Welfare Analysis of the Number and Locations of Local Public Facilities

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January 26, 2005; revised Aug. 1, 2005

Abstract

We develop a model with a finite number of households and congestable local public goods where the level of provision, the number of facilities and their locations are all endogenously determined. We prove that an equal-treatment identical-provision second-best optimum exists, where all households are required to reach the same utility level, the provision of local public good is required to be the same at all facilities, and all facilities must serve the same number of consumers. Such an optimal public facility configuration may be concentrated (single site) or dispersed (multiple sites), depending on congestability, commuting cost and household preference parameters.

JEL Classification Numbers: H41, R13, D61

Keywords: Congestable Local Public Goods, Optimal Public Facility Configurations

Acknowledgments: We are grateful for valuable comments and suggestions from Richard Arnott, Ted Bergstrom, Noel Campbell, Christopher Colburn, John Conley, Masa Fujita, Hideo Konishi, Richard Jensen, Andy McLennan, Tomoya Mori, David Wildasin, three anonymous referees and seminar participants at Academia Sinica, Kyoto, Vanderbilt, the North American Meetings of the Regional Science Association International, the Midwest Economic Theory and International Trade Conference, the Public Choice Society Meetings and the International Conference on Public Finance and Welfare. The first author acknowledges financial support from NSF grant No. SBR-9523940. The second author acknowledges support from NSC grant No. 89-2415-H-001-026 and Fulbright Foundation grant No. 21994, and also appreciates the hospitality of the Department of Economics, Washington University in St. Louis. We acknowledge financial support from the Institute of Economics of Academia Sinica, which facilitated international collaboration between the authors.

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1. Introduction

There are many papers examining the effects of local public goods (LPGs) on the underlying spatial structures. These studies focus primarily on how the level of LPG provision and the locations of public facilities influence the locational choices of households and firms.[1] The presence of public goods serves as an agglomerative force, leading to the clustering of households and firms around a public facility. While this positive analysis is interesting, it is equally important to examine public facility configuration from a normative point of view. The number and locations of public facilities are crucial in determining the level of economic welfare in a community. However, due to technical barriers, this important task has not been explored completely. In this paper, we attempt such an endeavor by performing a general-equilibrium welfare analysis using a model with a finite number of households and congestable LPGs in which the number of facilities and their locations, as well as the level of public good provision, are all endogenously determined.[2]

The joint determination of the optimal number and locations of public facilities and the optimal level of public good provision is therefore the central feature of the present paper. As in the literature, we restrict our focus to "identical provision" of the LPG with the same level of the LPG provided in each site where each site is used by the same number of people. For example, Fujita (1986) and Sakashita (1987) feature identical provision where the locations of public facilities are endogenous. However, the number of public facilities is exogenously given as one in Fujita and as one or two in Sakashita; there is no endogenous mechanism pinning down the number of public facilities within the system. One may wonder whether the provision of a large central park/library or a few small community parks/libraries is more efficient in a spatial economy. Furthermore, the existing literature on public facilities in a spatial context assumes that the level of provision

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[1]For example, see Arnott and Stiglitz (1979), Brueckner (1979), Ellickson (1979), Scotchmer (1986), Wildasin (1987), Thisse and Wildasin (1992), and those papers cited in a survey by ReVelle (1987).

[2]The discrete model is the original urban model of Alonso (1964). He proposes models with land prices that are additive across parcels, where no arbitrage is possible in equilibrium, as well as models with land prices that are not additive across parcels, such as pricing according to the value of the point in a parcel closest to the city center, where arbitrage is possible in equilibrium.

is exogenously fixed. In contrast, we endogenize the optimal allocation of resources between the public and the private goods but establish a sufficient condition such that the level of provision of the LPG is independent of the configuration of public facilities and the allocation of the private good.

The second special feature of our paper is the consideration of locally congestable public goods. Most previous studies focus on the non-exclusive nature of LPGs (e.g., see Fujita 1986, Sakashita 1987 and Peng 1996), ignoring the possibility that public goods may generate local congestion as a function of the locationally dependent number of users. Local congestability is certainly observed in reality, as in the case of parks, public schools and libraries, where aggregate usage would affect the quality of services. Theoretically, it is also interesting to investigate how the presence of local congestability may affect the welfare properties of allocations in the spatial context.

The third distinctive feature is that our paper is the only one, to our knowledge, based on the concept of Pareto optimality in determining the optimal public facility configuration. In particular, we construct a concept of "equal-treatment" identical-provision optimality that has all households achieving the same level of utility.[3] Fujita (1986) and Sakashita (1987) employ representative consumer utility maximization, which may be reinterpreted as an equally weighted social welfare maximum of households with concave utility functions. It may be noted that in Sakashita, land rents exit the system and thus his command optimal outcomes need not be consistent with Pareto optimality. While our equal-treatment identical-provision optimality concept is well-defined in the general equilibrium sense, the feature of equal treatment allows us to compare our results with the contributions of other studies.

In contrast with most previous studies in the area of LPGs and spatial economics, where models with a continuum households are used, our paper employs a discrete-household framework.[4] This is an important

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[3]We will show that with identical provision of LPGs, our concept of optimality generally forces a second-best allocation since non-identical provision of LPGs may be Pareto dominant.

[4]A drawback to our discrete model is inelastic demand for land. Nonetheless, it is a normative model of public facility location with a fixed stock of housing. Our model structure can be regarded as an extension of the discrete model of Berliant and Fujita (1992) to incorporate congestable LPGs with the level of provision, the number and the locations of facility sites endogenously determined.

departure because in conventional location models with a continuum of agents, one can prove that 0 = 1; see Berliant (1985). Most importantly, it is typical in the congestable local public good literature that a finite number of consumers patronize a given facility.[5] It is unclear how to represent this in a spatial context with a continuum of consumers.[6] To circumvent theoretical shortcomings in continuum models, we adopt the discrete framework for conducting mathematically rigorous general-equilibrium analysis .

The main findings of this paper are as follows. First, to construct a well-defined and more complete normative analysis of the configuration of facilities providing congestable LPGs, we prove, under proper assumptions, the existence of an equal-treatment identical-provision optimum. We also establish a sufficient condition for the level of public good provision to be independent of the public facility configuration and other endogenous variables, thus enabling a tractable analytic solution. Second, given a single public facility, the equal-treatment optimal location of the public facility may not be at the geographic center and such an optimal location need not be unique, in contrast with both Fujita (1986) and Sakashita (1987). Third, with the number of public facilities endogenously determined, a sufficiently high degree of both congestability and commuting cost and a sufficiently low household valuation of the public good imply that the equal-treatment identical-provision optimal public facility configuration is always dispersed. These main findings are then shown to be robust to the number of households. Finally, we illustrate the possibility that an equal-treatment identical-provision optimum is generally Pareto inferior to a non-identical provision feasible allocation. This is because the latter has more flexible travel assignments and public good allocations that enables lower aggregate commuting costs and less congestion. This suggests that the notion of optimality used frequently in the literature yields second-best allocations.

In Section 2, we develop a simple spatial model of locally congestable public goods. Section 3

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[5]For example, see Conley and Wooders (2001) and Ellickson, Grodal, Scotchmer and Zame (forthcoming).

[6]It has been suggested that the number of public facilities be allowed to take any nonnegative real value instead of nonnegative integer values. It would be interesting but unusual to solve for the location of a fractional, perhaps irrational, facility.

proves the existence of an equal-treatment identical-provision optimum. In section 4, we characterize optimal public facility configurations with two or more households, using a utility function allowing for the optimal level of public good provision to be independent of public facility configuration and private good allocation. Section 5 examines the case of non-identical provision of the LPG. Section 6 concludes the paper and provides avenues for future research.

2. The Economy

Consider a community accommodating N identical households that must locate in a location $i \in X \equiv \{1, 2, ..., N\}$, where $N < \infty$. Each residential region has a unit length in a one-dimensional interval $L \equiv [0, N]$, and is of homogeneous quality (except for the aspect of public goods). Each household is required to reside in a location. In the absence of vacant land or multiple occupancy, land allocation in our spatial economy becomes a simple assignment problem in which each lot is occupied by only one household.[7] Households are homogenous with respect to their taste (preferences). Thus, household i consumes the interval $(i -1, i]$ and is conveniently labeled by the "front-door" location of its residence. The aggregate endowment of composite private good is $E > 0$.

Consumers enjoy consuming both composite private good and the LPG provided by the community at certain assigned ***public facility*** locations.[8] At a given location, a household has the nonnegative orthant of $\mathbb{R}^3$, $\mathbb{R}^3_+$, as its consumption set. There are three types of goods: (i) a composite consumption good, (ii) a LPG, whose facility service is valued by each household, and (iii) land. To simplify the analysis, we assume that the demand for land is perfectly inelastic - i.e., each household consumes a fixed, unit quantity of land. Thus, the consumption set reduces to $\mathbb{R}^2_+$.

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[7]This is similar to the Koopmans and Beckmann (1957) assignment problem, meaning that it is infeasible to have more than one household residing in the same lot.

[8]For notational simplicity, we follow the conventional wisdom, assuming that each location contains only one resident. More realistically, one may imagine that each residential region x_i is occupied by a finite number of identical households. This generalization only requires minor modification of the definition of land partition without changing any of our main results.

The technology used by the community government to produce the single public good is constant returns to scale, generating one unit of public good for each unit of composite private good used as an input.[9] Thus, we let K denote both the composite consumption good used in production as well as the total supply of public good. This total supply is divided among the n facilities. The public good may be provided at one (***concentrated***) or more than one (***dispersed***) location and (*ex ante*) more than one facility is allowed to be at the same location. Denote the set of public facility locations as $H \subset L$. When there is a single public facility site at $\eta \in L$, we have $H \equiv \{\eta\}$. All households patronize this facility. When there are n sites, we have $H \equiv \{\eta_1, \eta_2, ..., \eta_n\}$ where $\eta_j \in L$ with a corresponding level of the LPG denoted by K_j, and where M_j is the number of households patronizing site j (j = 1, ..., n) (obviously, $\sum_{j=1}^{n} K_j = K$ and $\sum_{j=1}^{n} M_j = N$). Thus, by denoting the cardinality of a discrete set as $|\cdot|$, we have $n = |H|$ and the pair (n,H) summarizes the configuration of public facilities, i.e., both the number and locations of local public facility sites. Examples of such LPGs include schools, libraries, museums, parks, theater halls, public swimming pools, exercise fields or other recreational facilities.

