TRADE,
EXPROPRIATION
AND ALLOCATION

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Abstract

Allocation rules map preference profiles into allocations, whereas trading rules map preference profiles and allocations into allocations. It is shown that no allocation rule can derive from a trading rule based on voluntary trade and satisfying a weak efficiency condition. If the trading rule allows compulsory trade then the only allocation rules that can derive from a trading rule satisfying certain additional mild conditions are those having a hierarchy of dictators. These results contribute to accentuate the difference between centralized and decentralized allocation mechanisms.

Key words: Allocation rules, Hierarchy of dictators, Trading rules, Compulsory trade.

JEL Classification: D61, D71

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1. Introduction

Arguably, the analysis of allocation problems constitutes the task that has contributed most to shape economic theory. Solutions to allocation problems range between two extreme solutions, the private and the public one. The private solution (the "market solution") relies on mechanisms of decentralized allocation: agents are given property rights over the objects to be allocated and voluntary trade generates the final allocation. The public solution (the "State solution") relies on mechanisms of centralized allocation: there is no interaction among individuals determining the final allocation but instead some collective authority has the power to choose and implement the final allocation.

This note is concerned with the extent to which a centralized allocation mechanism can replicate the outcome of a decentralized allocation mechanism. Since a decentralized mechanism is typically more complex than a centralized one, the possibility that the latter could be a perfect substitute for the former is at least worth considering from a metaeconomic point of view: in principle, if two mechanisms lead to the same result, the simpler one should be preferable. In this respect, it is odd that textbooks resort to a centralized allocation mechanism (the Walrasian auctioneer) to justify the outcomes (prices and quantities) generated by the reference decentralized allocation mechanism (the perfectly competitive market).

The model in which the connection between centralized and decentralized allocation mechanisms is explored is taken from Shapley and Scarf (1974, pp. 24-25). Their model represents the essential elements to analyze exchange and allocation: agents, indivisible objects and the agents' preferences over the objects. Several recent contributions to the analysis of the allocation of indivisible objects rely on this model; see Svensson (1994, 1999), Pápai (2000, 2001, 2003) and Ehlers, Klaus and Pápai (2002).

At a conceptual level, it is presumed that the existence of property rights over the objects to be allocated is the basic divide between centralized and decentralized mechanisms. In view of this, allocation rules are assumed to represent centralized allocation mechanisms, as such rules map profiles of the agents' preferences over the objects into allocations, without taking explicitly into account the existence of property rights over the objects. On the other hand, trading rules are supposed to represent decentralized allocation mechanisms, because a trading rule maps preference profiles and allocations (allocation that could be interpreted as expressing property rights) into allocations. In this framework, the question of whether a centralized mechanism can mimic a decentralized one becomes, roughly speaking, the question of whether (and, if so, under which conditions) an allocation rule can simulate a trading rule.

The note presents two results, Propositions 3.2 and 4.4. The first one expresses the impossibility of having an allocation rule derive or simulate a trading rule in which trade is voluntary. Allowing compulsory exchange in a trading rule leads to the second result, that shows that the only allocation rules that can derive from a trading rule (satisfying certain additional mild conditions) are those having a hierarchy of dictators, that is, there is a fixed ordering of the set of agents such that an agent chooses the most preferred object not already chosen by the previous agents in the ordering. The final section of the note suggests an interpretation of the two results.

2. Framework

Let $N = \{1, \dots, n\}$ be a non-empty finite set whose $n \geq 2$ elements represent agents and A a non-empty finite set, with also n elements, representing objects, tasks or anything that could be assigned to the agents. An allocation is a bijection $\alpha : N \longrightarrow A$ that assigns a different object to each agent. Denote by A^* the set of all allocations and, for $i \in N$, let α_i stand for $\alpha(i)$. A preference on A is identified with a sequence $(x_1, \dots, x_n)$ of members of A such that $\{x_1, \dots, x_n\} = A$, the interpretation being that x_s is strictly preferred to x_t if, and only if, $s < t$. The fact that x is strictly preferred to y in preference p is expressed as $x\ p\ y$. For preference $p = (x_1, \dots, x_n)$ and $k \in \{1, \dots, n\}$, define ${}^k p := x_k$ to be the kth most preferred object according to p. Denote by L the set of all preferences on A and by L^n the set of all preference profiles. For $i \in N$, $Q_i \in L$ and $P \in L^n$, (Q_i, P_{-i}) designates the preference profile R such that $R_i = Q_i$ and, for $j \in N \backslash \{i\}$, $R_j = P_j$.

Definition 2.1. An <u>allocation rule</u> is a mapping $f : L^n \longrightarrow A^*$.

An allocation rule determines, from the agents' preferences on A represented by a preference profile, how the n objects in A are allocated among the n agents. The object $i \in N$ receives under allocation rule f and preference profile $P \in L^n$ is denoted by $f_i(P)$.

