Market Mechanisms for Fair Division with Indivisible Objects and Money[*]

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Abstract

A fair division problem with indivisible objects and money consists of a set of agents, a set of objects, a value matrix which shows the value of each agent for each object and a money endowment to be shared among agents. Each agent has a quasi-linear utility function over objects and money. A solution to a fair division problem is an allocation which assigns an object and a money share to each agent. In this paper, we introduce a family of mechanisms, namely market mechanisms, which determine allocations for each fair division problem. We prove that a market mechanism converges to an envy-free, efficient, and individually rational solution for each fair division problem using the principles of a tâtonnement process. We formulate underdemanded objects, overdemanded objects and perfectly demanded objects to define the formal money adjustment process in the domain of fair division problems. We find some of the interesting mechanisms in this domain.

Keywords: Fair division problem, market mechanism, tâtonnement process.

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1 Introduction

The market mechanism is considered as the central notion in finding a competitive allocation in an exchange economy. In this paper, we propose a class of market mechanisms for "fair division problems with indivisible objects and money." A fair division problem with indivisible objects and money consists of a set of agents, a set of indivisible objects, a fixed amount of money endowment, and utility profiles of agents on objects and money. In a solution to the problem, each object shall be assigned to an agent and each agent will get a share from the money endowment. We assume that each agent's utility function is quasi-linear in money shares.

In real life, there are several applications of fair division problems. One example is parking space and benefit allocation at a workplace. In this problem, each employee shall get a parking space and a share from a fixed benefits package. Another example is allocation of a bequest consisting of houses and money so that each inheritor shall get a house and a share from the money. A third example is job allocation among a group of employees. In this problem, each employee shall be assigned a job and a money compensation. Another application is a room assignment-rent division problem. In this problem, a group of agents shall rent a house. Each agent shall get a room and pay a share of the rent of the house. In this problem, the money endowment is a negative amount, where as in the applications we mentioned above there is a positive money endowment.

A solution of a fair division problem is a matching which assigns each agent an object and a money distribution vector which attaches a money share for each object. Such a solution is called an allocation. An allocation mechanism finds an allocation for each fair division problem. In this paper, we propose a class of allocation mechanisms which use the principal ideas of the "tâtonnement process."

As it is well-known, tâtonnement process can be iteratively used to find a competitive allocation in an exchange economy starting from an arbitrary price vector. We keep the sum of the prices of goods constant instead of having a numéraire good. We first formulate "overdemand," "underdemand," and "perfect demand" in the domain of fair division problems. The dynamic mechanism we propose mimics the market mechanism starting from an initial money distribution vector. After we find the demand of each agent at the initial money distribution, we determine the set of underdemanded objects, the set of overdemanded objects, and the set of perfectly demanded objects using a well-known result in combinatorial optimization theory, Gallai (1963, 1964) -Edmonds(1965) Decomposition Lemma. We then apply the following money adjustment process: (i) money shares attached to the underdemanded objects are increased by an equal small amount; (ii) money shares attached to the overdemanded objects are decreased by an equal small amount; (iii) money shares attached to the perfectly demanded objects are changed by an equal small amount that is no larger than the increment for the underdemanded objects and no smaller than the decrement for the overdemanded objects. We choose the amount of changes such that the sum of money changes for all objects is equal to zero. This adjustment gives a new "feasible" money distribution vector such that the sum of money shares is equal to the money endowment. We determine the demands of all agents, the supplies of all objects, the set of overdemanded objects, the set of underdemanded objects, and the set of perfectly demanded objects at the new money distribution vector. Then we repeat the above money adjustment process. We iteratively continue adjusting money shares until there are no underdemanded objects left. We show that at this point, we can assign each agent a matching in her demand set and such a matching can be constructed by Edmonds' (1965) algorithm.

An allocation is "envy-free" (Foley, 1967) if nobody prefers the object and money share of another agent to her allocation share. An allocation is "efficient" if the summation of indirect utilities of all agents under this allocation is no smaller than the sum under any other allocation.[1] An allocation

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[1] Efficiency and Pareto efficiency are equivalent concepts in this domain of problems.

is "individually rational" if the indirect utility of no agent is less than the reservation utility, which is zero. We show that an outcome of a market mechanism is envy-free, efficient, and individually rational.

Since there are various applications of fair division problems, different market mechanisms can be used to solve different applications.

In a room assignment-rent division problem it is important to find an allocation with "non-positive money shares." If an allocation has positive money shares, it involves the compensation of an agent by the other renters. However, the other renters will be better off by keeping this agent out of their coalition and leaving her room empty. However, there may not exist envy-free allocations with non-positive money shares (Maskin, 1987). We propose a market mechanism to solve this problem whenever such an allocation exists. If the change of money shares of the perfectly demanded objects is equal to the increment of money shares of the underdemanded objects, then the induced market mechanism will find an envy-free allocation with non-positive money shares whenever such an allocation exists. We show this result by the equivalence of this particular market mechanism to the Abdulkadiroğlu, Sönmez and Ünver (2004) mechanism, which has the mentioned property.

On the other hand, in a bequest allocation problem it is important to find an allocation with "non-negative money shares." If an allocation has negative money shares, it involves taxation of an agent by the other inheritors. However, there may not be additional money owned by the inheritor to pay the tax share. We propose a market mechanism for solving this problem whenever such an allocation exists. If the change of money shares of the perfectly demanded objects is equal to the decrement of the money shares of the overdemanded objects, then the induced market mechanism will find an envy-free allocation with non-negative money shares whenever such an allocation exists.

Note that the outcome of the first mechanism above can be found using a standard Vickrey-type auction such as Demange, Gale, and Sotomayor's (1986): Find the buyer-optimal competitive price/money distribution and outcome using an auction for the induced auction market and then balance the budget by equally subsidizing/taxing owner of each object. The dual of this procedure can be used to find the outcome of the second mechanism (another algorithm for this mechanism is suggested by Aragones (1995)). However, the outcome of none of the other market mechanisms we introduce in this paper can be found using an auction already introduced in the literature. An interesting example of these is as follows: The two above mechanisms are "egalitarian" in the sense that they minimize the maximum money share and maximize the minimum money share among all envy-free allocations, respectively. We introduce a compromise between the two above mechanisms: we treat underdemanded and overdemanded objects symmetrically in money adjustments and we do not adjust the prices of perfectly demanded objects in the process. We refer to this mechanism as the "compromised egalitarian mechanism."

1.1 Literature Background

In the literature, there are many studies on fair division problems with indivisible objects and money. Studies including Svensson (1983), Quinzii (1984), Maskin (1987) and Alkan, Demange and Gale (1991) derive properties of envy-free allocations and Walrasian equilibrium in fair division problems. Tadenuma and Thomson (1991, 1995) and Svensson and Larsson (2002) present axiomatic approaches for fair division problems. Aragones (1995), Su (1999), Klijn (2000), Brams and Kilgour (2001), Haake, Raith and Su (2002), Potthoff (2002), Abdulkadiroğlu, Sönmez and Ünver (2004) (ASÜ, from now on) propose different allocation mechanisms for fair division problems. One of these papers is closely related to our current study.

In a recent paper, ASÜ formulated an allocation procedure based on the principles of the market mechanism. They propose a natural mechanism for a room assignment-rent division problem. A room assignment-rent division problem is a fair division problem with a negative amount of money endowment. They show that an informal tâtonnement process can be formalized in such a way that the induced market mechanism finds an envy-free allocation with a non-positive money distribution whenever possible. They only formulate the set of overdemanded objects. By decreasing the money share of the objects in this set by a small amount, and by increasing the money shares of the remaining objects by a small amount, they obtain a money adjustment process. This is the key to their market mechanism. We generalize this idea to generate a general class of mechanisms to solve various fair division problems. We introduce the set of underdemanded objects and the set of perfectly demanded objects using a well-known result in combinatorial optimization theory, known as Gallai (1963, 1964)-Edmonds (1965) Decomposition Lemma. We then propose the generalized money adjustment process that is outlined in the previous section. It turns out th'at the ASÜ mechanism is one of the mechanisms in this class.

The choice of the tâtonnement process does not usually have an effect on the choice of the competitive allocation in exchange economies. However, in the fair division problems, selection of the tâtonnement process bears additional significance. As we show in this paper, the choice of money updating method changes the properties of the outcome extensively. It follows from the definition of our mechanism, ASÜ Theorem 2 and the dual of this theorem that two of the mechanisms in our class find extreme envy-free allocations with the smallest of the money shares maximized or with the largest of the money shares minimized among all envy-free allocations.[2] There are various other market mechanisms, which can be used to find solutions to other applications such as the newly introduced "compromised egalitarian mechanism."

In the next section, we introduce our model.

2 Fair Division Problems

A **fair division problem** is a quadruple $\langle I, A, V, m\rangle$ where $I = \{i_1, \dots, i_n\}$ is a set of agents, $A = \{a_1, \dots, a_n\}$ is a set of objects, $V = [v_a^i]_{i\in I, a\in A}$ is a value matrix where $v_a^i \in \mathbb{R}$ denotes the value of object $a \in A$ for agent $i \in I$, and $m \in \mathbb{R}$ is the money endowment. Each agent $i \in I$ can use one and only one object.[3] Let $\mathbb{F}$ be the set of all fair division problems. We will fix a problem $\langle I, A, V, m\rangle \in \mathbb{F}$ for the rest of our analysis.

A **money distribution** $t = (t_{a_1}, ..., t_{a_n})$ is a list such that $\sum_{a\in A} t_a = m$ where **money share** $t_a \in \mathbb{R}$ shows the money share attached to object $a \in A$ under money distribution t. Let $\mathcal{T}$ be the set of money distributions.

The **utility** of each agent $i \in I$ is a function $u_i : A \times \mathbb{R} \to \mathbb{R}$ which is defined as

$$u_i(a, t_a) = v_a^i + t_a$$

for all objects $a \in A$ at all money shares $t_a \in \mathbb{R}$. Each agent has a reservation utility, which is equal to zero, denoting her outside options.