Each household must incur a commuting cost to travel to the assigned public facility locations.[10] For simplicity, it is assumed that the amount of land required by the facility is negligible.[11] Moreover, we consider throughout the paper ***identical provision*** (henceforth called **IP**) of the LPG in the sense that when there are multiple public facility sites, the same level of LPG is provided at each site and the same number of

[9]In practice, in most countries there are higher level authorities (city, state, region, federal). The higher tier may seek to influence or constrain the activities of local communities by direct methods (local authorities may be forbidden from enacting certain restrictive legislation or compelled to provide a minimum standard of public goods), or by taxation and subsidies (e.g., a matching grant). Also, it is possible that the cost of LPG provision may vary across locations, especially if it requires land as an input and if land is heterogeneous or the population distribution is not uniform. Nevertheless, these issues are far beyond the scope of the present paper.

[10]Fujita (1986) denoted this as a traveled-for public good.

[11]Fujita (1986), Sakashita (1987) and Peng (1996) make a similar assumption. The model can easily be modified to allow public facilities to use land, provided that the land requirement for the provision of K units of public good is fixed. In this case, the division of a larger public facility into several small units would not change total land usage.

households use each of them. Thus, when there are n sites, the identical provision of the public good implies that each site is equipped with a LPG of size $K_j = K/n$ and shared by a population of households of $M_j = N/n$ (for all j = 1, ..., n). While the feature of identical provision allows us to compare our results with those in these previous studies, we would like to alert the reader that in general, a non-identical provision of the public good may be Pareto dominant (see Section 5 for a detailed illustration).

Let $Y_i \subset$ H be a subset of public facilities to which household i travels and denote $y_i \in Y_i$ as a travel destination of household i. In general, household i may be assigned to randomize, with probability $\theta(i,\eta_j) \in [0,1]$ and $\sum_{\eta_j \in H} \theta(i,\eta_j) = 1$, to travel to a particular destination η_j where $\theta(i,\eta_j) > 0$ for $\eta_j \in Y_i$ with $|Y_i| \leq n$ and $\theta(i,\eta_j) = 0$ for $\eta_j \notin Y_i$.[12] Obviously, n is at most N under this setup and hence $n \in$ X. This randomization framework is designed to be consistent with the consideration of identical public good provision.[13]

The pecuniary cost of commuting, $T_i(y_i)$, varies with distance from the public facility location. Since any single point has zero measure, we do not have to differentiate between open and closed intervals and can utilize the ***inner-Hausdorff metric*** (or closest-point distance), denoted d, to measure the distance between a single point and a half open interval.[14] Specifically, the distance between household i's residential region in L and a public facility location y_i, denoted $d(i, y_i)$, is defined as: (i) $(y_i - i)$, if $y_i > i$ [rightward traveling pattern]; (ii) 0, if $i - 1 \leq y_i \leq i$ [non-traveling]; (iii) $(i - 1) - y_i$, if $y_i < i - 1$ [leftward traveling pattern], respectively. In this spatial economy, we assume that the commuting cost schedule (measured in terms of the composite

[12]To be more concrete, consider the case of $N = 3$ and $n = 2$ where the two facility sites are at $\eta_1 = 1$ and $\eta_2 = 2$. Thus, a possible travel pattern is: households 1 and 3 commute to $y_1 = \eta_1$ and $y_3 = \eta_2$, respectively, whereas household 2 randomizes by commuting to $y_2 \in Y_i = \{\eta_1, \eta_2\}$ with $\theta(2,\eta_1) = \theta(2,\eta_2) = 1/2$.

[13]An alternative is to follow Lucas (1990), assuming there is a continuum of household members in each household and the household head makes a joint decision for all members. In this case, for the example in footnote 12, we simply assume that half the members of household 2 travel to one site and the remaining half to the other.

[14]Importantly, the standard Euclidean metric is not applicable in this case. For further discussion, the reader is referred to Klein and Thompson (1984, p. 39) and Berliant and Wang (1993, pp. 130-131). In Section 4.2 below, we will discuss the consequences of using mid-point distance in place of closest point distance.

good), T, satisfies:

(**A1**) (Commuting cost) $T_i = T(d(i,y_i))$ where T is twice continuously differentiable, $T(0)=0$, $T'>0$ and $T''\geq 0$.

While (A1) is imposed in obtaining general theorems, we consider a simple linear commuting cost schedule frequently used in the literature (e.g., see Alonso 1964 and Fujita 1986) to characterize optimal public facility configurations (in Sections 4 and 5 below):

$$T_i = t\,d(i,y_i),\ t > 0. \tag{1}$$

Denote the total commuting cost as $C(n,H) \equiv \sum_{i=1}^{N}\sum_{y_i\in Y_i}\theta(i,y_i)T(d(i,y_i))$, where $Y_i \subset H \subset L$ for all i = 1,...,N. For given $H = \{\eta_1,\ldots,\eta_n\}$, $n = |H|$ and $K_j = K/n$, a set of random travel assignments $\Theta = \{\theta(i,\eta_j)\}_{\eta_j\in H;\, i=1,\ldots,N;\, j=1,\ldots,n}$ is called ***cost-minimizing*** if $Y_i \equiv \mathit{argmin}_{\eta_j\in H}\ d(i,\eta_j)$ for all $i \in$ X (i.e., every household is assigned to the closest public facility sites). For given n and $K_j = K/n$, a public facility configuration associated with a set of public facility sites H and a set of random travel assignments $\Theta = \{\theta(i,\eta_j)\}_{\eta_j\in H;\, i=1,\ldots,N;\, j=1,\ldots,n}$ is called ***aggregate-cost-minimizing*** if for all $H'\subset$ L and $\Theta' = \{\theta'(i,\eta_j)\}_{\eta_j\in H';\, i=1,\ldots,N;\, j=1,\ldots,n}$, $C(n,H) \equiv \sum_{i=1}^{N}\sum_{y_i\in Y_i\subset H}\theta(i,y_i)T(d(i,y_i)) \leq \sum_{i=1}^{N}\sum_{y_i'\in Y_i'\subset H'}\theta'(i,y_i')T(d(i,y_i'))$.

Let the quantity of composite good consumption and of public good consumption be given by z and G, respectively. Let α denote the ***degree of congestability*** of the public good. The amount of public service embodying congestion can therefore be specified as:[15]

(**A2**) (Consumption of public good) $G(K_j, M_j) = K_j (M_j)^{-\alpha}$, $\alpha \in [0,1]$.

Under identical provision, the LPG allocated to each site is $K_j = K/n$ and each site is patronized by $M_j = N/n$ households. We can thus rewrite: $G = (K/n)/(N/n)^{\alpha}$. Notably, $\alpha = 0$ implies a pure public good, whereas $\alpha = 1$ implies a private good. In a special case where all households share one concentrated LPG at a single facility

[15]A LPG subject to the external effect of congestion is defined as a congestable LPG. For detailed discussion of congestable public goods, see Starrett (1991).

of size K (i.e., $n = 1$), our setup reduces to that considered by Hochman (1982), Thisse and Wildasin (1992) and Peng (1996): $G = K(N)^{-\alpha}$. Notably, under our normative analysis (Pareto optimality), whether a household or the local government bears the commuting costs is irrelevant. Thus, the type of public goods considered in this paper can include police or fire protection where the services are taken to the consumers. However, our congestable LPG does not permit either positive externalities (where one may enjoy consuming the good with others) or the possibility of a variable degree of congestability (which may depend on the number of users).

(**A3**) (Well-behaved preferences of households) $u(z, G)$ is twice continuously differentiable, $\partial u/\partial z > 0$, $\partial u/\partial G > 0$, and is strictly concave.[16]

An ***allocation*** is defined as a list of (i) nonnegative quantities of composite good $z_i \in \mathbb{R}_+$ consumed by each household i (i =1, ..., N), (ii) a configuration of public facilities described by a set of public facility sites $H \equiv \{\eta_1, \eta_2, ..., \eta_n\} \subset L$ with the number of sites $n = |H| \in X$, (iii) a nonnegative level of provision of the LPG $K_j \in \mathbb{R}_+$ at each location $\eta_j \in L$ (j =1, ..., n and $\sum_{j=1}^{n} K_j = K$), with the number of households patronizing each public facility site $M_j \in \mathbb{R}_+$ (j =1, ..., n and $\sum_{j=1}^{n} M_j = N$), (iv) a subset of location indexes $Y_i \subset H$ (i =1, ..., N) containing the public facility sites that household i patronizes, and (v) a randomization probability $\theta(i,\eta_j) \in [0,1]$ assigning the probability with which household i travels to a particular destination $\eta_j \in H$ with $\sum_{\eta_j \in H} \theta(i,\eta_j) = 1$, $\sum_{i=1}^{N} \theta(i,\eta_j) = M_j$, and $\theta(i,\eta_j) = 0$ for $\eta_j \notin Y_i$. In short, an allocation can be written as: $a = (\{z_i, Y_i\}_{i=1,...,N}, \{K_j, M_j\}_{j=1,...,n}, \{\theta(i,\eta_j)\}_{i=1,...,N;\, j=1,...n}, n, H)$.

Given $y_i \in Y_i \subset H$ (i = 1, ..., N), the ***material balance condition*** for the composite good is:

$$\sum_{i=1}^{N} [z_i + \sum_{y_i \in Y_i} \theta(i,y_i)\, T(d(i,y_i))] = E - K. \qquad \textbf{(2)}$$

We now define a ***feasible allocation*** as an allocation $a = (\{z_i, Y_i\}_{i=1,...,N}, \{K_j, M_j\}_{j=1,...,n}, \{\theta(i,\eta_j)\}_{i=1,...,N;\, j=1,...n}, n, H)$

[16]It is necessary to assume strict concavity (rather than quasi-concavity) in order to ensure the sufficiency of the Lindahl-Samuelson condition to be derived in equation (5) below.

satisfying the material balance condition (2).