Definition 2.2. Allocation rule f has a <u>hierarchy of dictators</u> if there is a bijection $\pi : N \longrightarrow N$ such that, for all $P \in L^n$: (i) $f_{\pi(1)}(P) = {}^1P_{\pi(1)}$; and (ii) for all $i \in \{2, \dots, n\}$, $f_{\pi(i)}(P) = {}^kP_{\pi(i)}$, where k is the smallest $r \in \{1, \dots, n\}$ such that ${}^rP_{\pi(i)} \in A \backslash \{f_{\pi(1)}(P), \dots, f_{\pi(i-1)}(P)\}$, so that $f_{\pi(i)}(P)$ is the first member in the preference $P_{\pi(i)}$ belonging to the set $A \backslash \{f_{\pi(1)}(P), \dots, f_{\pi(i-1)}(P)\}$.

An allocation rule has a hierarchy of dictators if there exists a fixed ranking $(i_1, i_2, i_3, \dots, i_n)$ of the n agents such that, for every preference profile P: the object x_1 the first agent i_1 in the ranking receives is his most preferred object ${}^1P_{i_1}$; the second agent i_2 receives the most preferred object x_2 on the set of remaining objects $A\backslash\{x_1\}$ according to P_{i_2}; i_3 receives his most preferred object on $A\backslash\{x_1, x_2\}$ according to P_{i_3}; and so on.

Definition 2.3. A trading rule is a mapping $F : L^n \times A^* \longrightarrow A^*$. Denoting by $F_i(P, \alpha)$ the object $i \in N$ is assigned in allocation $F(P, \alpha)$, a bilateral trading rule is a trading rule $F : L^n \times A^* \longrightarrow A^*$ such that, for all $(P, \alpha) \in L^n \times A^*$ and $i \in N$, if $F_i(P, \alpha) \neq \alpha_i$ then there is $j \in N\backslash\{i\}$ such that $F_i(P, \alpha) = \alpha_j$ and $F_j(P, \alpha) = \alpha_i$.

A trading rule represents a mechanism that, given the agents' preferences P and some initial allocation α of the objects among the agents, determines a new allocation $F(P, \alpha)$. The interpretation is that agents own the corresponding object in the initial allocation and that the final allocation is obtained through some exchange process. A bilateral trading rule makes this process more specific by forcing trade to be based on bilateral exchange: if some agent i receives a new object from another agent j then i and j are permuting their objects. For a bilateral trading rule F, the allocation $F(P, \alpha)$ does not always represent the final allocation reached when agents trade with preference profile P and initial allocation α, as it may be that further exchange takes place with preference profile P and initial allocation $F(P, \alpha)$. In fact, $F(P, \alpha)$, $F(P, F(P, \alpha))$, $F(P, F(P, F(P, \alpha)))$, … would represent the trading sequence induced by F starting from (P, α). It is therefore convenient to associate with a bilateral trading rule F another summarizing trading rule that associates with each (P, α) the first allocation β in the above sequence at which trade stops, namely, $F(P, \beta) = \beta$.

Definition 2.4. For bilateral trading rule F, all $P \in L^n$ and $\alpha \in A^*$, define $F^1(P, \alpha) := F(P, \alpha)$ and, for $t \geq 2$, $F^t(P, \alpha) := F(P, F^{t-1}(P, \alpha))$. The summarizing trading rule associated with a bilateral trading rule $F : L^n \times A^* \longrightarrow A^*$ is the trading rule $\overline{F} : L^n \times A^* \longrightarrow A^*$ such that $\overline{F}(P, \alpha) := \beta$ if, and only if, there exists $t \geq 1$ such that $F^t(P, \alpha) = F^{t+1}(P, \alpha) = \beta$ and, when $t \geq 2$, for all $r \in \{1, 2, \dots, t-1\}$, $F^r(P, \alpha) \neq F^{r+1}(P, \alpha)$.

The mapping $\overline{F}$ is a trading rule summarizing the bilateral trading rule F, so that $\overline{F}(P, \alpha)$ is the allocation $F^t(P, \alpha)$ such that t is the smallest r with $F(P, F^r(P, \alpha)) = F^r(P, \alpha)$. The allocation $\overline{F}(P, \alpha)$ is obtained from the first trading rule F^t in the sequence $(F^1(P, \alpha), F^2(P, \alpha), F^3(P, \alpha), \dots)$ having a fixed point.

3. First result

Trading rules are chosen to represent decentralized allocation mechanisms, since the initial allocation can be interpreted as the specification of the agents' property rights over the objects. On the other hand, as property rights play no direct role in allocation rules, they will be chosen as representations of centralized allocation mechanisms. As a way to relate mechanisms of centralized allocation with mechanisms of decentralized allocation, the aim of this note is to state a correspondence result between bilateral trading rules and allocation rules. Specifically, the motivating questions are two: (i) when can an allocation rule accurately replicate the outcome of a trading rule?; and (ii) which types of allocation rules are successful in this replication?

