Discrete Public Goods with Incomplete Information*

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November 1998

Abstract

We analyze simultaneous discrete public good games with incomplete information and continuous contributions. To use the terminology of Admati and Perry (1991), we consider contribution and subscription games. In the former, contributions are not refunded if the project is not completed, while in the latter they are. We provide necessary conditions that increasing equilibria of contribution games and subscription games must satisfy for general distribution functions. We then characterize a symmetric equilibrium of the subscription game when valuations are uniformly distributed. Our analysis shows that there is a significant difference between the equilibrium outcomes of the two games. In fact, we show that for the contribution game, "contributing zero" is the only equilibrium for a general family of distributions.

JEL Classification:D79, D89, H89

Key Words: private provision of public goods; contribution and subscription games; incomplete information.

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[*] We would like to thank John Conley, Charles Kahn, Paul Pecorino, Todd Sandler and Harris Schlesinger for helpful discussions. Menezes and Monteiro gratefully acknowledge the financial support of CNPq. Temimi acknowledges the financial support of the College of Business Administration at the University of Alabama.

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1 Introduction

The literature on private provision of public goods can be divided into two broad categories.[1] The first branch of the literature focuses on the provision of continuous public goods. Papers include Warr (1982, 1983), Cornes and Sandler (1984), Bergstrom, Blume and Varian (1986), Andreoni (1988), Gradstein, Nitzan and Slutsky (1994) and others. A standard result in this literature is that public goods are underprovided by voluntary contributions due to free riding behavior.[2] One might conjecture that the government can solve this underprovision problem by providing some of the good and financing it by imposing taxes on contributors. However, Warr (1982) and Roberts (1984), in two influential papers, show that government contributions result in a dollar to dollar reduction in private contributions if the tax on contributors does not change the set of contributing individuals. Bergstrom, Blume and Varian (1986) show that the crowding out effect is only partial if one allows for the taxes that pay for the government contribution to be also collected from non contributors. Hence the literature suggests that underprovision is a robust conclusion if the public good level is endogenously determined by voluntary contributions.[3]

The second strand of the literature focuses on discrete public goods where a fixed level of a public good is provided if enough contributions are collected to cover its cost c. Otherwise, the good is not provided. Typical examples include building a bridge, a library of a certain size, public radio fund raising to finance a certain program. In all of these examples, if enough money is raised to cover the cost of the public good, then the good is provided, otherwise the good is not provided.

Palfrey and Rosenthal (1984) developed the first modern treatment of private provision of discrete public goods. More specifically they analyze contribution and subscription games — to use the terminology of Admati and Perry (1991) — for a discrete public good under complete information where players' strategies are restricted to either contribute zero or an exogenous positive amount. In a contribution game, contributions are not refunded if the sum of the contributions does not cover the cost of the public good c, while in a subscription game players

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[1] In this paper, we focus on private provision. There is an extensive mechanism design literature that treats public provision. Papers in this literature develop efficient mechanisms for the provision of public goods. These mechanisms are in general complex and require a central authority to implement them and hence would be best decsribed as mechanisms for public provision of public goods. Gradstein (1994) examines efficient mechanisms for discrete public goods. Other papers that treat public provision include Maskin (1977), d'Aspremont and Varet (1979, 1982), Palfrey and Srivastava (1986). Cornelli (1996) examines an optimal mechanism for a monopolist which produces an excludable good that has large fixed costs.

[2] The free riding problem becomes worse in the presence of incomplete information. Gradstein (1992) considers a dynamic model of private provision for a continuous public good with incomplete information with the restriction that players either contribute zero or an exogenous positive amount. He concludes that in addition to the standard underprovision results, inefficiency occurs because of a delay in contributions. This inefficiency does not disappear as the population becomes large.

[3] Andreoni (1997) analyzes the role of seed money in the presence of nonconvexities in the production of the public good. He shows that a small amount of seed money can generate a substantial amount of voluntary contributions.

get their money back if the project is not completed. An example with the main features of a subscription game can be given as follows. The Wisconsin Governor has recently pledged \$27 million in state bonds to finance a new \$72 million basketball arena on the condition that the rest of the money be raised by private donations. (Andreoni, 1997). That is, the Governor will provide \$27 million as long as the remaining \$45 million is raised, otherwise the offer is cancelled. Other examples can be found in the fund-raising literature. Examples of contribution games include public radio and TV fund-raising efforts where contributors do not get their money back if the program is not provided. Other examples of contributions games are situations where contributions take the form of physical labor, in this case volunteers cannot recover their effort if the project is not completed.