We shall assume that

$$\sum_{a\in A} v_a^i + m \geq 0$$

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[2] Alkan, Demange and Gale (1991) prove the existence of extreme allocations of this sort. In different domains of problems, there are extreme allocations and mechanisms of this sort as well. For example, in "two-sided matching markets" Gale and Shapley (1962) propose two mechanisms which find optimal stable matchings for each side of the market.

[3] In reality, there may be more agents than objects. In this case, we create $|I| - |A|$ new **dummy** objects, add these to set A, and make the number of objects equal to number of agents. We modify the value vector of each agent by adding value $v_a^i = 0$ (the reservation utility) for any dummy object a. In a solution (that is described below) of the problem, some agents will be assigned dummy objects, or no real objects, and they will be only compensated/taxed by money shares.

for each agent $i \in I$. This assumption is fairly standard in the literature.[4] It will ensure individual rationality of the mechanisms that we will introduce. There are two interpretations of this assumption: (i) Each agent thinks that the total disutility of the objects can be compensated by the money endowment. It is natural to assume that if an agent thinks it is not worth to obtain the objects then she will not be in this coalition of agents. (ii) If an agent would like to form a coalition with agents who are exact copies of her, then this coalition of agents will be willing to obtain these objects.

A **matching** $\mu = \{\{i_1, \mu_{i_1}\}, ..., \{i_n, \mu_{i_n}\}\}$ is a list of the assignments of objects to agents such that each object is assigned to one agent and each agent is assigned one object and component $\mu_i \in A$ denotes the assignment of agent $i \in I$ under matching μ. Let $\mathcal{M}$ denote the set of matchings. That is, for any $i \in I$, $\mu_i \in A$, and for any $\{i, j\} \subseteq I$, we have $\mu_i \neq \mu_j$.

A solution of the fair division problem is a matching - money distribution pair $(\mu, t) \in \mathcal{M} \times \mathcal{T}$. Such a pair is called an **allocation**. At an allocation $(\mu, t) \in \mathcal{M} \times \mathcal{T}$ each agent $i \in I$ obtains object μ_i and money share t_{μ_i}.

An **allocation mechanism** is a systematic procedure which finds a set of allocations for each fair division problem.

3 A Market Approach

We adopt a market approach in our analysis. The **demand** of each agent $i \in I$ is a correspondence $D_i : \mathcal{T} \twoheadrightarrow A$ that is defined as

$$D_i(t) = \left\{a \in A : u_i(a, t_a) \geq u_i(a', t_{a'}) \ \forall \ a' \in A\right\}$$

for all $t \in \mathcal{T}$. By definition, $D_i(t) \neq \emptyset$ for any $i \in I$ and $t \in \mathcal{T}$.

The **demand profile** at money distribution $t \in \mathcal{T}$ is defined as $D(t) = (D_i(t))_{i \in I}$.

The **supply** of each object $a \in A$ is a correspondence $S_a : \mathcal{T} \twoheadrightarrow I$ which is defined as

$$S_a(t) = \{i \in I : a \in D_i(t)\}$$

for all $t \in \mathcal{T}$.

The **supply profile** at money distribution $t \in \mathcal{T}$ is defined as $S(t) = (S_a(t))_{a \in A}$.

The **indirect utility** function of each agent $i \in I$, $\widetilde{u}_i : \mathcal{T} \to \mathbb{R}$, is defined as

$$\widetilde{u}_i(t) = \max_{a \in A} \ u_i(a, t_a)$$

for all $t \in \mathcal{T}$.

For any $t \in \mathcal{T}$, a matching $\mu \in \mathcal{M}(t)$ **clears the market** if $\mu_i \in D_i(t)$ for any $i \in I$.

In order to formulate our market approach, we need to define the objects in excess supply, the objects in excess demand, and the objects in perfect demand. This will be crucial for the definition of the "tatônnement" process.

3.1 Gallai-Edmonds Decomposition

We use a graph theoretic approach to define the notions of overdemand, underdemand and perfect demand.

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[4] See Su (1999), Brams and Kilgour (2001), Haake, Raith and Su (2002), and Abdulkadiroğlu, Sönmez and Ünver (2004).

We say that a set $\{i, a\}$ consisting of an agent $i \in I$ and an object $a \in A$ is a **link** at money distribution $t \in \mathcal{T}$ if and only if $a \in D_i(t)$. The **set of links** at a money distribution $t \in \mathcal{T}$ is a correspondence $L : \mathcal{T} \twoheadrightarrow 2^{I \cup A}$ such that

$$L(t) = \{\{i, a\} : i \in I,\ a \in A, \text{ and } a \in D_i(t)\}$$

for each $t \in \mathcal{T}$.

The **demand-supply graph** at money distribution $t \in \mathcal{T}$ is a pair $\langle I \cup A, L(t)\rangle$ where agents in I and objects in A are the nodes of the graph and links in $L(t)$ are the arcs of the graph. We denote the demand-supply graph at money distribution $t \in \mathcal{T}$ by $\mathcal{G}(t)$.

Fix $t \in \mathcal{T}$. A **market assignment at** t is a set of links $Q \subseteq L(t)$ such that any agent receives either no object or one object in her demand and any object is assigned to either nobody or one agent in its supply. That is, $\{i, a\} \in Q$ and $\{j, b\} \in Q \setminus \{\{i, a\}\}$ implies $i \neq j$ and $a \neq b$. Component Q_i is the assigned object of agent $i \in I$ under market assignment Q : if agent i is assigned an object under Q then $Q_i \in A$, and if agent i is not assigned any object under Q then $Q_i = \emptyset$. Let $\mathcal{Q}(t)$ be the set of market assignments at t. For any $x \in A \cup I$ and $Q \in \mathcal{Q}(t)$ we say that x is **unmatched under** Q if there is no $\{i, a\} \in Q$ such that $x \in \{i, a\}$. A market assignment $Q \in \mathcal{Q}(t)$ is **maximal** if for all $R \in \mathcal{Q}(t)$ we have $|Q| \geq |R|$. Let $\mathcal{M}(t)$ be the set of maximal market assignments at t. The following observations are trivial, and yet crucial for our market approach:

Observation 1: A market assignment $Q \in \mathcal{Q}(t)$ is a matching if and only if $|Q| = n$.

Observation 2: A matching $\mu \in \mathcal{M}$ clears the market at t if and only if it is a market assignment at t.

Since $|Q| \leq n$ for any $Q \in \mathcal{Q}(t)$, if a market assignment $Q \in \mathcal{Q}(t)$ is a matching then it is maximal.

Consider the following partitions $\{UD(t), OD(t), PD(t)\}$ of A and $\{US(t), OS(t), PS(t)\}$ of I defined at t:

Let $UD(t) \subseteq A$ denote the set of objects such that $a \in UD(t)$ implies that there is a maximal market assignment that leaves a unmatched.

Let $US(t) \subseteq I$ denote the set of agents such that $i \in US(t)$ implies that there is a maximal market assignment that leaves i unmatched.

Let $OD(t) \subseteq A$ denote the set of objects such that $a \in OD(t)$ implies that there exists an agent $i \in US(t)$ with $\{i, a\} \in L(t)$.

Let $OS(t) \subseteq I$ denote the set of agents such that $i \in OS(t)$ implies that there exists an object $a \in UD(t)$ with $\{i, a\} \in L(t)$.

Let $PD(t) = A \setminus (UD(t) \cup OD(t))$ and $PS(t) = I \setminus (US(t) \cup OS(t))$.

Set $UD(t)$ ($US(t)$) is the set of objects (agents) for each of which (whom) there is a maximal market assignment leaving it (her) unmatched at t. Set $OD(t)$ ($OS(t)$) is the set of objects (agents) each of which (whom) is matched under any maximal market assignment at t and has a link to an agent in $US(t)$ (an object in $UD(t)$). Set $PD(t)$ ($PS(t)$) is the set of objects (agents) each of which (whom) is matched under any maximal market assignment at t and does not have any link to agents in $US(t)$ (objects in $UD(t)$). These sets are crucial for the structure of maximal market assignments, which is well-studied in the combinatorial optimization literature. Gallai (1963, 1964) and Edmonds (1965)) Decomposition (GED) Lemma for bipartite graphs is about the structure of maximal market assignments.[5]

Gallai-Edmonds Decomposition (GED) Lemma for Bipartite Graphs: Let $t \in \mathcal{T}$. Let $Q \in \mathcal{M}(t)$ be a maximal market assignment at t.

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[5] See Bogomolnaia and Moulin (2004) and Roth, Sönmez and Ünver (2004) for use of the Gallai-Edmonds Decomposition Lemma in other allocation problems.

1. Every object in $OD(t)$ is assigned to an agent in $US(t)$ under Q.
2. Every agent in $OS(t)$ is assigned an object in $UD(t)$ under Q.
3. Every object in $PD(t)$ is assigned to an agent in $PS(t)$ and every agent in $PS(t)$ is assigned an object in $PD(t)$ under Q.[6]

Based on the GED Lemma, we refer to $OD(t)$ as the **set of overdemanded objects**, $UD(t)$ as the **set of underdemanded objects**, and $PD(t)$ as the **set of perfectly demanded objects** at money distribution t. Similarly, we refer to $OS(t)$ as the **set of oversupplied agents**, $US(t)$ as the **set of undersupplied agents**, and $PS(t)$ as the **set of perfectly supplied agents** at money distribution t.[7] We refer to the partitions $\{UD(t), OD(t), PD(t)\}$ of A and $\{US(t), OS(t), PS(t)\}$ of I as the **Gallai-Edmonds Decomposition (GED)** of the problem at t. The next result is a direct corollary of the GED Lemma and the definitions:

Corollary 1: Let $t \in \mathcal{T}$. We have

1. If $UD(t) \neq \emptyset$ then $|UD(t)| > |OS(t)|$, otherwise $OS(t) = \emptyset$.
2. If $US(t) \neq \emptyset$ then $|US(t)| > |OD(t)|$, otherwise $OD(t) = \emptyset$.
3. $|PD(t)| = |PS(t)|$.