In relation to the first question, the problem is to make precise in which sense an allocation rule can be seen as reproducing the outcome of a trading rule. The difficulty lies in the fact that an allocation rule associates a unique allocation with a preference profile, whereas a trading rule could associate, depending on the initial allocation, several allocations with a preference profile.

Definition 3.1. Allocation rule f derives from a bilateral trading rule F if, for all $P \in L^n$, $i \in N$ and $Q_i \in L$, $f(Q_i, P_{-i}) = \overline{F}((Q_i, P_{-i}), f(P))$.

Definition 3.1 suggests a possible way of inferring that an allocation rule is mimicking how a certain bilateral trading rule operates. The presumption is that the allocation rule implicitly assigns property rights to the agents. This presumption is justified by interpreting the allocation rule in a dynamic context. By way of illustration, consider profile P, allocation rule f and the corresponding allocation $f(P)$. Suppose that some agent i changes his preference from P_i to Q_i. This leads to a situation in which the preference profile is (Q_i, P_{-i}) and in which $f(P)$ could be considered a starting allocation from which the final allocation $f(Q_i, P_{-i})$ is obtained. Therefore, if there is some binary trading rule F such that $f(Q_i, P_{-i})$ agrees with the terminal allocation $\overline{F}((Q_i, P_{-i}), f(P))$ reached by applying the trading rule F then it could be interpreted that f has determined $f(Q_i, P_{-i})$ by considering the final result in the trading process that F embodies. If this occurs for every preference profile P, every agent i, every preference Q_i and for the same F then it could be inferred that f allocates by resorting to F. In this respect, f would be reproducing the outcomes of F (or, more specifically, $\overline{F}$). Proposition 3.2 will next show that it is impossible for an allocation rule to operate in this way when the trading rule relies on voluntary trade. The assumptions of this result are stated below.

E0. For all $P \in L^n$, $i \in N$ and $\alpha \in A^*$, if ${}^1P_i = \alpha_i$ then $F_i(P, \alpha) = \alpha_i$.

E0 expresses a basic implication of free and voluntary trade: if an agent already has his most preferred object then he will not participate in any exchange and will therefore retain his initial object. For $P \in L^n$, let $\alpha^P \in A^*$ be such that, for all $i \in N$, $\alpha^P{}_i = {}^1P_i$.

E1. For all $P \in L^n$ and $\alpha \in A^*$, if, for all $i \in N$ and $j \in N \backslash \{i\}$, ${}^1P_i \neq {}^1P_j$ then $\bar{F}(P, \alpha) = \alpha^P$.

E1 states another likely implication of free and voluntary trade: if all agents have a different object as the most preferred one then, no matter the initial allocation, the trading process embodied in the binary trading rule must converge to the allocation in which every agent obtains his most preferred object. Observe that such an allocation turns out to be the only Pareto efficient allocation given the preference profile ($\alpha \in A^*$ is Pareto efficient given $P \in L^n$ if, for all $\beta \in A^*$ and $i \in N$, $\beta_i \; P_i \; \alpha_i$ implies that, for some $j \in N \backslash \{i\}$, $\alpha_j \; P_j \; \beta_j$).

Proposition 3.2. There is no allocation rule f that derives from a binary trading rule F satisfying E0 and E1.

Proof. Suppose not: f derives from a trading rule F that satisfies E0 and E1. Choose $x \in A$, $y \in A \backslash \{x\}$, $i \in N$ and $j \in N \backslash \{i\}$. With $A \backslash \{x, y\} = \{x_1, x_2, \ldots, x_{n-2}\}$ and $N \backslash \{i, j\} = \{i_1, i_2, \ldots, i_{n-2}\}$, consider any $P \in L^n$ such that ${}^1P_i = x$, ${}^1P_j = y$ and, for $r \in \{1, \ldots, n-2\}$, ${}^1P_{i_r} = x_r$. By E1, for all $\alpha \in A^*$, $\bar{F}(P, \alpha) = \alpha^P =: \beta$. Given this and the fact that f derives from F, $f(P) = \beta$. Let $Q_i \in L$ satisfy ${}^1Q_i = y$. As f derives from F, $f(Q_i, P_{-i}) = \bar{F}((Q_i, P_{-i}), f(P)) = \bar{F}((Q_i, P_{-i}), \beta)$. By E0, for all $k \in N \backslash \{i\}$, $F_k((Q_i, P_{-i}), \beta) = \beta_k$. Therefore, $F_i((Q_i, P_{-i}), \beta) = \beta_i$ and $F((Q_i, P_{-i}), \beta) = \beta$. In view of this, $\bar{F}((Q_i, P_{-i}), \beta) = \beta$ and $f(Q_i, P_{-i}) = \beta$.