In contrast with the standard underprovision result for continuous public goods, Palfrey and Rosenthal (1984) obtain the startling conclusion that efficient provision of a discrete public good is a possible outcome of both contribution and subscription games even though inefficient equilibria also exist. This is rather intuitive since the coordination problem becomes easier when the level of public good is fixed and common knowledge to all the players. Bagnoli and Lipman (1989) extend Palfrey and Rosenthal model by allowing individuals to make continuous contributions. They show that the set of undominated perfect equilibrium outcomes of the subscription game is not only efficient but coincides with the core of the economy. They also show that with a dynamic version of the subscription game, it is possible to obtain efficient outcomes even if the level of the public good is not binary as long as the number of units of the public good is countable.[4] It follows that the general conclusion from the theoretical literature is that private provision of discrete public goods with complete information *can* be efficient. Moreover, efficient provision is a possible outcome for both subscription and contribution games.

An important question that has not been addressed in the literature on private provision of discrete public goods is to what extent these results generalize to a model where players have incomplete information about other players' valuation of the public good. Moreover, there is both casual evidence from the fund-raising literature of the superiority of subscription games and experimental evidence from Bagnoli and McKee (1991) and Cadsby and Maynes (1997) that a refund increases the chance of providing the good. More specifically, Cadsby and Maynes consider a discrete public goods experiment. They provide experimental evidence showing that provision is encouraged in a subscription game vis-a-vis a contribution game. They also provide evidence that a high c discourages provision in the contribution game but not in the subscription game. Our results will confirm these findings.

More specifically, we analyze contribution games and subscription games for

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[4] Admati and Perry (1991) consider a dynamic private provision model for a discrete public good. They analyze contribution and subscription games in a Rubinstein type framework with complete information where players alternate in making contributions to the public good. They show that the equilibrium of the subscription game is efficient while the equilibrium of the contribution game is inefficient.

a discrete public good in the presence of incomplete information about preferences.[5] Our model also relaxes the binary contribution restriction imposed in the literature.[6] While there are important instances where binary contributions are relevant due to transaction costs, in general players can give any amount of money they desire (alumni donation, donations to a library, etc.). Moreover, in a continuous contributions framework, individuals make contributions that best match their preferences as opposed to a discrete contribution model. Cadsby and Maynes (1997) provide experimental evidence showing that allowing continuous rather than binary "all-or-nothing " contributions facilitates provision.

Within this framework, we first characterize a solution to subscription and contribution games for the class of public goods where it is feasible for a player to provide the public good by himself. Common examples include opening a window in a hot room, rescuing an injured person in a traffic accident, etc. We show that when the cost of provision c is not prohibitively high as to prevent a single player from providing the good, there always exists an equilibrium where a player with a sufficiently large valuation provides the good. If the cost of provision is sufficiently high so that provision by a single person is not possible, this problem with incomplete information, continuous contributions, and nonconvexities is extremely difficult when we have more than two players. To see this, consider for example the contribution game with more than two players. If players $i = 2, 3, \ldots, N$ follow the contribution function $b(\cdot)$ and player 1 with valuation v contributes $x \geq 0$, his expected surplus is

$$\phi(x) = v \Pr\left(x + b(v_2) + \ldots + b(v_N) \geq c\right) - x.$$

We see from the above that if $N > 2$, to write $\phi(x)$ as a function of the distribution of individual valuations, F, detailed information on $b(.)$ is needed; it does not suffice to know that $b(.)$ is increasing in the private valuation. This is the source of the technical complexity of this problem if we were to characterize equilibria that are strictly increasing in the players' valuations for the public good. However, we are able to directly characterize equilibrium strategies that are nondecreasing.