The next lemma is crucial for our analysis and our understanding of the conditions of a market clearing money distribution:

Lemma 1: Let $t \in \mathcal{T}$ and $Q \in \mathcal{M}(t)$.

1. $|Q| = n$ if and only if $UD(t) = \emptyset$.
2. $|Q| = n$ if and only if $US(t) = \emptyset$.

Proofs of all results are given in Appendix B.

Sets $UD(t)$ and $US(t)$ can be constructed in polynomial time complexity in number of agents n using several well-known algorithms in combinatorial optimization literature. For example, **Edmonds'** (1965) **algorithm** finds a maximal market assignment Q in $O(n^3)$ complexity, and then construction of sets $UD(t), OD(t), PD(t)$ and $US(t), OS(t), PS(t)$ is straightforward. These are explained in Appendix B.

4 Market Mechanisms

A market mechanism incrementally increases the money shares of the underdemanded objects. It incrementally decreases the money shares of the overdemanded objects. It changes the money shares of the perfectly demanded objects by no greater than the increment for the underdemanded objects and no smaller than the decrement for the overdemanded objects such that the sum of the money shares of all objects is equal to m. We are ready to state a market mechanism "informally" as follows:

Step 0: Initially set the money share of each object $a \in A$ to $t_a^0 = \frac{m}{n}$ such that $t^0 = (\frac{m}{n}, \frac{m}{n}, ..., \frac{m}{n})$. Find a maximal market assignment Q^0.

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[6] See Bogomolnaia and Moulin (2004) for an alternative formulation of the GED Lemma for bipartite graphs.

[7] For alternative approaches in constructing overdemanded and oversupplied sets using Hall's (1935) Theorem, see Demange, Gale and Sotomayor (1986) and de Vries, Schummer and Vohra (2005).

(a) If $\left|Q^0\right| = n$ then Q^0 is a matching such that $Q_i^0 \in D_i(t^0)$ for each agent i by Observation 1 and $UD\left(t^0\right) = \emptyset$ by Lemma 1. We terminate the procedure and $\left(Q^0, t^0\right)$ is the outcome.
(b) If $\left|Q^0\right| < n$ then $UD\left(t^0\right) \neq \emptyset$ by Lemma 1 and we proceed to the next step.

In a general step s,

Step s: Construct GED at t^{s-1} using Q^{s-1}. Let $t^s \in \mathcal{T}$ be defined as $t_a^s = t_a^{s-1} + \alpha$ for any $a \in UD(t^{s-1})$, $t_a^s = t_a^{s-1} + \beta$ for any $a \in PD(t^{s-1})$, and $t_a^s = t_a^{s-1} + \gamma$ for any $a \in OD(t^{s-1})$ such that $\alpha \to 0$, $\beta \to 0$, $\gamma \to 0$ with
(i) $\alpha \geq \beta \geq \gamma$ and
(ii) $\left|UD(t^{s-1})\right| \alpha + \left|PD(t^{s-1})\right| \beta + \left|OD(t^{s-1})\right| \gamma = 0$.
Construct a maximal market assignment Q^s at t^s.
(a) If $|Q^s| = n$ then Q^s is a matching such that $Q_i^s \in D_i(t^s)$ for each agent i by Observation 1 and $UD\left(t^s\right) = \emptyset$ by Lemma 1. We terminate the procedure and (Q^s, t^s) is the outcome.
(b) If $|Q^s| < n$ then $UD\left(t^s\right) \neq \emptyset$ by Lemma 1 and we proceed to the next step.

The above money adjustment rule definition is in the flavor of an informal "tâtonnement" process in exchange economies. In the next subsection, we will make a formal definition of the money adjustment rule and be precise about the properties of increments (or decrements) α, β, γ.

4.1 Formalization of a Market Mechanism

We claim that the crucial money distribution levels in an informal tâtonnement process are those at which a new object joins the demand of an agent. Because that is only when the set of overdemanded objects, the set of underdemanded objects and the set of perfectly demanded objects can change. We prove this with a lemma.

Lemma 2: Let $\{t^s\}$ be the money distribution sequence of a market mechanism. If we have $D_i(t^{s+1}) \subseteq D_i(t^s)$ for all $i \in I$ for a step s, then $UD(t^{s+1}) = UD(t^s)$, $PD(t^{s+1}) = PD(t^s)$, and $OD(t^{s+1}) = OD(t^s)$.

This result shows that as long as new objects do not join the demands of agents, overdemand, underdemand and perfect demand will not change in a market mechanism. Note that even if some objects are dropped from demands of agents during this process, overdemand, underdemand and perfect demand will stay put.

We can directly find the next money distribution when a new object joins the demand of an agent. This new money distribution may possibly induce a change in overdemand, underdemand and perfect demand.

Let $t \in \mathcal{T}$ be a money distribution reached in the market mechanism. We continuously increase or decrease money shares. Let $t' \in \mathcal{T}$ be the first money distribution level reached after t where the demand of an agent includes a new object.

We will distinguish momentary rates α,β and γ and the discrete changes achieved in money distribution after a certain amount of money adjustments. Let $\alpha(t)$ be a discrete increment for the money shares of the underdemanded objects at money distribution t, $\beta(t)$ be a discrete increment or decrement for the money shares of the perfectly demanded objects at money distribution t, and $\gamma(t)$ be a discrete decrement for the money shares of the overdemanded objects at money distribution t such that

$$\begin{aligned} t'_a &= t_a + \alpha(t) && \forall\, a \in UD(t), \\ t'_a &= t_a + \beta(t) && \forall\, a \in PD(t), \text{ and} \\ t'_a &= t_a + \gamma(t) && \forall\, a \in OD(t). \end{aligned}$$

The agent who demands a new object at t' is necessarily a member of $US\,(t)$ or $PS(t)$. That is, because (i) each agent in $OS(t)$ demands an object in $UD(t)$ at money distribution t and the money shares of objects in $UD(t)$ are increasing uniformly at the highest rate, and (ii) utilities are quasi-linear in money. Three observations hold for the discrete changes $\alpha\,(t)$, $\beta\,(t)$, and $\gamma\,(t)$:

A new object can join the demand of an agent in $US\,(t)$ in two ways. Either this object is a perfectly demanded object at money distribution t and its money share is rising faster than money shares of the overdemanded objects or this object is an underdemanded object.

We define

$$x(t) = \begin{cases} \min_{i\in US(t)} \left(\tilde{u}_i\,(t) - \max_{a\in UD(t)} u_i(a, t_a)\right) & \text{if } UD(t) \neq \emptyset \text{ and } US\,(t) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$y(t) = \begin{cases} \min_{i\in US(t)} \left(\tilde{u}_i\,(t) - \max_{a\in PD(t)} u_i(a, t_a)\right) & \text{if } PD(t) \neq \emptyset \text{ and } US\,(t) \neq \emptyset \\ x(t) & \text{otherwise} \end{cases}.$$

Consider objects $a, b, c \in A$ such that $a \in UD(t)$, $b \in PD(t)$, and $c \in OD\,(t)$. The money share of a increases at the rate α, the money share of b increases at the rate β, and the money share of c increases at the rate γ initially at t until t' is reached.

- The money differential $t_a - t_c$ increases at the same momentary rate, $\alpha - \gamma$, for any pair of such objects until a new object joins the demand of an agent. This may happen when an agent in $US\,(t)$ demands an underdemanded object and $x(t)$ is the minimum differential for that to occur. Therefore, we need $\alpha\,(t) - \gamma\,(t) \leq x\,(t)$ to have the same momentary rates, α and γ, to use in our money adjustments between t and t' by Lemma 2.
- The money differential $t_b - t_c$ increases at the same momentary rate, $\beta - \gamma$, for any pair of such objects until a new object joins the demand of an agent. This may happen when an agent in $US\,(t)$ demands a perfectly demanded object and $y(t)$ is the minimum differential for that to occur. Therefore, we need $\beta\,(t) - \gamma\,(t) \leq y\,(t)$ to have the same momentary rates, β and γ, to use in our money adjustments between t and t' by Lemma 2.

A new object can join the demand of a member of $PS(t)$, if it is an underdemanded object and money shares of the underdemanded objects is rising faster than money shares of the perfectly demanded objects. We define

$$z(t) = \begin{cases} \min_{i\in PS(t)} \left(\tilde{u}_i\,(t) - \max_{a\in UD(t)} u_i(a, t_a)\right) & \text{if } UD(t) \neq \emptyset \text{ and } PS\,(t) \neq \emptyset \\ x(t) & \text{otherwise} \end{cases}.$$

Consider objects $a, b \in A$ such that $a \in UD(t)$, $b \in PD(t)$. The money share of a increases at the rate α, the money share of b increases at the rate β initially at t until t' is reached.

- The money differential $t_a - t_b$ increases at the same momentary rate, $\alpha - \beta$, for any pair of such objects until a new object joins the demand of an agent. This may happen when an agent in $PS\,(t)$ demands an underdemanded object and $z(t)$ is the minimum differential for that to occur. Therefore, we need $\alpha\,(t) - \beta\,(t) \leq z\,(t)$ to have the same momentary rates, α and β, to use in our money adjustments between t and t' by Lemma 2.

One of these three situations will occur before the others causing a new object to join the demand of an agent. Using this information we can construct a “discrete algorithm” to find the outcome of a market mechanism.

To summarize, we have

$$\begin{aligned} &\alpha(t) - \gamma(t) \leq x(t), \\ &\beta(t) - \gamma(t) \leq y(t), \text{ and} \\ &\alpha(t) - \beta(t) \leq z(t) \end{aligned} \tag{1}$$

and, since at money distribution t' a new object joins the demand of an agent, one of the above inequalities is binding.

Since t' is a money distribution level reached in a market mechanism, we have

$$\gamma(t) \leq \beta(t) \leq \alpha(t) \tag{2}$$

and

$$|UD(t)|\,\alpha(t) + |PD(t)|\,\beta(t) + |OD(t)|\,\gamma(t) = 0\ . \tag{3}$$

Note that these equations imply that $\alpha(t) > 0$ and $\gamma(t) < 0$ whenever $UD(t) \neq \emptyset$.

