting $s^* = \frac{2+t-\sqrt{4+t^2}}{2t}$,

$$\begin{aligned}
u_1 - u_2 &= [\frac{1}{s^*} - \frac{1-2s^*}{2s^*(1-s^*)^2}(1 - s^*)](1 - s^*) + ts^* - 1 - \ln(\frac{1}{s^*} - 1) \\
&= \frac{1-s^*}{s^*} - \frac{1-2s^*}{2s^*} + ts^* - 1 - \ln(\frac{1}{s^*} - 1) \\
&= \frac{1}{2s^*} + ts^* - 1 - \ln(\frac{1}{s^*} - 1) \\
&= \frac{t}{2+t-\sqrt{4+t^2}} + \frac{2+t-\sqrt{4+t^2}}{2} - 1 - \ln(\frac{2t}{2+t-\sqrt{4+t^2}} - 1) \\
&= [\frac{t}{2+t-\sqrt{4+t^2}} - \frac{\sqrt{4+t^2}}{2}] + [\frac{t}{2} - \ln(\frac{2t}{2+t-\sqrt{4+t^2}} - 1)]
\end{aligned}$$

We claim that this expression is positive for $0 < t < 2$. In fact, we prove that each bracketed expression is positive. To begin, consider the first bracketed expression. Notice that $4+t^2 \leq 4+t^2+\frac{t^4}{16}$, so $\sqrt{4+t^2} < \sqrt{4+t^2+\frac{t^4}{16}} \leq 2+\frac{t^2}{4}$, and therefore multiplying both sides by $2+t$, $(2+t)\sqrt{4+t^2} \leq 4+2t+\frac{t^2}{2}+\frac{t^3}{4}$. Furthermore, since $t < 2$, $\frac{t}{4} < \frac{1}{2}$ so $\frac{t^3}{4} < \frac{t^2}{2}$, and thus $(2+t)\sqrt{4+t^2} < 4+2t+t^2$, or $2t > (2+t)\sqrt{4+t^2} - 4 - t^2$. Division of both sides by $2+t-\sqrt{4+t^2}$ (which is positive as $(2+t)^2 = 4+t^2+4t > 4+t^2$, $2+t > \sqrt{4+t^2}$) establishes the positivity of the first bracketed term, $\frac{t}{2+t-\sqrt{4+t^2}} - \frac{\sqrt{4+t^2}}{2}$. The second bracketed term, $\frac{t}{2} - \ln(\frac{2t}{2+t-\sqrt{4+t^2}} - 1)$, is also positive as we will prove now. This expression tends to 0 for t tending to 0 by application of l'Hôpital's rule to $\frac{2t}{2+t-\sqrt{4+t^2}}$. Hence it suffices to show that its derivative is positive. Now $\frac{d}{dt}[\frac{t}{2} - \ln(\frac{2t}{2+t-\sqrt{4+t^2}} - 1)] = \frac{1}{2} - \frac{1}{\frac{2t}{2+t-\sqrt{4+t^2}}-1}[\frac{2}{2+t-\sqrt{4+t^2}} - 2t\frac{1-\frac{t}{\sqrt{4+t^2}}}{(2+t-\sqrt{4+t^2})^2}]$. Multiplied by $(\frac{2t}{2+t-\sqrt{4+t^2}}-1)(2+t-\sqrt{4+t^2}) = 2t-(2+t-\sqrt{4+t^2}) = t-2+\sqrt{4+t^2} > 0$, the derivative becomes $\frac{t}{2} - 1 + \frac{1}{2}\sqrt{4+t^2} - 2 + 2t\frac{1-\frac{t}{\sqrt{4+t^2}}}{2+t-\sqrt{4+t^2}}$. Multiplied further by $(2+t-\sqrt{4+t^2})\sqrt{4+t^2} = t(\sqrt{4+t^2}-t) + 2(\sqrt{4+t^2}-2) > 0$, this expression is positive if and only if $[2 - \frac{t}{2} - \frac{1}{2}(\sqrt{4+t^2}-2)][t(\sqrt{4+t^2}-t) + 2(\sqrt{4+t^2} - 2)] < 2t(\sqrt{4+t^2}-t)$. Expanding and collecting terms, this inequality is $4\sqrt{4+t^2} - t^2 - 8 < 0$, which is true.