For two-person games, we first provide necessary conditions that strictly increasing equilibria to contribution and subscription games must satisfy for general distribution functions. We then completely characterize a symmetric equilibrium of the subscription game when valuations are uniformly distributed. We show that efficiency may no longer be obtained when we introduce incomplete information. This inefficiency stems from the difficulties arising in coordinating to overcome the free-rider problem in the presence of incomplete information.

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[5] It is important to note that there are several papers that introduce incomplete information in one form or another but do not address the above questions. Palfrey and Rosenthal (1988) analyze the provision of a discrete public good when individuals have incomplete information about the degree of altruism of other players under the restriction that players are only allowed to make discrete contributions. Nitzan and Romano (1990) show that when the cost of the discrete public good is uncertain to the players, then efficiency is no longer obtained.

[6] Bagnoli and Lipman (1989) also considers continuous contributions. However their model is with complete information.

Although this type of inefficiency is well known in problems with incomplete information, we go beyond that by explicitly characterizing the trade-off involved when information is incomplete. We explicitly show the trade-off between free riding behavior and the probability of provision of the public good.

Moreover, we measure how inefficient the provision is and we are able to do that since we are able to explicitly solve for the contribution functions of the players. Our analysis shows that there is a significant difference in the equilibrium outcome between subscription and contribution games. We show that for the contribution game, for a general class of distribution functions, the only equilibrium outcome is where both players contribute zero regardless of their valuations. This confirms the evidence from the experimental literature of the superiority of subscription games over contribution games. In contrast to the mechanism design literature, our approach of characterizing equilibrium behavior in existing mechanisms enables us to provide results which can be tested in a laboratory environment or by using empirical data.

Finally, we note that the formulation of the problem that a potential donor faces appears at first to be similar to that of a bidder in a single object auction. However, the nature of these two problems are quite distinct and, therefore, so are there solutions. In an auction, there is a single winner who obtains the entire prize and, consequently, bidders want to outbid their opponents. In the private-provision-public-good game, the good is provided only if the sum of the contributions is at least equal to the cost of provision c. Thus, an individual's pledge or contribution affects the utility of other individual's positively. As a result, the solution of the public-good game is more complex than the solution of auctions-the latter is characterized by an ordinary differential equation while the former is characterized by a system of differential equations.

2 The Model

Before we present the general model, we consider an example with discrete distributions.

2.1 Example

Two players, 1 and 2, have the following valuations for a discrete public good:

$$v_i = \begin{cases} 0, \text{ with probability } \frac{1}{2} \\ 1, \text{ with probability } \frac{1}{2} \end{cases} , i = 1, 2$$

Valuations are private information. That is, each individual knows his own valuation but only the distribution of his opponent's valuation. The public good will be provided if the threshold $1 < c < 2$ is met through private contributions.

We first consider a subscription game where the money contributed by both players is returned if the sum of the pledges is less than c. If the sum of pledges is greater or equal than c, the good is provided but any resources above c are not returned to individuals.[7] This game has a symmetric Bayesian Nash equilibrium where an individual pledges \$0 if his value is 0 and \$$\frac{c}{2}$ if his value is 1.[8] Note that in this equilibrium, the sum of the pledges never exceeds c, individuals never pledge more than their values, and all equilibria are efficient, as the good is always provided whenever it is socially efficient to do so.

Consider now a contribution game where the contributions are not returned to individuals when the threshold c is not met. Recall that $1 < c < 2$. We examine this game from the perspective of Player 1. (As the game is symmetric, it does not really matter who we choose). Player 1 will of course contribute zero if his value is zero. The question is what will he contribute if his value is equal to one. We can readily verify that the equilibrium of the subscription game where each player contributes $\frac{c}{2}$ if his value is equal to one and zero if his value is equal to zero does not emerge in this game. If Player 2 follows this strategy, player 1 will make negative profits by following it as well as his profits equal $\frac{1}{2} - \frac{c}{2} < 0$ in this case. It follows (by symmetry) that a player cannot contribute $\frac{c}{2}$ or more and make positive profits. (The maximum any player could possibly contribute and make nonnegative profits is clearly $\frac{1}{2} \leq \frac{c}{2}$). Therefore, the only pure strategy equilibrium of this game is for both players to contribute zero no matter what their values are. That is, the good is never provided in equilibrium.[9] In this paper, we investigate to what extent the intuition emerging from the above example generalizes to a situation where each individual believes his opponent's value can be any of a continuum.