Let $Q_j \in L$ satisfy ${}^1Q_j = x$ and $R := (Q_j, P_{-j})$. As f derives from F, $f(R) = \bar{F}(R, f(P)) = \bar{F}(R, \beta)$. By E0, for all $k \in N \backslash \{j\}$, $F_k(R, \beta) = \beta_k$. Consequently, $F_j(R, \beta) = \beta_j$ and $F(R, \beta) = \beta$. As a result, $\bar{F}(R, \beta) = \beta$ and $f(R) = \beta$. With $S := (Q_i, R_{-i})$, by E1, for all $\alpha \in A^*$, $\bar{F}(S, \alpha) = \alpha^S =: \gamma$. In view of this, by the assumption that f derives from F, $f(S) = \gamma$. Observe that $\gamma \neq \beta$. In fact, γ differs from β only in that $\gamma_i = \beta_j = y$ and $\gamma_j = \beta_i = x$. Since f derives from F, $f(P_j, S_{-j}) = \bar{F}((P_j, S_{-j}), f(S)) = \bar{F}((P_j, S_{-j}), \gamma)$. By E0, for all $k \in N \backslash \{j\}$, $F_k((P_j, S_{-j}), \gamma) = \gamma_k$. Thus, $F_j((P_j, S_{-j}), \gamma) = \gamma_j$ and $F((P_j, S_{-j}), \gamma) = \gamma$. Accordingly, $\bar{F}((P_j, S_{-j}), \gamma) = \gamma$ and $f(P_j, S_{-j}) = \gamma$. But $(P_j, S_{-j}) = (Q_i, P_{-i})$, so $f(P_j, S_{-j}) = f(Q_i, P_{-i})$ must be case. Nevertheless, $f(Q_i, P_{-i}) = \beta$, $f(P_j, S_{-j}) = \gamma$ and $\beta \neq \gamma$: contradiction.□

The problem causing the impossibility can be better illustrated in the case $n = 2$. With $N = \{1, 2\}$, $A = \{x, y\}$ and f deriving from some F satisfying E0 and E1, let xy represent the preference (x, y) in which x is preferred to y and yx represent the reverse preference. Consider the preference profile (xy, yx) in which 1 prefers x to y and 2 prefers y to x. By E1, $f(xy, yx) = (x, y)$, namely, 1 obtains x and 2 obtains y. If, starting from this situation, 1 reverses his preference then E0 ensures that 2 retains y. Thus, $f(yx, yx) = (x, y)$.

If, starting from the original situation (xy, yx) and $f(xy, yx)$, 2 reverses his preference then E0 ensures that 1 retains x, so that $f(xy, xy) = (x, y)$. If, starting from this situation, it is 1 who reverses his preference then, by E1, $f(yx, xy) = (y, x)$. Finally, if, starting from this situation, 2 reverses his preference then, by E0, 1 retains y and $f(yx, yx) = (y, x)$: contradiction.

The source of this contradiction is that two ways of reaching the preference profile (yx, yx) lead to two different allocations. The allocation rule therefore appears to generate two conflicting claims over y under preference profile (yx, yx): reaching (yx, yx) through (xy, yx) grants the right to obtain y to agent 2; reaching (yx, yx) through (yx, xy), grants it to agent 1. Since y must be allocated to just one agent in (yx, yx), it seems that this agent should be given the right to expropriate y when necessary in order to preserve the consistency of the allocation rule. The next section shows that, by allowing this sort of compulsory trade in a binary trading rule, it is possible to have allocation rules that derive from binary trading rules. Nonetheless, though the impossibility is removed, it also follows that a very specific type of allocation rule is consistent with this demand.

4. Second result

Dropping E0 is the strategy followed to try to escape from the impossibility that Proposition 3.2 expresses. This means that some agent could “expropriate” another agent’s object despite the fact that the second agent is not willing to part with his object. As this possibility threatens voluntary trade, it will be restricted as much as possible. In particular, an agent will be able to expropriate, at most, another agent (if agent i can expropriate everybody then i is a dictator, in the sense that i always obtains his most preferred object). The following definition formalizes these considerations.

Definition 4.1. A bilateral trading rule $F : L^n \times A^* \longrightarrow A^*$ embodies an expropriation function $\varepsilon : N \longrightarrow N$ if, for all $P \in L^n$, $i \in N$, $j \in N$ and $\alpha \in A^*$, $\varepsilon(i) = j$ implies that it is not the case that $\overline{F}_j(P, \alpha)\ P_i\ \overline{F}_i(P, \alpha)$.