Next we restrict our attention to the case of identical provision of the LPG. An ***identical-provision* (IP) *feasible allocation*** is a feasible allocation such that $K_j = K/n$ and $M_j = N/n$ for all j = 1, ..., *n*. An ***identical-provision optimum*** is an IP feasible allocation such that there is no other IP feasible allocation making no household worse-off and at least one better-off. Under assumption (A2) together with household randomization and identical provision, the public good consumed inclusive of the congestion factor becomes:

$$G(n,K) = K/(N^{\alpha} n^{1-\alpha}), \qquad \textbf{(3)}$$

which is constant across all households. The term $N^{\alpha}n^{1-\alpha}$ can be regarded as a "net congestion factor," increasing in the population and the number of public facility sites. Since all consumers are identical, the assignment of consumers to parcels is arbitrary and thus land allocation is trivial. Identical LPG provision enables us to write $K_j = K/n$ and $M_j = N/n$ for all j, so the IP optimality problem is simply given by:

$$\underset{\{z_i, y_i, \theta(i,y_i)\}_{i=1,\dots,N}, K, n, H}{Max} \quad u(z_1, G(n,K)), \qquad \textbf{(4)}$$

$$\text{s.t.} \quad (i) \quad u(z_i, G(n,K)) \geq v_i, \; i=2,3,\dots,N;$$

$$(ii) \quad \textit{equation } (2);\; z_i \geq 0,\, K \geq 0,\, n \in X,\, Y_i \subset H,\; i=1,2,\dots,N;\; \sum_{i=1}^{N} \theta(i,\eta_j) = \frac{N}{n} \;\; \forall \eta_j \in H.$$

This states the IP optimality problem as one of trying to maximize the well-being of household 1 subject to required utility levels for others in the economy (constraint (i)) and feasibility (equation (2)) as well as the IP constraint in (ii).

In the existing literature, an "optimal" configuration of public facilities is based on the concept of "command optimality" in which the community government determines the public facility configuration subject to (i) the competitive determination of the land rents (that often exit the system without a landlord to formally close the model) and (ii) locational equilibrium (that equalizes households' utility). It is clear that such a concept is not well-defined in the Pareto sense - indeed, it is a mix of equilibrium and optimality concepts.[17] To be compatible with the literature, however, we modify the concept of IP optimality by considering ***equal treatment*** (**ET**) among all households. Specifically, an ***equal-treatment identical-provision optimum*** (in short, an **ETIP** optimum) is an IP optimum such that all households reach the same utility, i.e., we have $u(z_i,G(n,K)) = v_o \geq 0$ for all i = 1,..., N in problem (4).

Three comments are now in order. First, ETIP allocations may be justified by the equal protection clause of the U.S. Constitution, federal law, or by state constitutions. For example, the judicial system uses this clause and federal law to force equal school spending within jurisdictions (see Inman and Rubinfeld 1979, p. 1699 for application of the equal protection clause and pp. 1697-1701 for application of federal law) while state constitutions can require equal school spending across jurisdictions (see Inman and Rubinfeld 1979, pp. 1705-1708).[18] Second, an ETIP optimum is Pareto optimal provided that the provision of public good at all facilities is the same in an equal utility Pareto optimum and the number of consumers served at each facility is the same at the Pareto optimum. That is, an ETIP optimum may fail to be first-best if it is welfare enhancing to alter the ETIP optimum to serve different numbers of consumers or provide different levels of LPGs at different sites. As discussed in Section 5 below, there may be an allocation with non-identical public good provision that Pareto dominates the allocations restricted to identical provision. Third, our definition of ETIP optimality basically determines the optimal distribution to consumers for given aggregate endowments of the composite good and land (E and L, respectively), as well as optimal public facility configurations in terms of the number and locations of sites. Thus, it does not involve an equilibrium concept, contrasting with the command optimality concept in previous studies.

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[17] This problem has been pointed out by Berliant and ten Raa (1994, p. 638): "Most of the models [of public facility location] appear to use a mix of positive and normative concepts, equilibria and optima. At best, they can be interpreted as partial welfare analysis."

[18] Of course, this need not imply that in practice, school spending is equalized across jurisdictions.

3. Existence of an Equal-Treatment Identical Provision Optimum

In this section we establish theorems concerning the existence of an ETIP optimum and its general properties. An ETIP optimum can be obtained by solving the problem specified in (4) with $v_i = u(z_1, G(n,K)) = v_o$ for all i = 2,..., N and (4)(i) holding with equality.

To begin, we determine the optimal allocation between the public and the private composite good (see the Appendix): $\sum_{i=1}^{N}\left(\frac{\partial u_i/\partial G}{\partial u_i/\partial z_i}\right)\frac{\partial G}{\partial K} = 1$, which is in essence the *Lindahl-Samuelson condition*. With regard to ETIP optimality, we define $\Lambda(K,n,z) \equiv \frac{1}{N}\frac{\partial u/\partial z}{(\partial u/\partial G)(\partial G/\partial K)}$ and show that any ETIP optimal provision of the LPG satisfies the following necessary condition (see the Appendix):

$$\Lambda(K,n,z) = 1 \qquad \textbf{(5)}$$

Denote $MRS_{Gz} \equiv (\partial u/\partial G)/(\partial u/\partial z)$ as the marginal rate of substitution between the public and the private composite goods. It may be more intuitive to use (3) to rewrite (5) as: $\Lambda(K,n,z) = \frac{N^{\alpha}n^{1-\alpha}}{N(MRS_{Gz})} = 1$. That is, the sum of the marginal willingness-to-pay (i.e., $N\,(MRS_{Gz})$, under ETIP) is equal to the marginal cost of public good provision (i.e., the net congestion factor $N^{\alpha}n^{1-\alpha}$). Notably, the optimal composite good allocation, the optimal number of public facility sites and the optimal level of the public good are all jointly determined, making it difficult to characterize the ETIP optimum. We thus take the method adopted by Bergstrom and Cornes (1983), seeking a condition such that the optimal level of public good provision is not only uniquely determined but also *independent* of the public facilities configuration and the distribution of composite good. By examining (5), we impose:

Condition K: $\partial\Lambda/\partial n = \partial\Lambda/\partial z = 0,\ \partial\Lambda/\partial K > 0$.

Condition K requires that *ex ante* the sum of the marginal willingness-to-pay by all households is *not* affected by the number of public facilities or the allocation of the composite good. However, individual valuation of the public good is allowed to depend on the public facility configuration. Intuitively, the independence of the aggregate marginal willingness-to-pay with respect to the allocation of the composite good eliminates the income effect concerning the entire community as a whole, whereas that with respect to the number of public facility sites removes the spatial-distribution effect in aggregation. As it can be seen from Lemma 2 below, Condition K makes the system recursive, thereby greatly simplifying the analysis with respect to the optimal travel assignment.

Condition K is sufficient for (5) alone to pin down the optimal level of K, independent of n and z. An example satisfying Condition K will be provided shortly.

Lemma 1: *Under Assumptions (A2) and (A3) and Condition K, an ETIP optimum, if it exists, is associated with a uniquely determined level of public good K_0 that is independent of the public facility configuration.*

Proof: All proofs are relegated to the Appendix.

To guarantee a nonempty feasible set, we assume the aggregate endowment of composite private good E is sufficient to cover the spending on providing the LPG at the optimal level K_0:[19]

Condition E: $K_0 < E$.

For illustrative purposes, an example is now in order:

Example: Consider $u(z,G) = \gamma z + \ln(G)$ where $\gamma > 0$. Then we have $\Lambda = \gamma K/N$ and hence (5) implies $K = K_0 \equiv N/\gamma$, independent of any endogenous variables. It is clear that Condition E is met if $\gamma(E/N) > 1$, meaning that households' valuation of the average composite good endowment be greater than one.

Due to the endogenous number of public facilities, Bergstrom and Cornes' (1983) condition of a separable utility function that is quasi-linear in the composite good is insufficient in our model to guarantee an optimal level of aggregate public good provision independent of other endogenous variables. The imposition of Condition K simplifies our analysis greatly.

Under Assumptions (A1)-(A3) and Condition K and given the optimal number of facilities, we obtain another important necessary condition governing optimal travel assignments and public facility locations:

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[19] Condition E is not necessary – if it is not satisfied, the level of private good consumption is zero, implying a corner solution for the optimum.

Lemma 2: *Under Assumptions (A1)-(A3) and Condition K, an ETIP optimum with an optimal number of public facilities given by n^*, if it exists, is associated with an aggregate-cost-minimizing travel assignment among all ETIP feasible travel assignments and public facility configurations with $n = n^*$. That is, let $(\{z_i, Y_i\}_{i=1,\ldots,N}, \{K_j, M_j\}_{j=1,\ldots,n^*}, \{\theta(i,\eta_j)\}_{i=1,\ldots,N;\, j=1,\ldots n^*}, n^*, H)$ be an ETIP optimum. Then* $\forall H', |H'| = n^*, \forall \theta'(i,y_i'), y_i' \in Y_i' \subset H' \; (i=1,\ldots,N)$,

$$\sum_{i=1}^{N} \sum_{y_i \in Y_i \subset H} \theta(i,y_i)\, T(d(i,y_i)) \leq \sum_{i=1}^{N} \sum_{y_i' \in Y_i' \subset H'} \theta'(i,y_i')\, T(d(i,y_i')) \tag{6a}$$

Moreover, the number of households patronizing a particular public facility site $j = 1, \ldots, n^*$ satisfies:

$$M_j = \sum_{k=1}^{n} \frac{1}{k} |\{i : \eta_j \in Y_i \subset H, |Y_i| = k\}| = \frac{N}{n}. \tag{6b}$$

Lemma 2 says that given the optimal number of public facilities, an optimal set of sites must be such that the aggregate travel cost is minimized. This property is crucial for determining households' travel patterns as well as the optimal configuration of public facilities. Should Condition K be violated, the travel assignment generally interacts with the level of public good provision. As a consequence, the characterization of the optimal number and locations of public facilities becomes intractable even when focusing exclusively on ETIP feasible allocations.

At an ETIP optimum, public good *G* is uniform across households (as specified in (3)) and thus ET implies that the composite good consumption must be identical. Clearly, the total commuting cost $C(n,H)$ is non-increasing in n. The ETIP optimal composite good allocation to each household is then:

$$z = (E - K - C(n,H))/N. \tag{7}$$

Since C is a function of (n,H), G is a function of (n,K) and, from Lemma 1, $K = K_0$ (an exogenous constant), households' indirect utility must be a function of (n,H) alone. We denote this value by $v(n,H)$.

Theorem: (*Existence of ETIP Optimum*) *Under Assumptions (A1)-(A3) and Conditions K and E, there exists an equal-treatment identical-provision optimum.*

It is important to note that from the proof of the theorem, the Maximum Value Theorem can also be applied to the set of feasible allocations A (as A is nonempty and compact). Thus, the existence of a Pareto optimum (that is not necessarily ETIP or IP) can easily be established.