2.2 General Model

We now present the formal model. We consider a model where each individual i, $i = 1, ..., N$, knows his value for a certain discrete public good but only the distribution of his opponent's value. The valuations v_i are independently distributed. That is, each individual i knows that his opponent has a value v_j, $j \neq i$, that is drawn from a continuous distribution $F(\cdot)$. We denote the cost of providing the public good by c. Whatever the method to elicit donations is, the

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[7] This assumption is standard in the literature on discrete public goods. One can think of the the excess as accruing as "profits" to the provider of the public good (Nitzan and Romano 1990).

[8] "Always pledge zero" is another symmetric equilibrium. This equilibrium is known in the literature as the "strong free-riding equilibrium ". However this equilibrium involves weakly dominated strategies. As with the game of complete information, this game has infinitely many asymmetric equilibria both efficient and inefficient. In what follows we will focus on symmetric equilibria.

[9] In this footnote, we describe the equilibria for the contribution game and subscription game when information is complete. Consider a two-player provision game, where both individuals value for the public good are equal to one. The public good will be provided if the threshold $1 < c < 2$ is met through private contributions – any contributions above c are not returned to individuals. The subscription game has a continuum of equilibria, which includes the pairs $(0, 0)$, (ε, δ) — where ε and δ are such that $c - \varepsilon > 1$ and $c - \delta > 1$ — and all pairs of pledges (b_1, b_2) such that $b_1 + b_2 = c$, $b_1 \leq 1$ and $b_2 \leq 1$. Note that the set of Nash equilibria of the contribution game coincides with the set of Nash equilibria of the subscription games with the exception of the pairs of pledges (ε, δ) as described above.

good is only provided if at least \$$c$ are donated. If more than \$$c$ are donated, the additional money is not returned to the contributors.

We first characterize a solution for both contribution and subscription games when valuations for the public good can take values larger than the cost c, i.e. when $F(c) < 1$.

Theorem 1 *Suppose $F : [0, \infty) \to \mathbb{R}$ is a continuous distribution. Suppose there are $N \geq 2$ players for a project with cost $c > 0$ and that $F(c) < 1$. Then there exists an $\alpha > 0$ where α solves $\alpha F(\alpha)^{N-1} = c$ such that*

$$b(v) = \begin{cases} 0 & \text{if} \quad v \leq \alpha \\ c & \text{if} \quad v > \alpha \end{cases}$$

is an equilibrium strategy for both contribution and subscription games.

Proof. Suppose players $n = 2, ..., N$ play according to $b(\cdot)$. Let us find the best response of Player 1. If his value is $v \geq 0$ and his contribution is $x \geq 0$, his expected surplus in the contribution game is given by

$$g(x) = v \Pr(x + b(v_2) + ... + b(v_N) \geq c) - x.$$

Since $b(v_2) + ... + b(v_N)$ is either 0 or not less than c, it follows that if $x < c$ then $g(x) \leq g(0)$. Thus $\max\{g(x); x \geq 0\} = \max\{g(0), g(c)\} = \max\left\{v\left(1 - F(\alpha)^{N-1}\right), v - c\right\}$. Hence if $v < \frac{c}{F(\alpha)^{N-1}} = \alpha$, the best contribution is $x = 0$. If $v > \alpha$ the best contribution is $x = c$. And if $v = \alpha$ the player is indifferent between $x = 0$ and $x = c$. For the subscription game, the expected surplus is given by

$$f(x) = (v - x) \Pr(x + b(v_2) + ... + b(v_N) \geq c)$$

The proof is identical to the contribution game since $f(0) = g(0)$ and $f(c) = g(c)$.

Remark 1 *The solution is not unique in general. For example if $N = 2, F(x) = x, x \in [0, 1]$, then if $0 < c < 1/e$ another equilibrium strategy is $b(x) = \max\{c + k \log(x), 0\}$ where k is such that*[10] $k^k = e^{-c}$.

Remark 2 *This equilibrium is clearly not ex-post efficient. Although efficiency requires that the public good be provided whenever the sum of individual's valuations exceed c, the good is provided only if at least one individual's valuation exceeds α which is greater than c.*

Theorem 1 implies that when the cost of provision c is not prohibitively high as to prevent a single player from providing the good, there always exists an equilibrium where a player with a *sufficiently large valuation* provides the good by himself.