Definition 4.1 presumes that a certain function ε specifies whether an agent can expropriate another agent: if $\varepsilon(i) = i$ then i can expropriate no other agent; if $\varepsilon(i) = j \neq i$ then i can expropriate j. Hence, the no expropriation case dealt with in Section 3 corresponds to the identity expropriation function: for all $i \in N$, $\varepsilon(i)$. By Definition 4.1, the basic implication of $\varepsilon(i) = j$ is that, at the end of the trading process, it cannot be that j obtains an object that i prefers to the object i himself obtains, for in that case i would expropriate j the object j receives. As for the direct assumptions on a binary trading rule F, E1 from Section 3 is restated next in an equivalent form.

E1. For all $P \in L^n$ and $\alpha \in A^*$, if $\{{}^1P_1, \dots , {}^1P_n\} = A$ then $\bar{F}(P, \alpha) = \alpha^P$.

E2. For all $P \in L^n$, $i \in N$ and $\alpha \in A^*$, if (a) for all $j \in N$, $\alpha_j \, P_i \, \alpha_i$ implies $F_j(P, \alpha) = \alpha_j$ and (b) for all $j \in N$, $\alpha_i \, P_j \, \alpha_j$ implies $\varepsilon(j) \neq i$, then $F_i(P, \alpha) = \alpha_i$.

E2 replaces E0 as a rule determining when an agent does not trade: agent i retains the object x he is initially owning if: (a) every agent j having an object that i prefers to x does not trade (so i does not have a chance to obtain a more preferred object); and (b) every agent j preferring the object i has to his own object cannot expropriate i (so i can be sure that no other agent has the incentive and the power to strip him of x).

Definition 4.2. An expropriation function $\varepsilon : N \longrightarrow N$ is hierarchical if there exists a ranking $(i_1, i_2, \dots , i_n)$ of the members of N such that $\varepsilon(i_n) = i_n$ and, for all $t \in \{1, \dots , n - 1\}$, $\varepsilon(i_t) = i_{t+1}$.

In a hierarchical expropriation function, the agents are linearly ordered so that an agent can expropriate the next one in the order, nobody can expropriate the first agent and the last one can expropriate nobody.

Lemma 4.3. If allocation rule f derives from a binary trading rule F that embodies an expropriation function ε and satisfies E1 and E2 then ε is hierarchical.

Proof. With $A = \{x_1, x_2, \dots , x_n\}$, let $P \in L^n$ satisfy, for all $i \in N$ and $r \in \{1, \dots , n\}$, ${}^rP_i = x_r$. As $f(P)$ is such that, for all $x \in A$ there is a unique $i \in N$ with $f_i(P) = x$, let $(i_1, i_2, \dots , i_n)$ be the ranking of the members of N such that, for $r \in \{1, \dots , n\}$, $f_{i_r}(P) = x_{i_r}$. The proof consists of showing that $\varepsilon(i_n) = i_n$ and, for all $t \in \{1, \dots , n - 1\}$, $\varepsilon(i_t) = i_{t+1}$.

As f derives from F, there is $\beta \in A^*$ such that $f(P) = \bar{F}(P, \beta) = \alpha$. Clearly, for all $s \in \{1, \dots, n-1\}$ and $t \in \{s+1, \dots, n\}$, $\bar{F}_{i_s}(P, \beta)\ P_{i_t}\ \bar{F}_{i_t}(P, \beta)$. It then follows from the fact that F embodies the expropriation function ε that

$$\text{for all } s \in \{1, \dots, n-1\} \text{ and } t \in \{s+1, \dots, n\},\ \varepsilon(i_t) \neq i_s. \tag{1}$$

Therefore, for all $i \in N \backslash \{i_n\}$, $\varepsilon(i_n) \neq i$ and, consequently, $\varepsilon(i_n) = i_n$. Choose next $i_k \in \{i_1, \dots, i_{n-1}\}$. The proof concludes by showing that $\varepsilon(i_k) = i_{k+1}$. To this end, assume $\varepsilon(i_k) \neq i_{k+1}$. Recalling that $f(P) = \alpha$, choose $Q \in L^n$ such that, for all $r \in \{1, \dots, n\}$, ${}^1Q_{i_r} = \alpha_{i_r}$. As f derives from F, there is $\beta \in A^*$ such that $f(Q) = \bar{F}(Q, \beta)$. By E1, for all $\gamma \in A^*$, $\bar{F}(Q, \gamma) = \gamma^Q = \alpha$. Consequently, $f(Q) = \alpha$. Choose $R_{i_k} \in L$ with ${}^1R_{i_k} = \alpha_{i_{k+1}}$ and ${}^2R_{i_k} = \alpha_{i_k}$ and $R_{i_{k+1}} \in L$ with ${}^1R_{i_{k+1}} = \alpha_{i_k}$ and ${}^2R_{i_{k+1}} = \alpha_{i_{k+1}}$.