Apparently, all we need to establish the theorem are the continuity of the households' utility function and the commuting cost function, and that *G* is non-increasing in *n* (to ensure that *n* is finite, bounded by *N*), which follows from the assumption that *u* is increasing in *G* and *T* is increasing in $d(i,y_i)$. The assumptions regarding differentiability, specific functional forms, or interior solutions imposed in (A1)-(A3) are not needed here, but are made for a complete characterization of the ETIP optimum in Section 4.

Remark 1: Recall that H = $\{\eta_1, \ldots, \eta_n\}$ and hence H and *n* are inter-related. For each *n*, define $H^*(n) = \mathbf{argmin}_H\ C(n,H)$. While the total commuting cost $C(n,H^*(n))$ is non-increasing in *n*, the net congestion factor $N^\alpha n^{1-\alpha}$ is strictly increasing in *n* (for $\alpha < 1$). Thus, under assumptions (A2) and (A3) as well as Condition K, equations (7) and (3) together with Lemma 1 ($K = K_0$) imply that per capita composite good consumption *z* is non-decreasing in *n* and *G* is strictly decreasing in *n*. An immediate consequence of these conflicting effects on $u(z,G)$ is that an ETIP optimum may be associated with an intermediate value of $n \in \{2, \ldots, N\text{-}1\}$, i.e., an ETIP optimal public facility configuration need not be completely concentrated ($n = 1$) or completely dispersed ($n = N$).

Remark 2: There are in general a continuum of ETIP optimal public facility configurations, even when n is fixed. For example, consider the case of $n = 1$ and *N* as an odd number with a central region $[[N/2], [N/2]+1]$, where **[**D**]** is the Gauss operator of $D \in \mathbb{R}_+$, defined as the largest integer that is less than or equal to D. We will show in Section 4 that under assumptions (A1)-(A3) and Conditions K and E, any $\eta \in [\,[N/2], [N/2]+1\,]$ is ETIP optimal. When $n = 1$ and *N* is an even number, there is a unique ETIP optimum $\eta = N/2$, which is in between the front door of the site of the **[**$N/2$**]**-household and the back door of the site of the (**[**$N/2$**]**+1)-

household.

We would like to remind the reader that the main purpose of this paper is to perform a *normative* analysis of local public facility configuration, which contrasts sharply with the existing literature based on a *positive* analysis of command optimality. Nevertheless, for comparison, we focus primarily on ETIP allocations, which are conceptually close to the allocations considered in previous studies.

4. Characterization of Optimal Public Facility Configurations

We characterize ETIP optimal public facility configurations explicitly given a linear commuting cost schedule specified as in (1). We focus on three issues: (i) whether an ETIP optimal configuration is uniquely determined, (ii) whether public facilities should be concentrated in one location or dispersed in more than one location, and (iii) whether a concentrated public facility should be geographically centralized.

4.1. Optimal Location of a Public Facility: The Case of $n = 1$

It is important to note that the uniqueness/multiplicity properties of ETIP optimality depend crucially on the number of households. First we fix the number of facilities exogenously at 1.

Proposition 1: (ETIP optimal location of a public facility with $n = 1$) *Assume (A2), (A3), Conditions K and E, and the linear commuting cost schedule specified as in (1). Given $n = 1$, the set of* ETIP *optimal public facility configurations satisfies the following properties:*

(i) for even numbers N, it is unique and geographically centralized;

(ii) for odd numbers $N > 1$, it is a continuum consisting of any location in the central residential interval.

The results in Proposition 1 contrast with findings in the framework with a continuum of households of Fujita (1986) and Sakashita (1987) in two important dimensions. First, in our model, an ETIP optimal location of the public facility need not be at the geographic center. Second, we find that multiple ETIP optimal configurations are possible in the sense that the optimal location of the public facility need not be unique. These discrepancies are mainly due to the difference between the finite versus continuum of households setups and the concept of equal-treatment identical-provision optimality versus command optimality. It may be noted that as N goes to infinity (by expanding L to the entire extended real line), this multiplicity property is still robust.

4.2. Optimal Number and Locations of Public Facilities: the Case of $N = 3$

We next turn to endogenizing the number of public facilities. When $N = 1$ or $N = 2$, the ETIP optimal facility location problem is trivial. When $N \geq 3$, it is possible that the ETIP optimal number of public facilities is no longer one. For illustrative purposes, it suffices to consider the case of $N = 3$. Given $\alpha < 1$ (i.e., G is not a pure private good), we claim:

Lemma 3: *Assume (A2) with $\alpha < 1$, (A3), Conditions K and E, and the linear commuting cost schedule specified as in (1). Given $N = 3$, the ETIP optimal number of public facilities n can only be 1 or 2.*

Lemma 3 rules out the possibility of $n = 3$. Hence, there are three candidate ETIP optimal public facility configurations remaining to be examined with $N = 3$: (i) centrally concentrated (type-C), (ii) non-centrally concentrated (type-N) and (iii) dispersed (type-D). Notably, Proposition 1 implies that an ETIP optimum with $n = 1$ (concentrated) must have the public facility of size K_0 in the central residential interval. In this case, there are two possibilities: either $H = \{3/2\}$ (i.e., type-C) or $H = \{\eta\}$ with $\eta \in [1, 2] \setminus \{3/2\}$ (i.e., type-N). With regard to the dispersed case ($n = 2$), we claim:

Lemma 4: *Assume (A2), (A3), Conditions K and E, and the linear commuting cost schedule specified as in (1). Given N = 3, if the type-D public facility configuration is ETIP optimal, there are two locationally symmetric and identically provided public facilities of size $K_0/2$ located at $\eta = 1$ and 2.*

Remark 3: We would like to brief the reader on the consequences of using the mid-point distance measure rather than the closest-point distance for the calculation of commuting cost. Suppose that we measure each household's location by the middle point of their residential interval. Figure 3 summarizes the corresponding total commuting costs for N = 2 as well as N = 3 with either a concentrated or a dispersed public facility configuration. First, note that the multiplicity of ETIP optima still emerges, although it now occurs when N is an even number. Second, in the dispersed case with two public facilities, it is still true that the ETIP optimal locations are not necessarily at the first and third-quarter of the length of the landscape – they are at one-sixth and five-sixths in case of N = 3. Finally, as can be seen from panels (b) and (c) of Figure 3, the marginal total commuting cost is no longer constant. This will lead to greater complexity in characterizing the ETIP optimal configuration of public facilities in the general case.

We are now left to examine each candidate configuration and to derive the corresponding conditions to support each case. In order to compute a household's indirect utility, we further assume that the utility function of the households takes the following form (satisfying Condition K):[20]

$$u(z,G) = \gamma z + \ln(G) \tag{8}$$

where $\gamma > 0$. From Lemma 1, we can solve the optimal provision of the LPG as $K_0 = N/\gamma$. Utilizing Lemmas 2-4, we can substitute the composite good allocation (7), and the public good (3) under each type of facility configuration into the utility function (8) to obtain the household value for each type of public facility configuration as follows:

$$v^C = v^N = \gamma(E - K_0)/3 + \ln(K_0/3) + (1-\alpha)\ln 3 - \gamma t/3 \tag{9a}$$

$$v^D = \gamma(E - K_0)/3 + \ln(K_0/3) + (1-\alpha)\ln 3 - (1-\alpha)\ln 2 \tag{9b}$$

where the superscripts denote the types of public facility configurations and the corresponding aggregate commuting costs are: $C^C = C^N = t$ and $C^D = 0$.

To compare different types of public facility configurations, we need to check if they are Pareto rankable. This is easily done by comparing households' value for each case, as given by (9a) and (9b),

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[20] Those interested in the conventional Cobb-Douglas utility functional form are referred to the Appendix. Since in that case Condition K is not satisfied, we instead assume that the level of the public good provision K is exogenously fixed. Under this assumption, the main conclusions remain valid.

provided that $\gamma E > 3$, which is ensured by Condition E.

Proposition 2: (ETIP optimal number and locations of public facilities with $N = 3$) *Assume (A2) with α < 1, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). When* $N = 3$*, an ETIP optimal public facility configuration satisfies the following properties:*

(i) *if* $t > 3(\ln 2)(1-\alpha)/\gamma$, *then an ETIP optimal public facility configuration is always dispersed;*

(ii) *if* $t < 3(\ln 2)(1-\alpha)/\gamma$, *then an ETIP optimal public facility configurations is always concentrated, either centralized or non-centralized.*

Thus, the determination of the optimal public facility configuration depends crucially on the commuting cost (t), the degree of non-exclusiveness (1-α), and the household valuation of the public good (1/γ). Within our congestable LPG framework with α < 1, we can conclude:

Proposition 3: *Assume (A2) with α < 1, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). Given* $N = 3$*, an ETIP optimal public facility configuration is dispersed if the degree of congestability is large, the household valuation of the public good is low, and the unit commuting cost is sufficiently high; otherwise, an ETIP optimal public facility configuration is concentrated.*

Therefore, building a large central park (or library, or school) is not ETIP optimal in an economy with a high degree of public good congestability or a high commuting cost. A Pareto improvement results if the public good is provided at dispersed facility sites with smaller size. This justifies the existence of local community parks, libraries and schools in reality.

4.3. Optimal Number and Locations of Public Facilities: the General Case of $N > 3$ and $n \geq 1$

In Section 4.1, we considered the case $n = 1$ for any given N, whereas Section 4.2 examines the case of $N = 3$ for an endogenous determination of ETIP optimal n that need not be equal to one.[21] We now characterize the general case of $N > 3$ allowing for multiple public facility sites.

We begin by proving the following Lemma:

Lemma 5: *Assume (A2) with α < 1, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). In an ETIP optimum, the following properties always hold:*

(i) for any even number N, there should be no more than N/2 public facilities;

(ii) for any odd number N that is a multiple of 3, there can be no more than 2N/3 public facilities;

(iii) for an odd number N that is not a multiple of 3, there may be as many as N -1 public facilities.

Part (iii) of Lemma 5 is due in particular to the consideration of identical LPG provision that requires the number of consumers patronizing each site to be the same. Although the completely dispersed ($n = N$) public facility configuration can never be optimal, it is possible to have the maximum dispersed ($n = N - 1$) public facility configuration as a candidate for ETIP optimality. In general, for any number $N > 3$, a maximum dispersed public facility configuration can be ETIP optimal only if N is not a multiple of 2 or 3.

Before establishing the general proposition for $N > 3$ in Lemma 6 and Proposition 4 below, we would like to note that with regard to concentrated ($n = 1$) versus dispersed ($n > 1$) public facility configurations, Proposition 3 can be applied to the general case of $N > 3$. In particular, an ETIP optimal public facility configuration is always dispersed when there is a high degree of public good congestability, it is costly to commute and household valuation of the public good is low. Otherwise, there are multiple ETIP optimal public facility configurations where the optimal configurations could be concentrated or dispersed.