We are ready to introduce an iterative discrete algorithm that is used to compute the outcome of a market mechanism.

Step 0: Initially set the money share of each object to $\frac{m}{n}$. Let $t^0 = (\frac{m}{n}, \frac{m}{n}, ..., \frac{m}{n})$. Find a maximal market assignment $Q^0 \in \mathcal{M}(t^0)$ at t^0 using Edmonds' algorithm.
(a) If $|Q^0| = n$ then Q^0 is a matching such that $Q_i^0 \in D_i(t^0)$ for each agent i and $UD(t^0) = \emptyset$ by Lemma 1. We terminate the procedure and (Q^0, t^0) is the outcome.
(b) If $|Q^0| < n$ then $UD(t^0) \neq \emptyset$ by Lemma 1 and we proceed to the next step.

In a general step s,

Step s: Construct GE Decomposition at t^{s-1} using Q^{s-1}. Let $t^s \in \mathcal{T}$ be defined as $t_a^s = t_a^{s-1} + \alpha(t^{s-1})$ for all $a \in UD(t^{s-1})$, $t_a^s = t_a^{s-1} + \beta(t^{s-1})$ for all $a \in PD(t^{s-1})$, and $t_a^s = t_a^{s-1} + \gamma(t^{s-1})$ for all $a \in OD(t^{s-1})$. Find a maximal market assignment $Q^s \in \mathcal{M}(t^s)$ at t^s using Edmonds' algorithm.
(a) If $|Q^s| = n$ then Q^s is a matching such that $Q_i^s \in D_i(t^s)$ for each agent i and $UD(t^s) = \emptyset$ by Lemma 1. We terminate the procedure and (Q^s, t^s) is the outcome.
(b) If $|Q^s| < n$ then $UD(t^s) \neq \emptyset$ by Lemma 1 and we proceed to the next step.

We use the discrete algorithm outlined above for showing that a market mechanism converges to a market outcome and for computing the outcome of a market mechanism.

Note that one may uniquely define a market mechanism by choosing one of the functions α, β, and γ as a function of the remaining two. For instance, different choices of function β induce algorithms for different market mechanisms. Three interesting mechanisms are (i) the market mechanism with $\beta = \alpha$, (ii) the market mechanism with $\beta = 0$, and (iii) the market mechanism with $\beta = \gamma$. In the first mechanism, the money shares of the perfectly demanded objects increase at the same rate as the underdemanded objects. In the second mechanism, the money shares of the perfectly demanded objects are kept constant. In the third mechanism, the money shares of the perfectly demanded objects decrease at the same rate as the overdemanded objects. In the below example, we find functions α, β, and γ used in the algorithms for these market mechanisms.

Example 1: For each of these mechanisms, we find functions α, β, and γ using Equation System 1, Equation 2, and Equation 3:

- **Market mechanism with $\beta = \alpha$:**

$$\begin{aligned}\alpha(t) &= \frac{|OD(t)|}{n}\min\{x(t), y(t)\},\\ \gamma(t) &= -\frac{|UD(t)|+|PD(t)|}{n}\min\{x(t), y(t)\},\\ \beta(t) &= \frac{|OD(t)|}{n}\min\{x(t), y(t)\}\end{aligned}$$

  for all $t \in \mathcal{T}$.

- **Market mechanism with $\beta = 0$:**

$$\begin{aligned}\alpha(t) &= \min\{\frac{|OD(t)|}{|OD(t)|+|UD(t)|}x(t), \frac{|OD(t)|}{|UD(t)|}y(t), z(t)\},\\ \gamma(t) &= \max\{-\frac{|UD(t)|}{|OD(t)|+|UD(t)|}x(t), -y(t), -\frac{|UD(t)|}{|OD(t)|}z(t)\},\\ \beta(t) &= 0\end{aligned}$$

  for all $t \in \mathcal{T}$.

- **Market mechanism with $\beta = \gamma$:**

$$\begin{aligned}\alpha(t) &= \frac{|OD(t)|+|PD(t)|}{n}\min\{x(t), z(t)\},\\ \gamma(t) &= -\frac{|UD(t)|}{n}\min\{x(t), z(t)\},\\ \beta(t) &= -\frac{|UD(t)|}{n}\min\{x(t), z(t)\}\end{aligned}$$

  for all $t \in \mathcal{T}$.

◆

In Appendix C, we show how the discrete algorithm can be used to find the outcome of a market mechanism for a fair division problem with an example.

4.2 Convergence of a Market Mechanism

We will prove that a market mechanism converges to a market outcome. We will use the discrete algorithm to prove this result. The following proposition shows that summation of indirect utilities monotonically decreases in the discrete algorithm of a market mechanism.

Proposition 1: Let $\{t^s\}$ be the money distribution sequence of the discrete algorithm of a market mechanism. For every step $s \geq 0$ with $UD(t^s) \neq \emptyset$ we have $\sum_{i\in I}\tilde{u}_i(t^s) \geq \sum_{i\in I}\tilde{u}_i(t^{s+1}) + \alpha(t^s) - \gamma(t^s)$.

We can state our main convergence result.

Theorem 1: Let $\{t^s\}$ be the money distribution sequence in the discrete algorithm of a market mechanism. There is a finite step S such that $UD(t^S) = \emptyset$.

4.3 Characteristics of a Market Outcome

Envy-freeness and efficiency are central notions in fair division problems. An allocation $(\mu, t) \in \mathcal{M} \times \mathcal{T}$ is **envy-free** if and only if

$$u_i(\mu_i, t_{\mu_i}) \geq u_i(a, t_a) \qquad \forall\, a \in A.$$

Note that an allocation $(\mu, t) \in \mathcal{M} \times \mathcal{T}$ is envy-free if and only if $\mu_i \in D_i(t)$ for each agent $i \in I$. An allocation $(\mu, t) \in \mathcal{M} \times \mathcal{T}$ is **efficient** if and only if

$$\sum_{i \in I} u_i(\mu_i, t_{\mu_i}) \geq \sum_{i \in I} u_i(\lambda_i, x_{\lambda_i}) \qquad \forall\ (\lambda, x) \in \mathcal{M} \times \mathcal{T}.$$

Since the utility of an agent is quasi-linear in money shares, an allocation $(\mu, t) \in \mathcal{M} \times \mathcal{T}$ is efficient if and only if

$$\sum_{i \in I} v^i_{\mu_i} \geq \sum_{i \in I} v^i_{\lambda_i} \qquad \forall\ \lambda \in \mathcal{M}.$$

Therefore, money shares have no significance for efficiency considerations. Svensson (1983) and Alkan, Demange, and Gale (1991) show that if an allocation is envy-free then it is efficient as well in different preference domains.

Another central notion is individual rationality. An allocation $(\mu, t) \in \mathcal{M} \times \mathcal{T}$ is **individually rational** if and only if

$$u_i(\mu_i, t_{\mu_i}) \geq 0 \qquad \forall\ a \in A.$$

An individually rational allocation guarantees at least the reservation utility for each agent. Our market mechanisms find allocations, which satisfy the above properties.

Proposition 2: An outcome of a market mechanism is envy-free, efficient, and individually rational.

5 The Family of Market Mechanisms and the Compromised Egalitarian Mechanism

Different choices of function β in the discrete algorithm find outcomes of different market mechanisms. Three examples are as follows:

- Consider a fair division problem with a negative money endowment. An example is room assignment-rent division problem. Hence, a natural solution to this problem involves an allocation with non-positive money shares. Envy-free allocations with non-positive money shares may not exist (Maskin, 1987). For example, let $I = \{i_1, i_2\}$, $A = \{a_1, a_2\}$, $v^{i_1} = v^{i_2} = (2, 16)$ and $m = -10$. The unique envy-free money distribution is $t = (2, -12)$.

  The market mechanism with $\beta(t) = \alpha(t)$ for all $t \in \mathcal{T}$ finds an envy-free allocation with non-positive money shares whenever such an allocation exists. This mechanism is suggested by ASÜ for a room assignment-rent division problem using a different definition for $OD(t)$. In Appendix D, we prove that this definition is equivalent to our definition of $OD(t)$ in this paper. They also prove the mentioned property of this mechanism in their Theorem 2. The money distribution that this mechanism converges to is the money distribution that solves

  $$\min_{t \in \mathcal{T}} \left(\max_{a \in A} t_a \right).$$

- Consider a fair division problem with a positive money endowment. An example is the division of a bequest consisting of houses and money among inheritors. A natural solution has "non-negative" money shares for each inheritor. Envy-free allocations with non-negative money shares may not exist. An allocation with a negative money share involves taxation of an agent by the others. It directly follows from the dual statement of ASÜ Theorem 2 that the market mechanism with $\beta(t) = \gamma(t)$ for all $t \in \mathcal{T}$ finds an envy-free allocation with non-negative money shares whenever such an allocation exists. The money distribution that this mechanism converges to is the money distribution that solves

$$\max_{t\in\mathcal{T}}\left(\min_{a\in A} t_a\right).$$

Alkan, Demange and Gale (1991) refer to the above two solutions as money-Rawlsian envy-free allocations in a fair division problem. Aragones (1995) also presents a method which finds the last money-Rawlsian allocation described above in a fair allocation problem. We will refer to these mechanisms as **maximal-share** and **minimal-share egalitarian mechanisms**, respectively. Note that the outcome of these mechanisms can also be found using Demange, Gale and Sotomayor (1986) auction and its dual: apply a Vickrey auction and find the buyer- or seller-optimal competitive price/money distribution and balance the budget by equally subsidizing/taxing each agent. However, this is not true for other market mechanisms, including the following interesting mechanism:

- The market mechanism with $\beta = 0$ neither favors nor disfavors perfectly demanded goods in money adjustments. Hence, overdemanded and underdemanded objects are treated symmetrically in this mechanism. Therefore, this mechanism is a compromise between the two egalitarian mechanisms introduced above. We refer to this mechanism as the **compromised egalitarian mechanism**.

We can also start from arbitrary initial money distributions $t \in \mathcal{T}$ instead of equal shares as t^0. All our convergence results translate to this case without loss of generality. These can reflect object specific weights and these will induce new market allocations.