Let $S := (R_{i_k}, Q_{-i_k})$. As f derives from F, $f(S) = \bar{F}(S, f(Q)) = \bar{F}(S, \alpha)$. Consider $F(S, \alpha)$. Given that, for every $i \in N \backslash \{i_k\}$, there is no $j \in N$ such that $\alpha_j\ S_i\ \alpha_i$, it follows that (a) in E2 holds for all $i \in N \backslash \{i_k\}$. Since, for every $i \in N \backslash \{i_{k+1}\}$, there is no $j \in N$ such that $\alpha_i\ S_j\ \alpha_j$, it follows that (b) in E2 holds for all $i \in N \backslash \{i_{k+1}\}$. With respect to i_{k+1}, observe that i_k is the only $j \in N$ such that $\alpha_{i_{k+1}}\ S_j\ \alpha_j$. By assumption, $\varepsilon(i_k) \neq i_{k+1}$. Thus, (b) in E2 also holds for $i = i_{k+1}$. Accordingly, by E2, for all $i \in N \backslash \{i_k\}$, $F_i(S, \alpha) = \alpha_i$. Given this, $i = i_k$ also satisfies (a) in E2. By E2, $F_{i_k}(S, \alpha) = \alpha_{i_k}$. In sum, $F(S, \alpha) = \alpha$. In view of this, $\bar{F}(S, \alpha) = \alpha$ and, hence, $f(S) = \alpha$.

With $T := (R_{i_{k+1}}, S_{-i_{k+1}}) = (Q_{i_k}, T_{-i_k})$, as f derives from F, $f(T) = \bar{F}(T, f(S)) = \bar{F}(T, \alpha)$. By E1, $\bar{F}(T, \alpha) = \alpha^T$, so $f(S) = \alpha^T =: \delta$. Observe that α and δ differ only in that $\delta_{i_k} = \alpha_{i_{k+1}}$ and $\delta_{i_{k+1}} = \alpha_{i_k}$. Consider finally $V := (R_{i_{k+1}}, Q_{-i_{k+1}})$. Since f derives from F, $f(V) = \bar{F}(V, f(T)) = \bar{F}(V, \delta)$ but also $f(V) = \bar{F}(V, f(Q)) = \bar{F}(V, \alpha)$. The proof concludes by deriving the contradiction $\bar{F}(V, \delta) \neq \bar{F}(V, \alpha)$.

In fact, for every $i \in N \backslash \{i_{k+1}\}$, there is no $j \in N$ such that $\alpha_j\ V_i\ \alpha_i$, so that (a) in E2 holds for all $i \in N \backslash \{i_{k+1}\}$. Moreover, for every $i \in N \backslash \{i_k\}$, there is no $j \in N$ such that $\alpha_i\ V_j\ \alpha_j$, so that (b) in E2 holds for all $i \in N \backslash \{i_k\}$. As for i_k, notice that i_{k+1} is the only $j \in N$ such that $\alpha_{i_k}\ V_j\ \alpha_j$. By (1), $\varepsilon(i_{k+1}) \neq i_k$. Therefore, (b) in E2 holds as well for $i = i_k$. In view of this, by E2, for all $i \in N \backslash \{i_{k+1}\}$, $F_i(V, \alpha) = \alpha_i$. This result ensures that $i = i_{k+1}$ satisfies (a) in E2, too. By E2, $F_{i_{k+1}}(V, \alpha) = \alpha_{i_{k+1}}$. Summing up, $F(V, \alpha) = \alpha$. It finally follows from $F(V, \alpha) = \alpha$ that $\bar{F}(V, \alpha) = \alpha$.

On the other hand, for every $i \in N \backslash \{i_k\}$, there is no $j \in N$ such that $\delta_j\ V_i\ \delta_i$, so that (a) in E2 holds for all $i \in N \backslash \{i_k\}$. In addition, for each $i \in N \backslash \{i_{k+1}\}$, there is no $j \in N$ such that

$\delta_i\ V_j\ \delta_j$, so that (b) in E2 holds for all $i \in N\backslash\{i_{k+1}\}$. Concerning i_{k+1}, note that i_k is the only $j \in N$ such that $\delta_{i_{k+1}}\ V_j\ \delta_j$. By assumption, $\varepsilon(i_k) \neq i_{k+1}$ and, as a consequence, (b) in E2 holds for $i = i_{k+1}$. Thus, by E2, for all $i \in N\backslash\{i_k\}$, $F_i(V, \delta) = \delta_i$. Given this, $i = i_k$ also satisfies (a) in E2. By E2, $F_{i_k}(V, \delta) = \delta_{i_k}$. All in all, $F(V, \delta) = \delta$. This implies $\overline{F}(V, \delta) = \delta \neq \alpha$: contradiction.□

Proposition 4.4. An allocation rule *f* derives from a binary trading rule *F* that embodies an expropriation function ε and satisfies E1 and E2 if, and only if, *f* has a hierarchy of dictators.