More generally, we can characterize ETIP locations of public facilities for $N > 3$ given the linear commuting cost schedule as in (1) and the utility function specification as in (8). For illustrative purposes, we focus on a set of parameters such that an ETIP optimum features $n = 2$. We claim:

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[21] The case of $N = 2$ is trivial: the ETIP optimal configuration must be $n = 1$ and $\eta = 1$ (concentrated and centralized). It is therefore omitted from our discussion.

Lemma 6: *Assume (A2) with α < 1, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). Given N > 3, the set of parameters resulting in a cost-minimizing ETIP optimal number of public facilities equal to two is nonempty.*

Proposition 4: *Assume (A2) with α < 1, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). Given N>3 and a set of parameters such that* $\arg\min_n C(n, H^*(n)) = 2$*, ETIP optimal locations of public facilities under cost-minimizing travel assignments possess the following properties:*

(i) *when N is a multiple of 4,* $H = \{N/4, 3N/4\}$*;*

(ii) *when N is even but not a multiple of 4,* $H = \{\eta_1, \eta_2\}$ *with* $\eta_1 \in [\,[N/4], [N/4]+1\,]$ *and* $\eta_2 \in [\,N - [N/4] - 1, N - [N/4]\,]$*;*

(iii) *when N is an odd number,* $H = \{\,[(N+1)/4], N - [(N+1)/4]\,\}$.

5. On the Non-Identical Provision of the Local Public Good

Up to this point, we have restricted our attention to the case of identical LPG provision. In this Section, we study the consequences of relaxing the IP assumption. Recall that in Section 3 we have verified the existence of a Pareto optimum that is not necessarily IP. Thus, our focus here is on illustrating that a feasible allocation with non-identical provision of a LPG can Pareto dominate any IP feasible allocation. That is, we claim:

Proposition 5: *Assume (A2) with α < 1, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). An ETIP optimum may be Pareto inferior to a non-identical provision feasible allocation.*

Since equal treatment only involves reallocation of the composite good, Proposition 5 suggests that a non-IP feasible allocation can Pareto dominate any IP feasible allocation. By comparing the non-IP, ET allocation with the ETIP optimum in the proof of Proposition 5, the number, the locations and the levels of public good provision are identical. The lone difference is the travel assignment and hence the allocation of users of each public facility. Thus, an IP travel assignment is generally Pareto suboptimal.

In order to gain further insight towards understanding the intuition underlying the inferiority of IP feasible allocations, let us provide more examples. First, consider the case of $N = 5$ satisfying the following condition: $\frac{15(1-\alpha)}{2\gamma}\ln\left(\frac{3}{2}\right) \le t \le \frac{15(1-\alpha)}{\gamma}\ln\left(\frac{4}{3}\right)$, so that the ETIP optimum is a set of public facility sites H = {1, 5/2, 4}, a provision of the LPG at each site of $K_j = K_0/3$ with $K_0 = 5/\gamma$ and j = 1, 2, 3, a travel assignment θ(1, 1) = θ(5, 4) = θ(3, 5/2) = 1, θ(2, 1) = θ(4, 4) = 2/3 and θ(2, 5/2) = θ(4, 5/2) = 1/3, and an allocation of the composite good $z = [E - K_0 - t/3]/5$. It is interesting to see that this ETIP optimal allocation is always inferior to a non-IP, ET feasible allocation with H = {1, 5/2, 4}, the public good provision of $\{2K_0/5, K_0/5, 2K_0/5\}$ at each site, a cost-minimizing travel assignment θ(1,1) = θ(2,1) = θ(3, 5/2) = θ(4, 4) = θ(5, 4) = 1, and an allocation of the composite good $z_3 = \{E - K_0 + 4[(1-\alpha)/\gamma]\ln 2\}/5$ and $z_i = \{E - K_0 - [(1-\alpha)/\gamma]\ln 2\}/5$ (i = 1,2,4,5). Precisely, this alternative non-IP feasible allocation results in zero aggregate commuting cost (compared to $t/3$ at the ETIP optimum) with a net per capita utility gain of $\{\gamma t/3 + (1-\alpha)[4\ln 2 - 5\ln(5/3)]\}/5$ without requiring any additional conditions. It is therefore observed from this example that the inferiority of an IP feasible allocation can be a consequence of suboptimal provision of the public good at each site. For both this example and that in the proof of Proposition 5, however, the inferiority of an IP feasible allocation is mainly due to a higher aggregate commuting cost.

Second, we consider the case of $N = 3$ with $t \ge \frac{3(1-\alpha)}{\gamma}\ln 2$, so that the ETIP optimum is a set of public facility sites H = {1, 2}, a provision of the LPG at each site of $K_j = K_0/2$ with $K_0 = 3/\gamma$ and j = 1, 2, 3, a travel assignment θ(1, 1) = θ(3, 2) = 1 and θ(2, 1) = θ(2, 2) = 1/2, and an allocation of the composite good $z = (E - K_0)/3$ with C(2, H) = 0. Now, construct a Pareto superior, non-IP, ET feasible allocation with H = {1/2, 2}, the public good provision of $\{K_0/3, 2K_0/3\}$, a cost-minimizing travel assignment θ(1, 1) = θ(2, 2) = θ(3, 2) = 1, and an allocation of the composite good $z_1 = \{E - K_0 + 2[(1-\alpha)/\gamma]\ln 2\}/3$ and $z_i = \{E - K_0 - [(1-\alpha)/\gamma]\ln 2\}/3$ (i = 2,3). This alternative non-IP feasible allocation maintains zero aggregate commuting cost but generates a net per capita utility gain of $(1-\alpha)(5\ln 2 - 3\ln 3)/3$ without requiring any additional conditions. Thus, the inferiority of an IP feasible allocation in this case is not a result of greater aggregate commuting costs, but a more restrictive provision of LPGs. The alternative non-IP feasible allocation leads to an improvement upon the ETIP optimum because it reduces the congestion externality – one can see that the net utility gain vanishes as the degree of congestability diminishes (i.e., $\alpha \to 1$).

In summary, a non-IP feasible allocation can Pareto dominate any IP feasible allocation. On the one hand, a non-IP feasible allocation can improve welfare by reducing aggregate commuting costs through more flexible travel assignments and/or non-identical provision of public goods at different public facility sites. On the other, a non-IP feasible allocation can make individuals better-off by reallocating the users of public facilities to decrease the congestion externality associated with LPGs. This suggests that the command optimum considered in the literature is generally Pareto suboptimal. Hence, more research is needed to explore systematically non-IP feasible allocations in the interest of the community's welfare.

6. Concluding Remarks

We have examined a model with a finite number of households and with a congestable LPG where the number and locations of facility sites and the level of public good provision are endogenously determined. We have demonstrated the existence of an ETIP optimum, shown that there may be a continuum of ETIP optimal public facility configurations, and shown that an ETIP optimal public facility location need not be central geographically. An ETIP optimal public facility configuration must be dispersed with multiple facility sites if the degree of congestability and the unit commuting cost are high and the household valuation of the LPG is low. Otherwise, the ETIP optimal public facility configuration is concentrated. Due to its restrictive travel assignments and public good allocations, such an ETIP optimum usually results in a higher commuting cost or a greater congestion and is therefore Pareto inferior to some non-IP feasible allocations.

Along these lines, there are at least three possible avenues of interest for future research. First, one may explore the theoretical front, examining the nonemptiness of an ETIP core (thus refining the set of ETIP optima) and the validity of the first and second welfare theorems based on Lindahl or competitive spatial equilibrium. Second, one may relax the assumption of identical households to allow for, say, two types of households, with heterogeneous tastes. One may study whether an optimal public facility configuration is geographically biased toward the rich or a consumer who likes the public good. Finally, it is possible to consider more than one public good. When all public goods are necessities, it is interesting to examine whether a concentrated public facility configuration may still emerge where different public goods are provided at the same site.

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Figure 1: Public Facility Configuration and Total Commuting Costs with N = 2

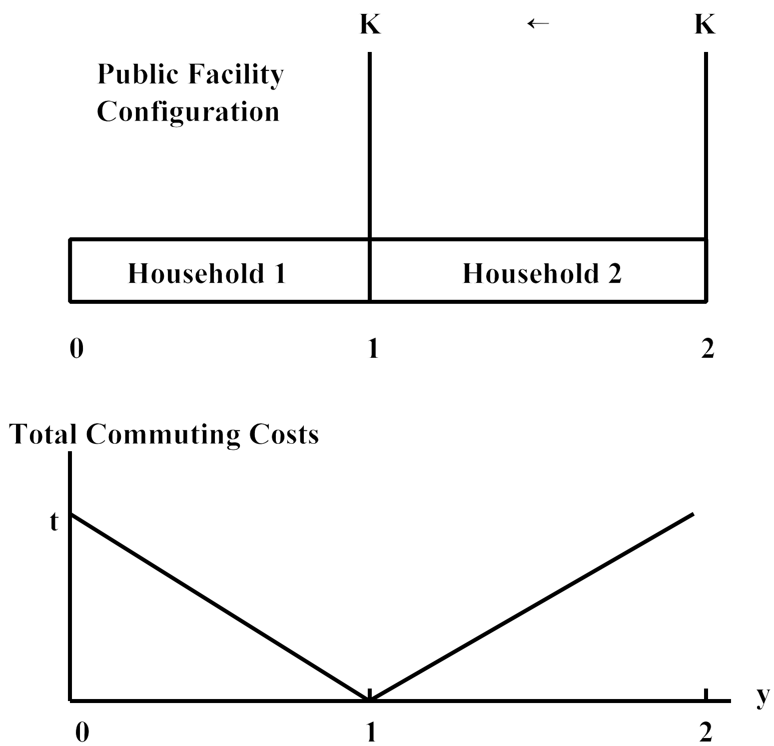
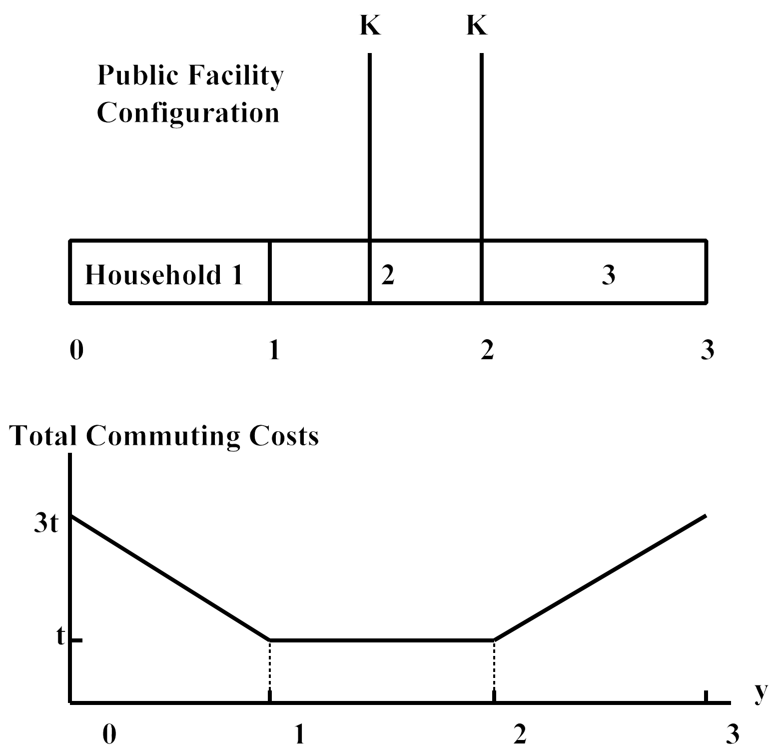