6 Conclusions

In closing, it is useful to comment on the strategic performance of the market mechanisms. As Alkan, Demange and Gale (1991) proved, there is no strategy-proof and envy-free mechanism. Since a market mechanism is envy-free, it is not strategy-proof. However, this result is not surprising, as the market mechanism in exchange economies is not strategy-proof, either. By giving up envy-freeness, it is straightforward to find a strategy-proof mechanism: (i) randomly order agents, (ii) pick a feasible money distribution vector, and (iii) use a serial dictatorship according to this ordering to assign objects in agents' demands determined according to this money distribution (break indifferences by assigning the smallest indexed object in current dictator's demand).[8]

7 Appendix A: Proofs of Results

Proof of Lemma 1: Let $t \in \mathcal{T}$ and $Q \in \mathcal{M}(t)$. First suppose $|Q| = n$, that is each agent is matched with an object in her demand set at t under Q. Then there is no agent and no object unmatched. Since each maximal market assignment has cardinality n, none of them leaves any agent or any object unmatched. Therefore, $UD(t) = US(t) = \emptyset$.

We will show that $UD(t) = \emptyset$ implies $|Q| = n$. Suppose $UD(t) = \emptyset$. Then by Corollary 1, $OS(t) = \emptyset$. Suppose $OD(t) \neq \emptyset$. We have $|OD(t)| < |US(t)|$ by Corollary 1. Moreover, $|PD(t)| = |PS(t)|$ by Corollary 1. Therefore,

$$\begin{aligned} n = |A| = |UD(t)| + |OD(t)| + |PD(t)| = |OD(t)| + |PD(t)| \\ < |US(t)| + |PS(t)| = |OS(t)| + |US(t)| + |PS(t)| = |I| = n, \end{aligned}$$

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[8] See Azacis (2004) for results on implementation of the max-min envy-free allocation using a version of the market mechanisms, Abdulkadiroğlu, Sönmez and Ünver (2004) mechanism.

a contradiction. We showed that $OD(t) = \emptyset$. Therefore, $PD(t) = A$ implying by GED Lemma Part 3 that every object in A is matched under Q and hence $|Q| = n$.

The proof of $US(t) = \emptyset$ implies $|Q| = n$ is the dual of the above proof. ◆

Proof of Lemma 2: Let $\{t^s\}$ be the money distribution sequence of a market mechanism. Let step s be such that $D_i(t^{s+1}) \subseteq D_i(t^s)$ for all $i \in I$. Let α, β, $\gamma \in \mathbb{R}$ satisfy

(a) $\alpha \geq \beta \geq \gamma$,

(b) $\alpha \to 0$, $\beta \to 0$, $\gamma \to 0$ such that $\alpha - \gamma \leq \tilde{u}_i(t^s) - u_i(a, t^s_a)$ for all $i \in I$ and $a \in A \backslash D_i(t^s)$, and

(c) $t^{s+1}_a = t^s_a + \alpha$ for all $a \in UD(t^s)$, $t^{s+1}_a = t^s_a + \beta$ for all $a \in PD(t^s)$, and $t^{s+1}_a = t^s_a + \gamma$ for all $a \in OD(t^s)$.

Let c^s be the cardinality of a maximal market assignment (i.e., number of agents that can feasibly be matched) at t^s. Define c^{s+1} similarly for t^{s+1}.

The agents in $US(t^s)$ only demand objects in $OD(t^s)$. They continue to demand the same objects at t^{s+1}, since no new object is in their demand and all the money shares attached to overdemanded objects decrease at the same rate γ. Clearly objects in $UD(t^s)$ continue to supply only the same agents in $OS(t^s)$ at t^{s+1}, since their money shares increase at the highest uniform rate α and no new agents are added to the supply at t^{s+1}. Therefore, whenever an agent or an object is underdemanded at t^s, she or it will remain underdemanded at t^{s+1}, implying $US(t^s) \subseteq US(t^{s+1})$ and $UD(t^s) \subseteq UD(t^{s+1})$. Moreover, the agents in $PS(t^s)$ will continue to demand the same objects in $PD(t^s)$ at t^{s+1}, since the money shares of objects at $PD(t^s)$ increase at the highest uniform rate β among all the objects they demand (objects in $PD(t^s)$ or in $OD(t^s)$).

Take $Q \in \mathcal{M}(t^s)$. By the GED Lemma, overdemanded objects (agents) are assigned to overdemanded agents (objects), and perfectly demanded objects are only assigned to perfectly supplied agents and vice versa under Q. Therefore, Q is a *feasible* market assignment at t^{s+1}, implying that $c^s \leq c^{s+1}$.

We will show that $US(t^s) \subsetneq US(t^{s+1})$ cannot hold. Suppose, there is some $i \in US(t^{s+1}) \backslash US(t^s)$. There is a maximal market assignment Q^{s+1} that leaves i unmatched at t^{s+1}. Since the demands of agents did not extend from t^s to t^{s+1}, Q^{s+1} is a market assignment at t^s. So $c^{s+1} \leq c^s$, implying with the above finding $c^s \leq c^{s+1}$ that we have $c^{s+1} = c^s$. Then Q^{s+1} is maximal at t^{s+1}. Since Q^{s+1} leaves i unmatched, $i \in US(t^s)$ contradicting the supposition. We showed that $US(t^{s+1}) = US(t^s)$.

Dual argument shows that $UD(t^{s+1}) = UD(t^s)$. These imply that $OD(t^{s+1}) = OD(t^s)$ and $PD(t^{s+1}) = PD(t^s)$; $OS(t^{s+1}) = OS(t^s)$ and $PS(t^{s+1})$ and $PS(t^s)$. ◆

Proof of Proposition 1: Let $\{t^s\}$ be the money distribution sequence of the discrete algorithm of a market mechanism. Let s be a step with $UD(t^s) \neq \emptyset$. We determine indirect utility $\tilde{u}_i(t^{s+1})$ for each agent $i \in I$. We consider agents in $US(t)$, $OS(t)$, and $PS(t)$ separately.

1. Let $i \in US(t^s)$ and $a \in D_i(t^s)$. By the construction of $US(t^s)$, we have $a \in OD(t^s)$. By the construction of t^{s+1}_a we have $t^{s+1}_a = t^s_a + \gamma(t^s)$. We obtain

$$u_i(a, t^{s+1}_a) = v^i_a + t^{s+1}_a = v^i_a + t^s_a + \gamma(t^s) = u_i(a, t^s_a) + \gamma(t^s). \tag{4}$$

We will show that $u_i(a, t^{s+1}_a) \geq u_i(b, t^{s+1}_b)$ for all $b \in A$. Let $b \in A \backslash \{a\}$. Three cases are possible:

   (a) $b \in UD(t^s)$. By the construction of $\alpha(t^s)$ and $\gamma(t^s)$, we have

$$\begin{aligned}\alpha(t^s) - \gamma(t^s) &\leq \min_{j \in US(t^s)} \left(\tilde{u}_j(t^s) - \max_{c \in UD(t^s)} u_j(c, t^s_c) \right) \leq \tilde{u}_i(t^s) - \max_{c \in UD(t^s)} u_i(c, t^s_c) \\ &\leq \tilde{u}_i(t^s) - u_i(b, t^s_b) = u_i(a, t^s_a) - u_i(b, t^s_b).\end{aligned}$$

This implies

$$u_i(b, t_b^s) + \alpha(t^s) \leq u_i(a, t_a^s) + \gamma(t^s). \tag{5}$$

By the construction of t_b^{s+1}, the money share of object b increases by $\alpha(t^s)$. We have $t_b^{s+1} = t_b^s + \alpha(t^s)$. This together with Equation 4 and Equation 5 implies

$$u_i(b, t_b^{s+1}) = v_b^i + t_b^{s+1} = v_b^i + t_b^s + \alpha(t^s) = u_i(b, t_b^s) + \alpha(t^s) \leq u_i(a, t_a^s) + \gamma(t^s) = u_i(a, t_a^{s+1}).$$

(b) $b \in PD(t^s)$. By the construction of $\gamma(t^s)$ and $\beta(t^s)$, we have

$$\begin{aligned}\beta(t^s) - \gamma(t^s) &\leq \min_{j \in US(t^s)} \left(\tilde{u}_j(t^s) - \max_{c \in PD(t^s)} u_j(c, t_c^s) \right) \leq \tilde{u}_i(t^s) - \max_{c \in PD(t^s)} u_i(c, t_c^s) \\ &\leq \tilde{u}_i(t^s) - u_i(b, t_b^s) = u_i(a, t_a^s) - u_i(b, t_b^s).\end{aligned}$$

This implies

$$u_i(b, t_b^s) + \beta(t^s) \leq u_i(a, t_a^s) + \gamma(t^s). \tag{6}$$

By the construction of t_b^{s+1}, the money share of object b increases by $\beta(t^s)$. We have $t_b^{s+1} = t_b^s + \beta(t^s)$. This together with Equation 4 and Equation 6 implies

$$u_i(b, t_b^{s+1}) = v_b^i + t_b^{s+1} = v_b^i + t_b^s + \beta(t^s) = u_i(b, t_b^s) + \beta(t^s) \leq u_i(a, t_a^s) + \gamma(t^s) = u_i(a, t_a^{s+1}).$$

(c) $b \in OD(t^s) \backslash \{a\}$. By the construction of t_b^{s+1}, the money share of object b increases by $\gamma(t^s)$. We have $t_b^{s+1} = t_b^s + \gamma(t^s)$. Since $a \in D_i(t^s)$, we have $u_i(b, t_b^s) \leq u_i(a, t_a^s)$. These together with Equation 4 imply

$$u_i(b, t_b^{s+1}) = v_b^i + t_b^{s+1} = v_b^i + t_b^s + \gamma(t^s) = u_i(b, t_b^s) + \gamma(t^s) \leq u_i(a, t_a^s) + \gamma(t^s) = u_i(a, t_a^{s+1}).$$

We showed that $a \in D_i(t^{s+1})$ and

$$\tilde{u}_i(t^{s+1}) = u_i(a, t_a^{s+1}) = u_i(a, t_a^s) + \gamma(t^s) = \tilde{u}_i(t^s) + \gamma(t^s). \tag{7}$$

2. Let $i \in OS(t^s)$ and $a \in D_i(t^s) \cap UD(t^s)$. By the construction of t_a^{s+1}, we have $t_a^{s+1} = t_a^s + \alpha(t^s)$. We have

$$u_i(a, t_a^{s+1}) = v_a^i + t_a^{s+1} = v_a^i + t_a^s + \alpha(t^s) = u_i(a, t_a^s) + \alpha(t^s). \tag{8}$$

We will show that $u_i(a, t_a^{s+1}) \geq u_i(b, t_b^{s+1})$ for all $b \in A$.