Proof. "⇐" If allocation rule *f* has a hierarchy of dictators $(i_1, \ldots, i_n)$, define the binary trading rule *F* as follows. For $P \in L^n$ and $\alpha \in A^*$, let *i* be the first member in the sequence $(i_1, \ldots, i_n)$ such that $\alpha_i \neq f_i(P)$, let $j \in N$ satisfy $\alpha_j = f_i(P)$ and let *k* be the immediate predecessor of *j* in the ranking $(i_1, \ldots, i_n)$. Then $F_k(P, \alpha) := \alpha_j$, $F_j(P, \alpha) := \alpha_k$ and, for all $r \in N\backslash\{j, k\}$, $F_r(P, \alpha) := \alpha_r$. This trading rule is binary and embodies the hierarchical expropriation function ε such that $\varepsilon(i_n) = i_n$ and, for all $t \in \{1, \ldots, n-1\}$, $\varepsilon(i_t) = i_{t+1}$.

Intuitively, $F(P, \alpha)$ identifies the first member in the hierarchy of dictators not having the object that he receives in the allocation determined by the hierarchy, identifies next the agent *j* who has that object and makes the immediate predecessor *k* of *j* in the hierarchy expropriate this object from *j*. This mechanism ensures that *i* will eventually receive the object he obtains in the corresponding hierarchical allocation. As a result, E1 holds and *f* derives from *F*. With respect to E2, it is plain that if the objects *i* prefers to his current object α_i have been already assigned by *F* in (P, α) and those agents *j* preferring *i*'s object to α_j cannot expropriate α_i from *i* then the above mechanism yields $F_i(P, \alpha) = \alpha_i$, which is the object that *i* receives in $f(P)$.

"⇒" By Lemma 4.3, there is a ranking $h := (i_1, \ldots, i_n)$ of the members of *N* such that $\varepsilon(i_n) = i_n$ and, for all $t \in \{1, \ldots, n-1\}$, $\varepsilon(i_t) = i_{t+1}$. Choose any $P \in L^n$. The proof amounts to showing that $f(P) = \alpha$, where α is the allocation determined by *h* when viewed as a hierarchy of dictators: $f_{i_1}(P) = {}^1P_{i_1}$ and, for all $t \in \{2, \ldots, n\}$, $f_{i_t}(P)$ is the first element of P_{i_t} in $A\backslash\{f_{i_1}(P), \ldots, f_{i_{t-1}}(P)\}$. Choose $Q \in L^n$ such that, for all $i \in N$, ${}^1Q_i = \alpha_i$. By the assumption that *f* derives from *F*, there is $\beta \in A^*$ with $f(Q) = \overline{F}(Q, \beta)$. By E1, for all $\gamma \in A^*$, $\overline{F}(Q, \gamma) = \gamma^Q = \alpha$. Therefore, $f(Q) = \alpha$.

Consider $R := (P_{i_1}, Q_{-i_1})$. Note that ${}^1P_{i_1} = {}^1Q_{i_1}$. Since *f* derives from *F*, $f(R) = \overline{F}(R, f(Q)) = \overline{F}(R, \alpha)$. Clearly, for every $i \in N$, there is no $j \in N$ such that $\alpha_j\ R_i\ \alpha_i$, so (a) in E2 holds for all $i \in N$. Furthermore, for every $i \in N$, there is no $j \in N$ such that $\alpha_i \; R_j \; \alpha_j$, so (b) in E2 holds for all $i \in N$. Thus, by E2, $F(R, \alpha) = \alpha$, which implies $\overline{F}(R, \alpha) = \alpha$. As a consequence, $f(R) = \alpha$.

Choose $i_k \in \{i_2, \dots , i_n\}$ and, arguing inductively, suppose that $f(P_{i_1}, \dots , P_{i_{k-1}}, Q_{i_k}, \dots , Q_{i_n}) = \alpha$. The proof concludes by showing that $f(P_{i_1}, \dots , P_{i_k}, Q_{i_{k+1}}, \dots , Q_{i_n}) = \alpha$. To this end, let $S := (P_{i_1}, \dots , P_{i_{k-1}}, Q_{i_k}, \dots , Q_{i_n})$ and $T := (P_{i_1}, \dots , P_{i_k}, Q_{i_{k+1}}, \dots , Q_{i_n})$. As f derives from F, $f(T) = \overline{F}(T, f(S)) = \overline{F}(T, \alpha)$, the last step by the induction hypothesis. Given that ${}^1P_{i_1} = \alpha_{i_1}$, (a) in E2 holds for $i = i_1$. The fact that, for all $j \in N$, $\varepsilon(j) \neq i_1$ guarantees that (b) in E2 holds for $i = i_1$. Hence, by E2, $F_{i_1}(T, \alpha) = \alpha_{i_1}$.