Figure 2: Public Facility Configuration and Total Commuting Costs with N = 3

(a) The Case of Concentrated Configuration



(b) The Case of Dispersed Configuration

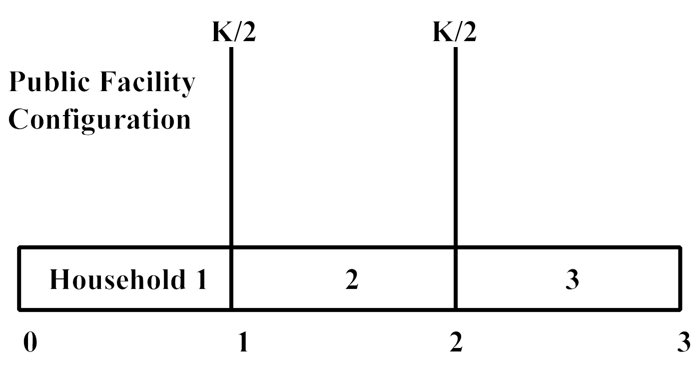
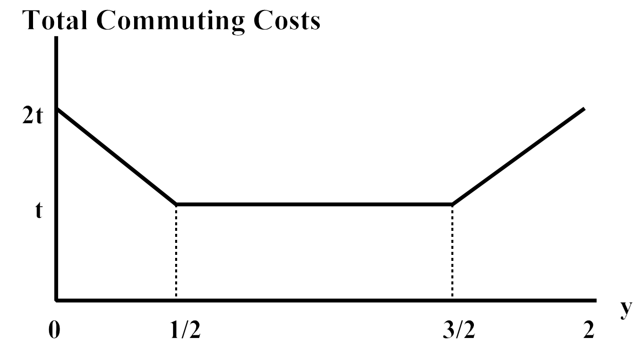
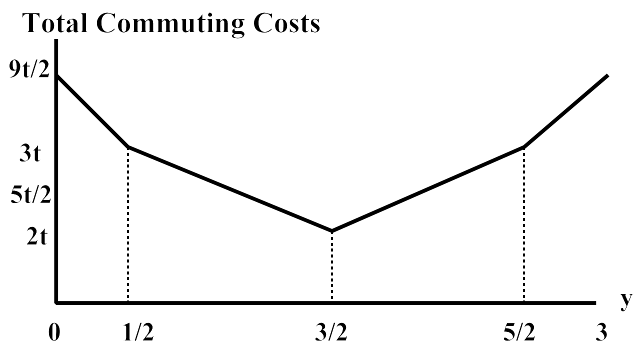


Figure 3: Total Commuting Costs under Mid-Point Distance Measure

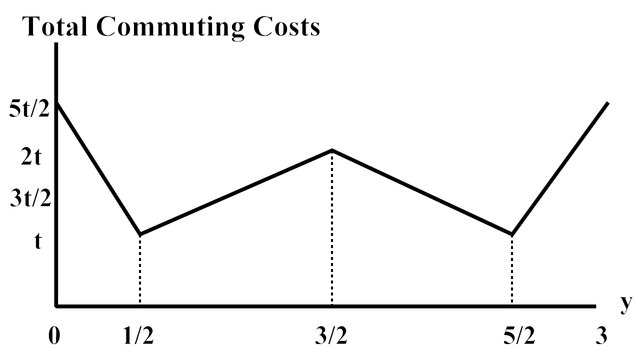
(a) The Case of N = 2



(b) The Case of N = 3 with Concentrated Configuration



(c) The Case of N = 3 with Dispersed Configuration



Appendix

Derivation of Equation (5) and Proof of Lemmas 1 and 2:

To solve problem (4) with $v_i = v_o$ for all i = 1,..., N and (4)(i) holding with equality, we note that random travel assignments have non-zero values only for sites $\eta_j \in Y_i$ (i = 1,...,N; j = 1,...,n) and thus we can focus exclusively on $\Theta(Y_i) \equiv \{\theta(i,y_i)\}_{y_i \in Y_i}$. We can set up the Lagrangian function as:

$$Ł(\{z_i,Y_i,\Theta(Y_i)\}_{i=1,....,N},K,n,H) = u(z_1,G(n,K)) + \sum_{i=2}^{N} \lambda_i^C\{u(z_i,G(n,K))\} + \lambda^Z\{E - K - \sum_{i=1}^{N}(z_i + T(d(x_i,y_i)))\} \quad \text{(A-1)}$$

where $G(n,K) = K/(N^{\alpha} n^{1-\alpha})$ and the multipliers λ^C and λ^Z are, in sequential order, associated with the two constraints given in (4).

The first-order conditions are:

$$\partial u_1/\partial z_1 - \lambda^Z = 0, \quad \text{(A-2a)}$$

$$\lambda_i^C(\partial u_i/\partial z_i) - \lambda^Z = 0, \quad i = 2,...,N \quad \text{(A-2b)}$$

$$(\partial u_1/\partial G)(\partial G/\partial K) + \sum_{i=2}^{N} \lambda_i^C(\partial u_i/\partial G)(\partial G/\partial K) = \lambda^Z, \quad \text{(A-2c)}$$

$$Ł(\{z_i,Y_i,\Theta(Y_i)\},K,n,H) \geq Ł(\{z_i,Y_i',\Theta'(Y_i')\},K,n',H'), \quad \forall\ Y' \subset H' \subset L,\ H' \neq H,\ n' = |H'| \in X,\ \Theta' \neq \Theta, \quad \text{(A-2d)}$$

For any n, (A-2a) and (A-2b) together imply:

$$(\partial u_1/\partial z_1)/(\partial u_i/\partial z_i) = \lambda_i^C. \quad \text{(A-3)}$$

Substituting (A-2a) and (A-3) into (A-2c) to eliminate λ^Z and λ_i^C, we obtain:

$$(\partial u_1/\partial G)(\partial G/\partial K) + \sum_{i=2}^{N} \frac{\partial u_1/\partial z_1}{\partial u_i/\partial z_i}(\partial u_i/\partial G)(\partial G/\partial K) = \partial u_1/\partial z_1, \quad \text{(A-4a)}$$

or, by rearranging terms,

$$\sum_{i=1}^{N} \frac{1}{\partial u_i/\partial z_i}(\partial u_i/\partial G)(\partial G/\partial K) = 1 \,. \tag{A-4b}$$

Under ETIP, all individuals have the same public good G and receive $u(z_i, G) = v_0$. Thus, z_i are the same for all i and (A-4b) becomes:

$$N\left(\frac{\partial u/\partial G}{\partial u/\partial z}\right)\frac{\partial G}{\partial K} = 1 \,. \tag{A-5}$$

Define $\Lambda(K,n,z) \equiv \frac{1}{N}\frac{\partial u/\partial z}{(\partial u/\partial G)(\partial G/\partial K)}$. Then, equation (5) follows immediately from (A-5).

Next, since $\partial\Lambda/\partial n = \partial\Lambda/\partial z = 0$, we can now write $\Lambda = \Lambda_0(K)$ with $d\Lambda_0/dK > 0$. Thus, (5) can be inverted to determine the optimal provision $K = K_0$ where $K_0 \equiv \Lambda_0^{-1}(1)$ is independent of n and z. Note that the original form of the first-order condition should be written as $1/\Lambda(\cdot) = 1$. Totally differentiating this expression (imposing ETIP) and utilizing the property that $\Lambda = \Lambda_0(K)$, the second-order condition reduces to: $-(d\Lambda_0/dK)/\Lambda_0^2 < 0$, which is ensured by Condition K.

Furthermore, from (A-2a), $\lambda^Z > 0$, which together with (A-1) implies $Ł(\{z_i, Y_i, \Theta(Y_i)\}, K, n, H)$ is strictly decreasing in the aggregate commuting cost. Thus, letting $(\{z_i, Y_i, \Theta(Y_i)\}_{i=1,\ldots,N}, \{K_j, M_j\}_{j=1,\ldots,n}, n, \mathrm{H})$ be an ETIP optimum, then $\forall H', \ \mu(H') = n, \ \forall\theta'(i,y_i), \ y_i \in Y_i' \subset H' \ (i=1,\ldots,N)$, the optimal choice given by (A-2d) satisfies:

$$\sum_{i=1}^{N} \sum_{y_i \in Y_i \subset H} \theta(i,y_i)\, T(d(i,y_i)) \leq \sum_{i=1}^{N} \sum_{y_i' \in Y_i' \subset H'} \theta'(i,y_i')\, T(d(i,y_i')) \tag{A-6}$$

This yields the necessary condition (6a) for the optimization problem (4). The remaining condition (6b) follows immediately under ETIP.

Note that under Condition E, (A-3) and (A-6) together with the second constraint of (4) jointly determine the optimal locations of public facilities and the indirect utility level $v(n,\mathrm{H})$ for a given n with $K = K_0$. By comparing $v(n,\mathrm{H})$ for each $n \in X$, the optimal number of public facilities n is therefore pinned down.