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[10] There are two solutions: $k_1 < 1/e < k_2$.

When a single player cannot provide the good by himself, then one would have to consider, for example, the possibility that at least two players are needed to provide the good or at least three players, and so on and so forth. This problem, as mentioned in the introduction, turns out to be technically very complex when the number of players $N > 2$, if we want to characterize equilibrium contribution functions that are strictly increasing in the players' valuations. However, we are able to directly characterize equilibria in which players strategies are nondecreasing .

For two-person games, we are able to provide necessary conditions that strictly increasing equilibrium strategies for subscription and contribution games must satisfy for any differentiable distribution function. Moreover we characterize the unique equilibrium of the contribution game for a general class of distribution functions and characterize a symmetric equilibrium of the subscription game for the uniform distribution. We turn to this in the rest of the paper.

3 The Subscription Game with N Players

In this section we analyze the subscription game i.e. where players make contributions that are refunded if the threshold is not met. Moreover we suppose the cost is such that no single player will supply the good alone. Thus we have $F(c) = 1$. For definiteness we suppose that Players' values are determined by independent draws from a continuous distribution $F : [0,1] \to \mathbb{R}$. The cost of the public good is $c \in [m, m+1)$ i.e. $m+1$ is the minimum number of players needed to provide the public good. In what follows, we characterize nondecreasing equilibria of N-Players subscription games.

Theorem 2 *Suppose that players's valuations are distributed in the* $[0,1]$ *interval. Suppose the cost of the public good is* $c \in [m, m+1)$ *and the number of players* $N \geq m+1$. *The following strategy is a symmetric equilibrium strategy of the subscription game:*

$$b(v) = \begin{cases} 0 & \text{if} \quad v \leq a; \\ \frac{c}{m+1} & \text{if} \quad a \leq v \leq 1. \end{cases}$$

where a *is such that*

$$a \frac{C_{N-1}^{m} (1 - F(a))^{m} F(a)^{N-1-m}}{\sum_{h=m}^{N-1} C_{N-1}^{h} (1 - F(a))^{h} F(a)^{N-1-h}} = \frac{c}{m+1}. \tag{1}$$

and $C_n^h = \frac{n!}{h!(n-h)!}$.

Proof. We first prove the existence of a. Define

$$h(a) = a \frac{C_{N-1}^{m} (1 - F(a))^{m} F(a)^{N-1-m}}{\sum_{h=m}^{N-1} C_{N-1}^{h} (1 - F(a))^{h} F(a)^{N-1-h}} = \frac{a C_{N-1}^{m}}{C_{N-1}^{m} + \sum\limits_{h=m+1}^{N-1} C_{N-1}^{h} (1 - F(a))^{h-m} F(a)^{m-h}}.$$

Note that

$$\lim_{a \to 0} h(a) = 0 \text{ and } \lim_{a \to 1} h(a) = \frac{C_{N-1}^{m}}{C_{N-1}^{m}} = 1$$

Since h is a continuous function, by the intermediate value theorem there exists an a such that $h(a) = \frac{c}{m+1} \in (0,1)$.

We now prove that $b(\cdot)$ is an equilibrium. Suppose $v < 1$ and player 1 pledges $x \geq 0$ and players $n = 2, \ldots, N$ follow $b(v_n)$. The expected utility of player 1 is given by

$$\phi(x) = (v - x) \Pr\left(\sum_{n=2}^{N} b(v_n) \geq c - x\right).$$

The range of $\sum_{n=2}^{N} b(v_n)$ is $\left\{0, \frac{c}{m+1}, \frac{2c}{m+1}, \ldots, \frac{(N-1)c}{m+1}\right\}$.