For $\rho = \frac{1}{s^*}$, $p(1) = 1$ and, substituting $s^* = \frac{2+t-\sqrt{4+t^2}}{2t}$ into equation (1), $u_1 - u_2 = \frac{1}{2s^*} - \frac{s^*}{2} + ts^* - 1 - \ln(\frac{1}{s^*} - 1) = \frac{t}{2+t-\sqrt{4+t^2}} - \frac{2+t-\sqrt{4+t^2}}{4t} + \frac{2+t-\sqrt{4+t^2}}{2} - 1 - \ln(\frac{2t}{2+t-\sqrt{4+t^2}} - 1)$. This expression is negative if $t \leq 0.9231$. Then, by the intermediate value theorem, there is a $\rho^* \in (\frac{1-2s^*}{s^*(1-s^*)^2}, \frac{1}{s^*})$ such that the utility levels match. The marginal willingness to pay for land of consumer 2 must be greater than or equal to the price:[11] $\frac{1}{z-s^*} \geq \frac{1}{s^*} - \rho(z-s^*)$ or $\frac{z-s^*}{s^*} - \rho^*(z-s^*)^2 \leq 1$. The left hand side of this inequality is initially 0, that is for $z = s^*$. The derivative of the left hand side of the inequality, $\frac{1}{s^*} - 2\rho^*(z-s^*)$, is nonnegative and remains nonnegative as long as $z < s^* + \frac{1}{2\rho^* s^*}$ which is automatic for $z \leq 1$. Consequently the left hand side of the inequality is maximal for $z = 1$. It follows that the marginal willingness to pay for land of consumer 2 exceeds price if $\frac{1-s^*}{s^*} - \rho^*(1-s^*)^2 \leq 1$, which is true for $\rho^* \geq \frac{1-2s^*}{s^*(1-s^*)^2}$.

[11] It might seem as though the marginal willingness to pay of consumer 2 should be required to be equal to price at the right endpoint of the city, 1. However, we are using a closed model here, so the right endpoint of the city is not variable. For example, there could be a lake beginning at 1. Thus, even though marginal willingness to pay exceeds price at 1, the consumer cannot buy land beyond 1. This is very similar to general equilibrium models with differentiated, indivisible objects.

In order to verify that this is really an equilibrium, we must show that composite good consumption is non-negative. We claim that this is true if $\omega \geq \frac{1}{8}$ and $t \leq 1$. For consumer 1, the calculation is as follows.

$$\begin{aligned}
& \omega + \frac{1}{2}[\frac{1}{s^*} - \frac{\rho^*}{2}(1-s^*)^2] - 1 \\
\geq \; & \omega + \frac{1}{2}[\frac{1}{s^*} - \frac{1}{2s^*}(1-s^*)^2] - 1 \\
= \; & \omega + \frac{1}{4s^*}[1 + s^*(2-s^*)] - 1 \\
\geq \; & \omega + \frac{1}{4s^*}[1 + s^* \cdot \frac{3}{2}] - 1 \\
\geq \; & \omega + \frac{1}{2} + \frac{3}{8} - 1 \\
\geq \; & 0
\end{aligned}$$

For consumer 2, the calculation is as follows.

$$\begin{aligned}
& \omega + \frac{1}{2}[\frac{1}{s^*} - \frac{\rho^*}{2}(1-s^*)^2] - [\frac{1}{s^*} - \frac{\rho^*}{2}(1-s^*)](1-s^*) - ts^* \\
= \; & \omega + \frac{1}{2s^*} - \frac{\rho^*}{4}(1-s^*)^2 - \frac{1}{s^*} + 1 + \frac{\rho^*}{2}(1-s^*)^2 - ts^* \\
= \; & \omega - \frac{1}{2s^*} + 1 + \frac{\rho^*}{4}(1-s^*)^2 - ts^* \\
\geq \; & \omega - \frac{1}{2s^*} + 1 + \frac{1-2s^*}{4s^*} - ts^* \\
= \; & \omega + 1 + \frac{1-4s^*}{4s^*} - ts^* \\
= \; & \omega + \frac{1}{4s^*} - ts^* \\
\geq \; & \omega + \frac{1}{2} - ts^* \\
\geq \; & 0
\end{aligned}$$