Proof of the Main Theorem:

Denote the set of feasible allocations as A = $\{(\{z_i, Y_i\}_{i=1,...,N}, \{K_j, M_j\}_{j=1,...,n}, \{\theta(i,\eta_j)\}_{i=1,...,N;\, j=1,...n}, n, H)$: $\{z_i, Y_i\}_{i=1,...,N}$ and $\{K_j\}_{j=1,...,n}$ satisfy (2), $Y_i \subset H\ \forall\ i = 1, ..., N$, $\sum_{k=1}^{n} \theta(i,\eta_j) = M_j\ \forall\ \eta_j \in H$, $\sum_{j=1}^{n} M_j = N$, $n = |H| \in X$ and $H = \{\eta_1, ..., \eta_n\} \subset L\}$. Under (A1) and noting that $|X| = N$ is finite, it is obvious that A is compact (closed and bounded). We next show that the set of IP feasible allocations is closed. Denote the set of IP feasible allocations as P = $\{ (\{z_i, Y_i\}_{i=1,...,N}, \{K_j, M_j\}_{j=1,...,n}, \{\theta(i, \eta_j)\}_{i=1,...,N;\, j=1,...n}, n, H) \in A$: each site $\eta_j \in H$ has a LPG quantity $K_j = K/n$ and $\{Y_i\}_{i=1,...,N}$ and $\{\theta(i,\eta_j)\}_{i=1,...,N;\, j=1,...n}$ are consistent with $\sum_{i=1}^{N} \theta(i,\eta_j) = M_j = \sum_{k=1}^{N} \frac{1}{k} |\{i : \eta_j \in Y_i, |Y_i| = k\}| = N/n\ \ \forall\ \eta_j \in H$, i = 1, ... , N, j = 1, ... , n }. The IP restrictions only involve randomization probabilities $\theta(i,\eta_j)$, which takes real values from a compact set [0,1], and quantities n, $y_i \in Y_i$ (i=1,...,N) and $\eta_j \in H$ (j=1,...,n), which all take values from 0 to N. Note that all variables are defined in Euclidean spaces and that more than one facility is allowed to locate at a given point in L when the set of feasible allocations A is constructed. Thus, the closedness property is trivial and P is compact as it is a closed subset of a compact set A. Further denote the set of ETIP feasible allocations as F = $\{ (\{z_i, Y_i\}_{i=1,...,N}, \{K_j, M_j\}_{j=1,...,n}, \{\theta(i,\eta_j)\}_{i=1,...,N;\, j=1,...n}, n, H) \in P : u(z_i,G(n,K)) = v_o \geq 0$ for all i = 1,..., N }. We claim that F is closed. Under ETIP and Condition K, we have $K = K_0$ by Lemma 1 and aggregate-cost-minimizing travel assignments by Lemma 2. It is sufficient to focus on continuous quantities $Z = (z_1,...,z_N, \theta(1,\eta_1),...,\theta(N,\eta_1), \theta(1,\eta_2),...,\theta(N-1,\eta_n), \theta(N,\eta_n))$. Consider now a sequence of ETIP allocations a^ℓ = $(\{z_i^\ell, Y_i\}_{i=1,...,N}, \{K_0, M_j\}_{j=1,...,n}, \{\theta(i,\eta_j)\}_{i=1,...,N;\, j=1,...n}, n, H) \in F$ for all ℓ = 0, 1, 2, ... and it has a limit point a = $(\{z_i, Y_i\}_{i=1,...,N}, \{K_0, M_j\}_{j=1,...,n}, \{\theta(i,\eta_j)\}_{i=1,...,N;\, j=1,...n}, n, H) \in P$. By construction, $u(z_i^\ell, G(n,K_0)) = v_o \geq 0$ for all i = 1,..., N. Since u is continuous by (A3), we have $u(z_i,G(n,K_0)) = v_o$ and hence $a \in F$, which proves the assertion. As F is a closed subset of a compact set P, F is compact.

Next, we show that F is nonempty. It is obvious that by constructing an IP public facility configuration (n,H) with $n = N$, $y_i = \eta_i \in (i\text{-}1, i]$, $K_i = K_0/N$, $\theta(i,\eta_j) = 1$ for all $\eta_i \in (i\text{-}1, i]$, $\theta(i,\eta_j) = 0$ for all $\eta_i \notin (i\text{-}1, i]$, and M_i =1 (i=1,...,N), the corresponding total commuting cost is $C(n,H)=0$ and $G(n,K_0) = K_0/N$. Under Condition E, we have $E\text{-}K_0 - C(n,H) > 0$ and by choosing $z_i = (E\text{-}K_0)/N$, such allocations are ETIP feasible, implying $F \neq \phi$.

Because n, $y_i \in Y_i$ (i=1,...,N) and $\eta_j \in$ H (j=1,...,n) are discrete and finite and u is continuous for each fixed ($\{Y_i\}_{i=1,...,N}$, n, H), the ETIP optimality problem has a continuous objective over a compact and nonempty domain F. By straightforward application of the Maximum Value Theorem implied by the Bolzano-Weierstrauss Theorem (Bartle 1976, pp. 154-155), an ETIP optimum exists.

Proof of Proposition 1:

With $n = 1$, the random travel assignment is trivial in that $|Y_i| = 1$ for all i $\in$ X. To prove part (i), consider an even number of households $N = 2$. As shown in Figure 1, for given unit commuting cost $t > 0$, we start with $\eta = 2$ (for the case of $\eta = 0$, the arguments are similar and hence omitted). In this case, the total commuting cost is $C = t$. Now, we move the public facility to $\eta = 1$. While the LPG G is unchanged, the total composite good available for allocation to households increases from $E\text{-}K_0\text{-}t$ to $E\text{-}K_0$, resulting in a Pareto improvement. Similar arguments apply to any $\eta \in (1, 2)$. Therefore, the allocation $(z_i,K,n,H) = (E/N\text{-}K_0/2,K_0,1,\{1\})$ is ETIP optimal, where K_0 solves (5). The corresponding total commuting cost schedule with respect to the public facility location is depicted in bottom panel of Figure 1 from which one can see that $\eta = 1$ is associated with minimum total commuting costs, satisfying the requirement in (8). This analysis and the uniqueness result apply to the case with $n = 1$ and any even number N.

We next prove part (ii) by considering N odd. Consider Figure 2. We claim that any allocation with H = {η} where $\eta \in [[N/2], [N/2]+1]$ (i.e., the central region) is ETIP optimal. Suppose not. Consider the public facility site on the right of the central region with $\eta = [N/2]+1 + \varepsilon$, where $\varepsilon > 0$. Then moving the site to $\eta = [N/2]+1$ would result in an improved allocation as it reduces the total commuting cost by $2\varepsilon t$. Similar arguments apply for the case with the public facility site on the left of the central region and for any $\varepsilon > 0$. Next, we show that the relocation of the public facility site within the central region leaves the total commuting cost unchanged. Consider moving the site from $\eta = [N/2]+1$ to $\eta = [N/2]+1 - \varepsilon$, where $\varepsilon \in (0, 1)$. While the commuting cost of each household residing on the left of the central region is reduced by εt, that of each household on the right is raised by εt (for illustrative purposes, we delineate the case of $N = 3$ and $\varepsilon = ½$ in Figure 2(a)). Since the total commuting cost C is unchanged, these allocations are all ETIP optimal.

Proof of Lemma 3:

This lemma can easily be proved by contradiction. Suppose $n = 3$ is ETIP optimal with one public facility of size $K_0/3$ in each residential interval (so no one commutes). Thus, $G(3,K_0) = K_0/3$. Consider an alternative ETIP allocation with $n = 2$, H = {1,2} and θ(1,1) = θ(3,2) = 1 and θ(2,1) = θ(2,2) = 1/2 in which $C(2,\{1,2\}) = 0$. Now public good provision becomes $G(2,K_0) = (3/2)^{1-\alpha}(K_0/3)$, strictly greater than $K_0/3$ for any $\alpha < 1$. Thus, an allocation with $n = 3$ cannot be an ETIP optimum.

Proof of Lemma 4:

To prove this lemma, we utilize Figure 2(b). In particular, we need to show that public facilities must be located at 1 and 2. Suppose not. Consider, say, an alternative ETIP feasible allocation with H = {1-ε, 2+ε}, θ(1,1-ε) = θ(3,2+ε) = 1, θ(2,1-ε) = θ(2,2+ε) = 1/2, and $z = (E - K_0 - \varepsilon t)/3$ where $\varepsilon \in (0,1)$. Then using the arguments in the proof of Proposition 1, we can obtain an improving reallocation with H = {1, 2}, θ(1,1) = θ(3,2) = 1, θ(2,1) = θ(2,2) = 1/2, and $z = (E - K_0)/3)$, which contradicts the definition of ETIP optimality. By similar arguments, one can show that any other ETIP feasible allocations with $n = 2$ and H = {1+ε, 2-ε} is Pareto dominated by that with H = {1,2} as the latter is the only ETIP feasible allocation associated with zero total commuting cost.

Proof of Lemma 5:

The proof of parts (i) and (ii) follows arguments similar to the proof of Lemma 3. When N is an even number, it is always Pareto dominant to have at least two adjacent households sharing one public facility at their back/front door location in which the public good G is higher than that with more than $N/2$ facilities. The proposed allocation has zero total commuting cost. When N is an odd number that is a multiple of 3, a configuration with three connected households sharing two public facilities as illustrated in Figure 2(b) always Pareto dominates an allocation with more than $2N/3$ facilities. Again, the proposed allocation has zero commuting cost.

For part (iii), it is sufficient to consider the case of $N = 5$. Since equal numbers of people must be served at all sites, there are five IP optimal public facility configuration candidates with n = 1, 2, 3, 4 and 5 and corresponding $H^*(1) \in [2, 3]$, $H^*(2) = \{1, 4\}$, $H^*(3) = \{1, 5/2, 4\}$, $H^*(4) = \{1, 2, 3, 4\}$ and $H^*(5) = \{\eta_j\}_{j=1,\ldots,5}$ with $\eta_j \in [j-1, j]$ where $C(1, H^*(1)) = 4t$, $C(2, H^*(2)) = t$, $C(3, H^*(3)) = t/3$, $C(4, H^*(4)) = C(5, H^*(5)) = 0$. It is easily seen that if $K_0 < E < K_0 + t/3$ (a sufficient but not necessary condition), the ETIP optimal configuration of public facilities must be associated with n = 4 or 5 (since any allocation with n = 1, 2 or 3 is not IP feasible). These arguments together with Lemma 3 imply that for an odd number that is not a multiple of 3, it is possible to have an ETIP optimal configuration with $n = N - 1$ if the unit commuting cost is sufficiently large relative to the aggregate endowment.