Let $b \in A \backslash \{a\}$. We have $u_i(a, t_a^s) \geq u_i(b, t_b^s)$, since $a \in D_i(t^s)$. By the construction of t_b^{s+1}, the money share of object b increases at most by $\alpha(t^s)$. We have $t_b^{s+1} \leq t_b^s + \alpha(t^s)$. These and Equation 8 imply

$$u_i(b, t_b^{s+1}) = v_b^i + t_b^{s+1} \leq v_b^i + t_b^s + \alpha(t^s) = u_i(b, t_b^s) + \alpha(t^s) \leq u_i(a, t_a^s) + \alpha(t^s) = u_i(a, t_a^{s+1}).$$

We showed that $a \in D_i(t^{s+1})$ and

$$\tilde{u}_i(t^{s+1}) = u_i(a, t_a^{s+1}) = u_i(a, t_a^s) + \alpha(t^s) = \tilde{u}_i(t^s) + \alpha(t^s). \tag{9}$$

3. Let $i \in PS(t^s)$ and $a \in D_i(t^s) \backslash OD(t^s)$. We have $a \in PD(t^s)$. By the construction of t_a^{s+1} we have $t_a^{s+1} = t_a^s + \beta(t^s)$. We obtain

$$u_i(a, t_a^{s+1}) = v_a^i + t_a^{s+1} = v_a^i + t_a^s + \beta(t^s) = u_i(a, t_a^s) + \beta(t^s). \tag{10}$$

We will show that $u_i(a, t_a^{s+1}) \geq u_i(b, t_b^{s+1})$ for all $b \in A$. Let $b \in A \backslash \{a\}$. Two cases are possible:

(a) $b \in UD(t^s)$. By the construction of $\alpha(t^s)$ and $\beta(t^s)$ we have

$$\begin{aligned}\alpha(t^s) - \beta(t^s) &\le \min_{j \in PS(t^s)} \left(\tilde{u}_j(t^s) - \max_{c \in UD(t^s)} u_j(c, t^s_c) \right) \le \tilde{u}_i(t^s) - \max_{c \in UD(t^s)} u_i(c, t^s_c) \\ &\le \tilde{u}_i(t^s) - u_i(b, t^s_b) = u_i(a, t^s_a) - u_i(b, t^s_b).\end{aligned}$$

We have

$$u_i(b, t^s_b) + \alpha(t^s) \le u_i(a, t^s_a) + \beta(t^s). \tag{11}$$

By the construction of t^{s+1}_b, the money share of object b increases by $\alpha(t^s)$. We have $t^{s+1}_b = t^s_b + \alpha(t^s)$. This together with Equation 10 and Equation 11 implies

$$u_i(b, t^{s+1}_b) = v^i_b + t^{s+1}_b = v^i_b + t^s_b + \alpha(t^s) = u_i(b, t^s_b) + \alpha(t^s) \le u_i(a, t^s_a) + \beta(t^s) = u_i(a, t^{s+1}_a).$$

(b) $b \in OD(t^s) \cup PD(t^s) \backslash \{a\}$. We have $b \in OD(t^s)$ or $b \in PD(t^s)$. By the construction of t^{s+1}_b, the money share of object b increases at most by $\beta(t^s)$. Hence, $t^{s+1}_b \le t^s_b + \beta(t^s)$. Since $a \in D_i(t^s)$, we have $u_i(b, t^s_b) \le u_i(a, t^s_a)$. These together with Equation 10 imply

$$u_i(b, t^{s+1}_b) = v^i_b + t^{s+1}_b \le v^i_b + t^s_b + \beta(t^s) = u_i(b, t^s_b) + \beta(t^s) \le u_i(a, t^s_a) + \beta(t^s) = u_i(a, t^{s+1}_a).$$

We showed that $a \in D_i(t^{s+1})$ and

$$\tilde{u}_i(t^{s+1}) = u_i(a, t^{s+1}_a) = u_i(a, t^s_a) + \beta(t^s) = \tilde{u}_i(t^s) + \beta(t^s). \tag{12}$$

Next, we will inspect the sum of indirect utilities at money distribution t^{s+1}. We have

$$\begin{aligned}\sum_{i \in I} \tilde{u}_i(t^{s+1}) &= \sum_{i \in OS(t^s)} (\tilde{u}_i(t^s) + \alpha(t^s)) + \sum_{i \in PS(t^s)} (\tilde{u}_i(t^s) + \beta(t^s)) + \sum_{i \in US(t^s)} (\tilde{u}_i(t^s) + \gamma(t^s)) \\ &= \sum_{i \in I} \tilde{u}_i(t^s) + \underbrace{|OS(t^s)|}_{\le |UD(t^s)|-1} \alpha(t^s) + \underbrace{|PS(t^s)|}_{=PD(t^s)} \beta(t^s) + \underbrace{|US(t^s)|}_{\ge |OD(t^s)|+1} \gamma(t^s) \\ &\le \sum_{i \in I} \tilde{u}_i(t^s) + (|UD(t^s)| - 1)\,\alpha(t^s) + |PD(t^s)|\,\beta(t^s) + (|OD(t^s)| + 1)\,\gamma(t^s) \text{ since } \alpha > 0 \text{ and } \gamma < 0 \\ &= \sum_{i \in I} \tilde{u}_i(t^s) + \gamma(t^s) - \alpha(t^s) + \underbrace{|UD(t^s)|\,\alpha(t^s) + |PD(t^s)|\,\beta(t^s) + |OD(t^s)|\,\gamma(t^s)}_{=0 \text{ by Equation 3}}\end{aligned}$$

This completes the proof of Proposition 1. ♦

Proof of Theorem 1: Let $\{t^s\}$ be the money distribution sequence in the discrete algorithm of a market mechanism. Consider any step s. For any $i \in I$ we have $\tilde{u}_i(t^s) \ge u_i(a, t^s_a)$ for all $a \in A$. We have

$$n\tilde{u}_i(t^s) \ge \sum_{a \in A} u_i(a, t^s_a) = \sum_{a \in A} v^i_a + \sum_{a \in A} t^s_a = \sum_{a \in A} v^i_a + m$$

for all $i \in I$. Hence,

$$\sum_{i \in I} \tilde{u}_i(t^s) \ge \frac{1}{n} \left(\sum_{i \in I} \sum_{a \in A} v^i_a \right) + m.$$

Therefore the sum of the indirect utilities of agents is bounded below in the market mechanism. By Proposition 1, the sum of the indirect utilities decrease monotonically at least by $\alpha(t^s) - \gamma(t^s)$ from step s to step $s+1$ as long as the set of underdemanded objects is non-empty. Moreover, for any step s such that $UD(t^s) \neq \emptyset$, $x(t^s)$, $y(t^s)$ and $z(t^s)$ are bounded away from zero, implying $\alpha(t^s) - \gamma(t^s)$ is bounded away from zero. Therefore, it should be true that there exists some finite step S such that $UD(t^S) = \emptyset$. ◆

Proof of Proposition 2: Identical to the proofs of ASÜ Proposition 1, Proposition 2 and Proposition 3. ◆

8 Appendix B: Construction of a Maximal Market Assignment, Sets of Underdemanded Objects and Undersupplied Agents

Fix $t \in \mathcal{T}$. Take any market assignment $Q \in \mathcal{Q}(t)$. Let $a \in A$. An **odd-length alternating path for Q from a** is a path $(a, i_1, a_1, ..., i_k, a_k, i)$ of distinct agents and objects such that $\{i_\ell, a_\ell\} \in Q$ for any $\ell \in \{1, 2, ..., k\}$; $\{a, i_1\}, \{a_k, i\} \in L(t) \backslash Q$, and $\{a_{\ell-1}, i_\ell\} \in L(t) \backslash Q$ for any $\ell \in \{2, 3, ..., k\}$.[9] That is, this path has $2k+1$ links in it (odd-length) and the first link *is not* in Q, the second link *is* in Q, the third link *is not* in Q,..., the last link *is* in Q (alternating). An odd-length alternating path from an agent is symmetrically defined. If Q has an odd-length alternating path, then it cannot be maximal: since we can have the assignments $\{a, i_1\}$,$\{a_1, i_2\}$,...,$\{a_k, i\}$ instead of the assignments $\{i_1, a_1\}$,$\{i_2, a_2\}$,..., $\{i_k, a_k\}$ and obtain $k+1$ agents matched instead of k without affecting the rest of the assignments under Q at t. Edmonds' algorithm starts from any market assignment, finds an odd-length alternating path in it, and then creates a new market assignment from the old one as explained above, and repeats the process until there are no odd-length alternating paths left. Berge (1957) showed that a market assignment is maximal if and only if it does not have any odd-length alternating path. Therefore, Edmonds' algorithm converges to a maximal market assignment. It has $O(n^3)$ time complexity, and faster algorithms than Edmonds' algorithm have been introduced.[10]

Once a maximal market assignment $Q \in \mathcal{M}(t)$ is determined, we can construct $US(t)$ and $UD(t)$ also in polynomial time complexity. Take an object a that is left unmatched under Q. Define an **even-length alternating path for Q from a** as a path $(a, i_1, a_1, ..., i_k, a_k)$ of distinct agents and objects such that $\{i_\ell, a_\ell\} \in Q$ for any $\ell \in \{1, 2, ..., k\}$; $\{a, i_1\} \in L(t) \backslash Q$ and $\{a_{\ell-1}, i_\ell\} \in L(t) \backslash Q$ for any $\ell \in \{2, 3, ..., k\}$. Note that (a) is a degenerate even-length alternating path with zero links. We define an even-length alternating path starting from an unmatched agent under Q in a symmetric manner. That is, this path has $2k$ links in it (even-length) and the first link *is not* in Q, the second link *is* in Q, the third link *is not* in Q..., the last link *is* in Q (alternating). Suppose that path $(a, i_1, a_1, ..., i_k, a_k)$ is an even-length alternating path from a. We can leave a_1 unmatched instead of a under a maximal market assignment: modify Q to include $\{a, i_1\}$ instead of $\{i_1, a_1\}$. Similarly, for any $\ell \in \{2, 3, ..., k\}$, we can leave a_ℓ unmatched instead of a under a maximal market assignment: modify Q to include links $\{a, i_1\}, \{a_1, i_2\}, ..., \{a_{\ell-1}, i_\ell\}$ instead of links $\{i_1, a_1\}, \{i_2, a_2\}, ..., \{i_\ell, a_\ell\}$. As explained in Goemans (2004), (i) the set of underdemanded objects is the union of all objects that can be reached by an even-length alternating path from at least one unmatched object under Q; and (ii) the set of undersupplied agents is the union of all agents that can be reached by an even-length alternating path from at least one unmatched agent under Q.