Choose $i_s \in \{i_2, \dots , i_k\}$ and, arguing inductively, suppose that, for all $i \in \{i_1, \dots , i_{s-1}\}$, $F_i(T, \alpha) = \alpha_i$. This and the way α has been defined imply that, for all $j \in N$ such that $\alpha_j \; T_{i_s} \; \alpha_{i_s}$, $F_j(T, \alpha) = \alpha_j$. This means that (a) in E2 holds for $i = i_s$. Besides, only members j of $N\backslash\{i_1, \dots , i_s\}$ can be such that $\alpha_{i_s} \; T_j \; \alpha_j$. But, being ε hierarchical, no such j satisfies $\varepsilon(j) = i_s$. That is, (b) in E2 holds for $i = i_s$. Thus, by E2, $F_{i_s}(T, \alpha) = \alpha_{i_s}$. It then follows by induction that, for all $i \in \{i_1, \dots , i_k\}$, $F_i(T, \alpha) = \alpha_i$.

As for $i \in N\backslash\{i_1, \dots , i_k\}$, it is plain that, owing to ${}^1Q_i = \alpha_i$, (a) in E2 holds for any such i. Moreover, (b) in E2 holds for any such i because, for $j \in N$, it is not the case that $\alpha_i \; T_j \; \alpha_j$. This is immediate for $j \in N\backslash\{i_1, \dots , i_k\}$, given that ${}^1Q_j = \alpha_j$. For $j \in \{i_1, \dots , i_k\}$, $\alpha_j \; T_j \; \alpha_i$ follows from the fact that j is before i in the hierarchy that determines the allocation α. Summing up, by E2, for all $i \in N\backslash\{i_1, \dots , i_k\}$, $F_i(T, \alpha) = \alpha_i$. The final conclusion is then $F(T, \alpha) = \alpha$.□

Proposition 4.4 is in line with several other results in which allocation is determined by a hierarchical structure when strategic considerations are taken into account; see, for instance, the theorems by Svensson (1999, p. 562) and Pápai (2000, p. 1425).

5. Concluding comments

Proposition 3.2 shows the impossibility of constructing an allocation rule (according to Definition 3.1) from an efficient (condition E1) trading rule based on voluntary trade (condition E0). Proposition 4.4 contributes to make precise how strong requiring an efficient allocation rule to derive from a binary trading rule is. On the one hand, by Proposition 3.2, if the efficiency condition E1 is retained, the binary trading rule must allow compulsory trade. By Lemma 4.3, even if the possibility of compulsory trade is restricted so that an agent can at most expropriate another agent, the binary trading rule is subject to a hierarchy of compulsory trade: agents form a chain in which an agent can only expropriate the agent coming immediately after him. On the other hand, by Proposition 4.4, the allocation rules that can derive from binary trading rules satisfying E1 and E2 distribute the power to determine the final allocation hierarchically. Paradoxically, the attempt to replicate what in principle is a decentralized allocation rule (represented by a bilateral trading rule) turns out to generate the arguably most centralized allocation rule, in which an agent always gets his most preferred object, a second agent always gets the most preferred object not already taken by the first agent, a third agent always gets the most preferred object not already taken by the first two agents and so on.

These results seem to suggest that there exists some sort of drastic divide between these two kinds of allocation mechanisms, in the sense that the degrees of freedom that can be associated with a decentralized allocation mechanism are not superfluous insofar as they make futile any attempt by a centralized allocation mechanism to, in general, replicate its outcomes. The decentralized mechanism must lost the sufficient degrees of freedom (in terms of allowing expropriation) for the centralized mechanism to be able to yield its outcomes.

As a matter of fact, Proposition 3.2 appears to indicate that property rights are the decisive degree of freedom establishing the divide. In a sense, Propositions 3.2 and 4.4 should not come as a surprise: if one wants a mechanism in which property rights are not a valuable input to reproduce the results of a mechanism in which property rights are valuable then the possibility of overcoming these rights must be allowed. Using expropriation functions to realize that possibility, the consistency of the allocation mechanism leads to a partition of the expropriation possibilities in the form of a hierarchy of expropriators: expropriation appears to be such a powerful and disturbing measure in a trading context that the power to expropriate has to be distributed in a non-conflicting way among the agents. Finally, as a by-product of the resulting hierarchical distribution of the expropriation power, the allocation of objects itself has to follow this hierarchical pattern.

The above results seem to suggest the following moral: an economic system's attempt to imitate the behaviour of a more complex economic system may force the first system to become simpler.

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