Proof of Lemma 6:

Recalling that $H^*(n) = arg\ \min_H C(n,H)$, substituting (3), (7) and $K = K_0$ into (8) and equating the utility values associated with n = 1 and n = 2, for N > 3, yield: $C(1,H^*(1)) - C(2,H^*(2)) = (\ln 2)(1-\alpha)/\gamma$. It is easily shown that for N = 4, $C(1,H^*(1)) - C(2,H^*(2)) = 2t$. Define $I_N \equiv \left\{2 + \sum_{J=5}^{N}\left(\left\lceil\frac{J-3}{2}\right\rceil - \left\lceil\frac{J-5}{4}\right\rceil\right)\right\}$. By induction, we can derive $C(1,H^*(1)) - C(2,H^*(2)) = tI_N$ for all $N \geq 5$. Let $\hat{t} = (\ln 2)(1-\alpha)/(\gamma I_N)$. One can then see that, for any t with $t - \hat{t}$ approaching to zero from above, n = 2 is ETIP optimal among all allocations with cost-minimizing travel assignments.

Proof of Proposition 4:

Case (i) is trivial. For case (ii), divide the entire region evenly into two where each half has a public facility site. By symmetry, we focus on the left half, $L_1 \equiv [0, N/2]$. Since N is not a multiple of 4, L_1 must contain an odd number of households. Since there is only one public facility in L_1, we can apply Proposition 1 (ii) to conclude that the ETIP optimal site must be in the central residential interval in L_1, which is $[\,[N/4], [N/4]+1\,]$. By symmetry, the site in the right half is obtained.

Next, consider case (iii).[22] Since it is always suboptimal to have both sites in the central region, we begin by removing the household residing in the central region (denoted i_m) and denoting $L_2 \equiv [0, [N/2]]$

[22] The arguments below are valid because we restrict our attention to allocations with cost-minimizing travel assignments.

which must contain one of the two sites. There are now two subcases to investigate. First, if L_2 contains an even number of households (i.e., **[**N/2**]** is even), by Proposition 1 (i), the ETIP optimal public facility location must be at the geographic center of L_2. Then we add back the middle household i_m. By randomization, only half of that household involves commuting to the public facility site in L_2, we claim that adding back i_m would not change the optimal public facility location. Suppose not, say, the facility site shifts toward i_m (it is trivial that a shift away from i_m can never be optimal). In this case, the reduction of the commuting cost incurred by i_m is half of the increased commuting cost by household 1, thus leading to a higher total commuting cost and contradicting the optimality condition. Therefore, the optimal location is at η = **[**N/2**]**/2. Second, if L_2 contains an odd number of households (i.e., **[**N/2**]** is odd), from case (ii) above, the optimal location is the central interval of L_2, [**[**N/4**]**, **[**N/4**]**+1]. Of course, when we add back i_m, only the upper bound of this interval is optimal, i.e., η = **[**N/4**]**+1. Notice that by the same arguments as above, the optimal allocation would not shift over to the adjacent interval toward i_m. Combining these two subcases, we obtain the ETIP optimal location in L_2 as η = **[**(N+1)/4**]** and that the analogous result holds for the right half as well.

Proof of Proposition 5:

It is sufficient to verify this by examining the special case of $N = 5$ where there are four possible IP feasible allocations with $n = 1, 2, 4$ and 5 which are consistent with cost-minimizing travel assignments and where all IP feasible allocations with $n = 3$ must have non-cost-minimizing travel assignments as a result of imposing the IP restriction. It is sufficient to show that an ETIP allocation with non-cost-minimizing travel assignments in the case of $n = 3$ is dominated by a non-IP feasible allocation, as ET can always be achieved by reallocation of the composite good under Condition K. More specifically, the IP feasible allocation with $n = 3$ involving lowest aggregate commuting cost is associated with a set of public facility sites H = {1, 5/2, 4}, a provision of the LPG at each site of $K_j = K_0/3$ with $K_0 = 5/\gamma$ and j = 1, 2, 3, a travel assignment $\theta(1, 1) = \theta(5, 4) = \theta(3, 5/2) = 1$, $\theta(2, 1) = \theta(4, 4) = 2/3$ and $\theta(2, 5/2) = \theta(4, 5/2) = 1/3$, and an allocation of the composite good $z = [E - K_0 - t/3]/5$. Assume that the unit commuting cost is sufficiently high, satisfying: $t \geq [3\alpha/\gamma][4\ln 2 - 5\ln(5/3)]$. Then, this allocation is inferior to a non-IP, ET feasible allocation with H = {1, 5/2, 4}, the public good provision of $K_j = K_0/3$, a cost-minimizing travel assignment θ(1, 1) = θ(2, 1) = θ(3, 5/2) = θ(4, 4) = θ(5, 4) = 1, and an allocation of the composite good $z_3 = \left[E - K_0 - 4(\alpha/\gamma)\ln 2\right]/5$ and $z_i = \left[E - K_0 + (\alpha/\gamma)\ln 2\right]/5$ (i = 1,2,4,5), because this alternative allocation results in $C(n,\mathrm{H}) = 0$ with a net per capita utility gain of $\{\gamma t/3 - \alpha\left[4\ln 2 - 5\ln(5/3)\right]\}/5$.

Next, we show that the ETIP feasible allocation with $n = 3$ proposed above weakly dominates other ETIP feasible allocations with $n \neq 3$. Denote the maximum indirect utility associated with these ETIP feasible allocations for given n as v_n. Defining $\bar{v} \equiv \gamma(E - K_0)/5 + \ln K_0 - \alpha \ln 5$, straightforward manipulations yield:

$$v_1 = \bar{v} - (4/5)\gamma t \tag{A-7a}$$

$$v_2 = \bar{v} - (1-\alpha)\ln 2 - (1/5)\gamma t \tag{A-7b}$$

$$v_3 = \bar{v} - (1-\alpha)\ln 3 - (1/15)\gamma t \tag{A-7c}$$

$$v_4 = \bar{v} - (1-\alpha)\ln 4 \tag{A-7d}$$

$$v_5 = \bar{v} - (1-\alpha)\ln 5 \tag{A-7e}$$

It is obvious from (A-7d) and (A-7e) that $v_4 > v_5$. Consider,

$$\frac{15(1-\alpha)}{2\gamma}\ln\left(\frac{3}{2}\right) \le t \le \frac{15(1-\alpha)}{\gamma}\ln\left(\frac{4}{3}\right) \tag{A-8}$$

From (A-7b), (A-7c) and (A-7d) and given $t > 0$ and $\alpha < 1$, the first inequality of (A-8) implies that $v_3 \geq v_2$ whereas the second inequality of (A-8) implies $v_3 \geq v_4$. Under (A-8), it must be true that $t \geq \left[5(1-\alpha)/(3\gamma)\right]\ln 2$ and hence we can show from (A-7a) and (A-7b) that $v_2 \geq v_1$. That is, condition (A-8) is sufficient to ensure the indirect utility ordering: $v_3 \geq \max\,\{v_1, v_2, v_4, v_5\}$. Since the non-IP, ET feasible allocation proposed above strictly dominates the ETIP feasible allocation achieving a level of indirect utility v_3, this non-IP, ET feasible allocation Pareto dominates any ETIP feasible allocations and hence any ETIP optima if the following condition is met: $\max\left\{\left(\frac{3\alpha}{\gamma}\right)\left[4\ln 2 - 5\ln\left(\frac{5}{3}\right)\right], \frac{15(1-\alpha)}{2\gamma}\ln\left(\frac{3}{2}\right)\right\} \le t \le \frac{15(1-\alpha)}{\gamma}\ln\left(\frac{4}{3}\right)$.

An Alternative Example – the Case of a Cobb-Douglas Utility Function:

Assume instead that the utility function of the households takes the Cobb-Douglas form, as in the conventional literature where K is fixed exogenously:

$$u(z,G(n))=z^{\gamma}[G(n)]^{1-\gamma}, \tag{A-9}$$

where $\gamma \in (0, 1)$ with $1-\gamma$ capturing the household valuation of the public good. The exogeneity of K is required since Condition K is no longer met under the Cobb-Douglas utility function specification. Consider $N = 3$. Utilizing Lemmas 2-4, we can combine (3), (7) and (A-9) to yield:

$$v^{C}=v^{N}=\frac{K^{1-\gamma}[(E-K)-t]^{\gamma}}{3^{\alpha(1-\gamma)+\gamma}} \tag{A-10a}$$

$$v^{D}=\frac{K^{1-\gamma}(E-K)^{\gamma}}{2^{(1-\alpha)(1-\gamma)}3^{\alpha(1-\gamma)+\gamma}}. \tag{A-10b}$$

where $C^{C}=C^{N}=t$ and $C^{D}=0$. Define:

$$B(\alpha,\gamma)\equiv 1-\left(\frac{1}{2}\right)^{(1-\alpha)(\frac{1-\gamma}{\gamma})} \quad \in (0,1) \tag{A-11}$$

which depends positively on the degree of non-exclusiveness ($1-\alpha$) and each household's valuation of the public good ($1-\gamma$). We now prove:

Proposition 2′: (ETIP optimal location of public facility with $N=3$) *Assume (A2) with $\alpha < 1$, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (A-9). When $N = 3$, an ETIP optimal public facility configuration satisfies the following properties:*

(i) if $B(\alpha,\gamma)(E - K) < t$, then an ETIP optimal public facility configuration is always dispersed;

(ii) if $B(\alpha,\gamma)(E - K) > t$, then an ETIP optimal public facility configuration is always concentrated, either centralized or non-centralized.

When $B(\alpha,\gamma)(E - K) = t$, an ETIP optimal public facility configuration may be dispersed or concentrated.

Proof: Comparing (A-10a) and (A-10b), one gets:

which completes the proof. Q.E.D.

$$v^C = v^N \gtrless v^D \Leftrightarrow B(\alpha,\gamma)(E-K) \gtrless t, \tag{A-12}$$

From (A-11), $B(\alpha,\gamma)$ depends positively on the degree of non-exclusiveness and the household valuation of the public good.

We next remind the reader that the characterization of the general case of $N > 3$ with multiple public facility sites (i.e., the properties in Lemma 5 and Proposition 4) remain valid under the Cobb-Douglas utility function specification, so long as K is exogenously fixed.

With regard to non-IP LPG provision, we can also show that Proposition 5 still holds for Cobb-Douglas utility if public good provision is fixed exogenously. The modification of the argument in the text is straightforward.