9 Appendix C: Example

Example 2: Consider the market mechanism in which the money shares of perfectly demanded objects are increased half as fast as the money shares of underdemanded objects, i.e. $\beta = \frac{1}{2}\alpha$.

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[9] It is also known as an **augmenting path**.

[10] See Korte and Vygen (2000) for an excellent reference on combinatorial optimization theory.

We find functions α, β, and γ for the discrete algorithm as

$$\alpha(t) = \min\left\{ \frac{|OD(t)|}{|OD(t)| + |UD(t)| + \frac{1}{2}|PD(t)|}x(t), \frac{|OD(t)|}{\frac{1}{2}|OD(t)| + |UD(t)| + \frac{1}{2}|PD(t)|}y(t), 2z(t) \right\},$$

$$\gamma(t) = -\frac{|UD(t)| + \frac{1}{2}|PD(t)|}{|OD(t)|}\alpha(t), \text{ and}$$

$$\beta(t) = \frac{1}{2}\alpha(t) \qquad \forall\, t \in \mathcal{T}.$$

Consider fair division problem $\langle I, A, V, m\rangle$ with agent set $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, object set $A = \{a_1, a_2, a_3, a_4, a_5\}$, the value matrix

$$V = \left[v^i_a\right]_{i\in I, a\in A} =$$

	a_1	a_2	a_3	a_4	a_5
i_1	37	62	13	14	12
i_2	-34	-47	1	-10	-24
i_3	58	-26	34	47	58
i_4	0	47	24	56	72
i_5	-36	47	-50	12	47
i_6	2	16	-81	-104	-69

,

and money endowment $m = 600$. Since $|A| < |I|$, we introduce a dummy object a_6 and set the value of each agent for this object to 0.

We will find the outcome of the market mechanism introduced above for this fair division problem using the discrete algorithm.

Step 0: We set the initial money distribution as

$$t^0 = (100, 100, 100, 100, 100, 100).$$

Below, we give the utility profile of agents at t^0. The indirect utilities of agents are highlighted in bold.

$$\left[u_i(a, t^0_a)\right]_{i\in I, a\in A} =$$

	a_1	a_2	a_3	a_4	a_5	a_6
i_1	137	**162**	113	114	112	100
i_2	66	53	**101**	90	76	100
i_3	**158**	74	134	147	**158**	100
i_4	100	147	124	156	**172**	100
i_5	64	**147**	50	112	**147**	100
i_6	102	**116**	19	-4	31	100

.

The set of links at t^0 is given as

$$L\left(t^0\right) = \left\{\{i_1, a_2\}, \{i_2, a_3\}, \{i_3, a_1\}, \{i_3, a_5\}, \{i_4, a_5\}, \{i_5, a_2\}, \{i_5, a_5\}, \{i_6, a_2\}\right\}.$$

We find

$$Q^0 = \left\{\{i_1, a_2\}, \{i_2, a_3\}, \{i_3, a_1\}, \{i_5, a_5\}\right\}$$

as a maximal market assignment.[11] Since $\left|Q^0\right| < 6 = n$, $US(t^0) \neq \emptyset$ and $UD\left(t^0\right) \neq \emptyset$ by Lemma 1 and hence, we proceed to the next step.

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[11] It is easy to find a maximal market assignment for this example, for more complicated problems we can use Edmonds' (1965) algorithm.

Step 1: We first find the GED of the problem at t^0. We find $UD\left(t^0\right)$ as follows: Q^0 leaves a_4 and a_6 unmatched, implying that a_4 and a_6 are underdemanded. There are no non-degenerate even-length alternating paths starting from either a_4 or a_6, implying $UD\left(t^0\right) = \{a_4, a_6\}$. We find $US\left(t^0\right)$ as follows: Q^0 leaves i_4 and i_6 unmatched implying i_4 and i_6 are undersupplied. Path $(i_4, a_5, i_5, a_2, i_1)$ is an even-length alternating path starting from i_4. Therefore, i_5 and i_1 are undersupplied. No other even-length alternating paths starting from i_4 or i_6 contain other agents. Hence, $US\left(t^0\right) = \{i_1, i_4, i_5, i_6\}$. The agents who have links with objects in $UD\left(t^0\right)$ are oversupplied. There are no links including either object a_4 or a_6 in $L\left(t^0\right)$, therefore, $OS\left(t^0\right) = \emptyset$. The objects which have links with agents in $US\left(t^0\right)$ are overdemanded. We have $OD\left(t^0\right) = \{a_2, a_5\}$. The rest of the objects are perfectly demanded, implying $PD\left(t^0\right) = I \backslash \left(UD\left(t^0\right) \cup OD\left(t^0\right)\right) = \{a_1, a_3\}$. The rest of the agents are perfectly supplied, implying $PS\left(t^0\right) = I \backslash \left(US\left(t^0\right) \cup OS\left(t^0\right)\right) = \{i_2, i_3\}$. In summary, we have

$$\begin{aligned} &UD\left(t^0\right) = \{a_4, a_6\}, \; OD\left(t^0\right) = \{a_2, a_5\}, \text{ and } PD\left(t^0\right) = \{a_1, a_3\}; \\ &US\left(t^0\right) = \{i_1, i_4, i_5, i_6\}, \; OS\left(t^0\right) = \emptyset, \text{ and } PS\left(t^0\right) = \{i_2, i_3\}. \end{aligned}$$

We determine $x(t^0)$, $y(t^0)$, and $z(t^0)$ in order to calculate money distribution t^1.

$$x(t^0) = \min_{i \in US(t^0)} \left(\tilde{u}_i\left(t^0\right) - \max_{a \in UD(t^0)} u_i(a, t^0_a) \right) = \tilde{u}_{i_6}\left(t^0\right) - u_{i_6}\left(a_6, t^0_{a_6}\right) = 116 - 100 = 16.$$

$$y(t^0) = \min_{i \in US(t^0)} \left(\tilde{u}_i\left(t^0\right) - \max_{a \in PD(t^0)} u_i(a, t^0_a) \right) = \tilde{u}_{i_6}\left(t^0\right) - u_{i_6}\left(a_1, t^0_{a_1}\right) = 116 - 102 = 14.$$

$$z(t^0) = \min_{i \in PS(t^0)} \left(\tilde{u}_i\left(t^0\right) - \max_{a \in UD(t^0)} u_i(a, t^0_a) \right) = \tilde{u}_{i_2}\left(t^0\right) - u_{i_2}\left(a_6, t^0_{a_6}\right) = 101 - 100 = 1.$$

We determine $\alpha(t^0)$, $\gamma(t^0)$, and $\beta(t^0)$ as

$$\begin{aligned} &\alpha(t^0) = \min\{\frac{2}{5}x\left(t^0\right), \frac{1}{2}y\left(t^0\right), 2z\left(t^0\right)\} = \min\{\frac{32}{5}, 7, 2\} = 2, \\ &\gamma(t^0) = -\frac{3}{2}\alpha(t^0) = -3, \text{ and} \\ &\beta(t^0) = \frac{1}{2}\alpha(t^0) = 1. \end{aligned}$$

We determine the new money distribution as

$$t^1 = (101, 97, 101, 102, 97, 102).$$

The utility matrix at t^1 is given below:

$$\left[u_i(a, t^1_a)\right]_{i \in I, a \in A} = \begin{array}{|c|c|c|c|c|c|c|} \hline & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \hline i_1 & 138 & \mathbf{159} & 114 & 116 & 109 & 102 \\ \hline i_2 & 67 & 50 & \mathbf{102} & 92 & 73 & \mathbf{102} \\ \hline i_3 & \mathbf{159} & 71 & 135 & 149 & 155 & 102 \\ \hline i_4 & 101 & 144 & 125 & 158 & \mathbf{169} & 102 \\ \hline i_5 & 65 & \mathbf{144} & 51 & 114 & \mathbf{144} & 102 \\ \hline i_6 & 103 & \mathbf{113} & 20 & \text{-}2 & 28 & 102 \\ \hline \end{array}.$$

The set of links at t^1 is given as

$$L\left(t^1\right)=\left\{\left\{i_1,a_2\right\},\left\{i_2,a_3\right\},\left\{i_2,a_6\right\},\left\{i_3,a_1\right\},\left\{i_4,a_5\right\},\left\{i_5,a_2\right\},\left\{i_5,a_5\right\},\left\{i_6,a_2\right\}\right\}.$$

We find

$$Q^1=\left\{\left\{i_1,a_2\right\},\left\{i_2,a_3\right\},\left\{i_3,a_1\right\},\left\{i_4,a_5\right\}\right\}$$

as a maximal market assignment at t^1. Since $\left|Q^1\right|<6=n$, $UD\left(t^1\right)\neq\emptyset$ and $US\left(t^1\right)\neq\emptyset$ by Lemma 1 and hence, we proceed to the next step.

Step 2: First, we find the GED of the problem at t^1 using Q^1. We determine $UD\left(t^1\right)$ as follows: Q^1 leaves a_4 and a_6 unmatched, implying a_4 and a_6 are underdemanded. Path (a_6,i_2,a_3) is an even-length alternating path starting from a_6. Therefore, a_3 is underdemanded, as well. There are no other non-degenerate even-length alternating paths starting from a_4 or a_6. Hence, $UD\left(t^1\right)=\{a_3,a_4,a_6\}$. We determine $US\left(t^1\right)$ as follows: Q^1 leaves i_5 and i_6 unmatched, implying i_5 and i_6 are undersupplied. Path (i_5,a_2,i_1) is an even-length alternating path starting from i_5, implying i_1 is undersupplied. Path (i_5,a_5,i_4) is an even-length alternating path starting from i_5, implying i_4 is undersupplied. Other even-length alternating paths starting from i_5 or i_6 do not include any other new agents. Therefore, $US\left(t^1\right)=\{i_1,i_4,i_5,i_6\}$. Set $OD\left(t^1\right)$ is the set of objects which have links to agents in $US\left(t^1\right)$. We have $OD\left(t^1\right)=\{a_2,a_5\}$. Set $OS(t^1)$ is the set of agents who have links to objects in $UD\left(t^1\right)$. We have $OS\left(t^1\right)=\{i_2\}$. The remaining objects are perfectly demanded: $PD\left(t^1\right)=A\backslash\left(UD\left(t^1\right)\cup OD\left(t^1\right)\right)=\{a_1\}$. The remaining agents are perfectly supplied: $PS\left(t^1\right)=I\backslash\left(US\left(t^1\right)\cup OS\left(t^1\right)\right)=\{i_3\}$. In summary, we have

$$\begin{aligned}&UD\left(t^1\right)=\left\{a_3,a_4,a_6\right\},\ OD\left(t^1\right)=\left\{a_2,a_5\right\},\text{ and }PD\left(t^1\right)=\left\{a_1\right\};\\&US\left(t^1\right)=\left\{i_1,i_4,i_5,i_6\right\},\ OS\left(t^1\right)=\left\{i_2\right\},\text{ and }PS\left(t^1\right)=\left\{i_3\right\}.\end{aligned}$$

We find,

$$x(t^1)=\min_{i\in US(t^1)}\left(\tilde{u}_i\left(t^1\right)-\max_{a\in UD(t^1)}u_i(a,t^1_a)\right)=\tilde{u}_{i_6}\left(t^1\right)-u_{i_6}\left(a_6,t^1_{a_6}\right)=113-102=11.$$

$$y(t^1)=\min_{i\in US(t^1)}\left(\tilde{u}_i\left(t^1\right)-\max_{a\in PD(t^1)}u_i(a,t^1_a)\right)=\tilde{u}_{i_6}\left(t^1\right)-u_{i_6}\left(a_1,t^1_{a_1}\right)=113-103=10.$$

$$z(t^1)=\min_{i\in PS(t^1)}\left(\tilde{u}_i\left(t^1\right)-\max_{a\in UD(t^1)}u_i(a,t^1_a)\right)=\tilde{u}_{i_3}\left(t^1\right)-u_{i_3}\left(a_4,t^1_{a_4}\right)=159-149=10.$$

We determine $\alpha(t^1)$, $\gamma(t^1)$, and $\beta(t^1)$ as

$$\begin{aligned}&\alpha(t^1)=\min\{\frac{4}{11}x\left(t^1\right),\frac{4}{9}y\left(t^1\right),2z\left(t^1\right)\}=\min\{4,\frac{40}{9},20\}=4,\\&\gamma(t^1)=-\frac{7}{4}\alpha(t^1)=-7,\text{ and}\\&\beta(t^1)=\frac{1}{2}\alpha(t^1)=2.\end{aligned}$$

We determine the new money distribution as

$$t^2=(103,90,105,106,90,106).$$

The utility matrix at t^2 is given below:

$$\left[u_i(a,t_a^2)\right]_{i\in I,a\in A} = \begin{array}{|c|c|c|c|c|c|c|}\hline & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \hline i_1 & 140 & \mathbf{152} & 118 & 120 & 102 & 106 \\ \hline i_2 & 69 & 43 & \mathbf{106} & 96 & 66 & \mathbf{106} \\ \hline i_3 & \mathbf{161} & 64 & 139 & 153 & 148 & 106 \\ \hline i_4 & 103 & 137 & 129 & \mathbf{162} & \mathbf{162} & 106 \\ \hline i_5 & 67 & \mathbf{137} & 55 & 118 & \mathbf{137} & 106 \\ \hline i_6 & 105 & \mathbf{106} & 24 & 2 & 21 & \mathbf{106} \\ \hline \end{array}.$$

The set of links at t^2 is given by

$$L\left(t^2\right) = \left\{\left\{i_1,a_2\right\},\left\{i_2,a_3\right\},\left\{i_2,a_6\right\},\left\{i_3,a_1\right\},\left\{i_4,a_4\right\},\left\{i_4,a_5\right\},\left\{i_5,a_2\right\},\left\{i_5,a_5\right\},\left\{i_6,a_2\right\},\left\{i_6,a_6\right\}\right\}.$$

A maximal market assignment is given by

$$Q^2 = \left\{\left\{i_1,a_2\right\},\left\{i_2,a_3\right\},\left\{i_3,a_1\right\},\left\{i_4,a_4\right\},\left\{i_5,a_5\right\},\left\{i_6,a_6\right\}\right\}.$$

We have $\left|Q^2\right| = 6 = n$, implying Q^2 is a market matching by Observation 1. We terminate the procedure and Q^2 is a matching that clears the market and $\left(Q^2,t^2\right)$ is an outcome of the market mechanism with $\beta = \frac{\alpha}{2}$ for fair division problem $\langle I, A, V, m\rangle$. ◆

10 Appendix D: Equivalence between Market Mechanism with $\beta = \alpha$ and ASÜ Mechanism

Fix $t \in \mathcal{T}$. Let $OD^*(t)$ be the *full set of overdemanded objects* at t as defined by ASÜ. It is constructed as follows: $B \subset A$ is **overdemanded** if $|\{i \in I : D_i(t) \subseteq B\}| > |B|$. A set of objects is a **minimal overdemanded set** if it is overdemanded and none of its proper subsets is overdemanded. ASÜ iteratively define the **full set of overdemanded objects** at t as follows: Given t find all minimal overdemanded sets. Remove these objects from the demand of each agent and find the minimal overdemanded sets for the modified demand profiles. Proceed in a similar way until there is no minimal overdemanded set for the modified demand profiles. The full set of overdemanded objects, $OD^*(t)$, is the union of each of the sets encountered in the procedure.

We prove that $OD^*(t) = OD(t)$:

- $OD^*(t) \subseteq OD(t)$: Let $a \in OD^*(t)$.
  - Suppose a is removed in the above construction in round 1 in minimal overdemanded set B_1. Let $J_1 = \{i \in I : D_i(t) \subseteq B\}$. Let $i \in S_a(t)$. We have $|J_1| > |B_1|$, since B_1 is overdemanded. Let $J \subseteq J_1 \backslash \{i\}$ be such that $|J| = |B_1|$. We have for all $B \subseteq B_1$, $|\{j \in J : D_j(t) \subseteq B\}| \leq |B|$, since B_1 is a *minimal* overdemanded set and none of its proper subsets are overdemanded. This implies by Hall's (1935) Theorem that all objects in B_1 can be distributed to agents in J. Since i only demands objects in B_1, i can remain unmatched in a maximal market assignment, implying $i \in US(t)$. Since $i \in S_a(t)$, we have $a \in OD(t)$.
  - Suppose a is removed in the above construction in round 2 in the remaining minimal overdemanded set B_2. We showed above that in a maximal market assignment all overdemanded objects removed in the first round can be matched to agents removed in the first round.

Let J_1 be the agents removed in first round, let B_1 be the objects removed in the first round. Let $J_2 = \{i \in I : D_i(t) \backslash B_1 \subseteq B_2\}$. Let $i \in S_a(t)$. Let $J \subseteq J_2 \backslash \{i\}$ be such that $|J| = |B_2|$. We have for any $B \subseteq B_2$, $|\{j \in J : D_j(t) \backslash B_1 \subseteq B\}| \leq |B|$, since B_2 is a *minimal* overdemanded set and none of its subsets are overdemanded after B_1 is removed, implying by Hall's Theorem that all objects in B_2 can be distributed to agents in J_2. Since all objects in B_1 and B_2 can be committed to agents in J_1 and J respectively, and since i does not demand any other objects (i.e. $D_i(t) \subseteq B_1 \cup B_2$), there is $Q \in \mathcal{M}(t)$ such that $Q_i = \emptyset$, implying $i \in US(t)$. Since $i \in S_a(t)$, we have $a \in OD(t)$.

Continuing in a similar iterative manner, we obtain $OD^*(t) \subseteq OD(t)$.

- $OD(t) \subseteq OD^*(t)$: Let $J^* = \{i \in I : D_i(t) \subseteq OD^*(t)\}$. There are no overdemanded sets in $A \backslash OD^*(t)$: for any $B \subseteq A \backslash OD^*(t)$ we have $|\{i \in I : D_i(t) \subseteq I\}| \leq |B|$. Since $|J^*| > |OD^*(t)|$, we have $|I \backslash J^*| < |A \backslash OD^*(t)|$. Therefore, by Hall's Theorem all agents in $I \backslash J^*$ can be matched with objects in $A \backslash OD^*(t)$. Since all agents $I \backslash J^*$ are matched under all maximal market assignments, $I \backslash J^* \subseteq I \backslash US(t)$, this in turn implies that $US(t) \subseteq J^*$. $OD(t)$ is the set of objects that agents in $US(t)$ demand, and $OD^*(t)$ is the set of objects that agents in J^* demand, implying $OD(t) \subseteq OD^*(t)$.

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