If $c - x \in (\frac{jc}{m+1}, \frac{(j+1)c}{m+1}]$, then

$$\phi(x) = (v - x) \Pr\left(\sum_{n=2}^{N} b(v_n) \geq \frac{(j+1)c}{m+1}\right) \leq \phi\left(c - \frac{(j+1)c}{m+1}\right)$$

Thus the optimal pledge $x^* \leq v$ is such that

$$x^* \in \left\{c - \frac{(j+1)c}{m+1}; j \geq -1\right\}.$$

Thus

$$x^* = c - \frac{(j^*+1)c}{m+1} \in [0,1),\ j^* \geq -1.$$

Therefore $j^* \in \{m, m-1\}$ since $\frac{2c}{m+1} \geq 2\frac{m}{m+1} \geq 1$. Finally we have $j^* = m - 1$ if and only if $\phi\left(\frac{c}{m+1}\right) \geq \phi(0)$ or equivalently if and only if

$$\left(v - \frac{c}{m+1}\right) \Pr\left(\sum_{n=2}^{N} b(v_n) \geq c - \frac{c}{m+1}\right) \geq v \Pr\left(\sum_{n=2}^{N} b(v_n) \geq c\right).$$

or

$$v \Pr\left(\sum_{n=2}^{N} b(v_n) = \frac{mc}{m+1}\right) \geq \frac{c}{m+1} \Pr\left(\sum_{n=2}^{N} b(v_n) \geq \frac{mc}{m+1}\right).$$

Since

$$\Pr\left(\sum_{n=2}^{N} b(v_n) = \frac{mc}{m+1}\right) = C_{N-1}^{m} (1 - F(a))^{m} F(a)^{N-1-m}$$

and

$$\Pr\left(\sum_{n=2}^{N} b(v_n) \geq \frac{mc}{m+1}\right) = \sum_{h=m}^{N-1} C_{N-1}^{h} (1 - F(a))^{h} F(a)^{N-1-h}$$

we conclude that $\phi\left(\frac{c}{m+1}\right) \geq \phi(0)$ if and only if $v \geq a$.

Remark 3 *Clearly, the above equilibrium is not ex-post efficient. Efficiency requires that the good be provided when the sum of the players' valuations exceeds c. In this equilibrium, if the good is provided then the sum of the player 's valuations is at least* $(m+1)a > c$.

Remark 4 *If* $N = m+1$, *then the equilibrium strategy has a very nice form:*

$$b(v) = \begin{cases} 0 & \text{if} \quad v \leq \frac{c}{m+1} \\ \frac{c}{m+1} & \text{if} \quad \frac{c}{m+1} \leq v \leq 1. \end{cases}$$

Thus each player consider the cost equally divided among the players and bids if and only if his value is at least his share.

4 The Subscription Game with Two Players

In this section, we characterize strictly increasing equilibrium strategies for two-players subscription games. Player 1's expected surplus given that he has a value v, contributes $x \geq 0$ and, given that $b : [0,1] \to \mathbb{R}$ is the strictly increasing contribution function of Player 2, is given by

$$\begin{aligned} \phi(x) &= (v-x)\Pr(b(v_2) \geq c-x) \\ &= (v-x)\Pr\left(v_2 \geq b^{-1}(c-x)\right) \\ &= (v-x)\left(1-F\left(b^{-1}(c-x)\right)\right). \end{aligned}$$

That is, Player 1's expected payoff is simply his surplus if the project is completed times the probability of completion. The first-order condition is

$$\phi'(x) = -\left(1-F\left(b^{-1}(c-x)\right)\right) + (v-x) f\left(b^{-1}(c-x)\right)\left(b^{-1}\right)'(v-x) = 0.$$

In a symmetric equilibrium $x = b(v)$, it follows that

$$(v-b(v)) f\left(b^{-1}(c-b(v))\right)\left(b^{-1}\right)'(c-b(v)) = 1-F\left(b^{-1}(c-b(v))\right).$$

This is not an ordinary differential equation since the function $b(v)$ appears inside the argument. To solve it, we first define the following change of variables $G(v) = b^{-1}(c-b(v))$. This implies that

$$\begin{aligned} b(G(v)) + b(v) &= c \\ (v-b(v)) f(G(v))\left(b^{-1}\right)'(b(G(v))) &= 1-F(G(v)). \end{aligned}$$

Since

$$\left(b^{-1}\right)'(b(G(v))) = 1/b'(G(v))$$

substituting this in the last equations and using the fact that $G(G(v)) = v$, we get

$$b'(v) = \frac{(G(v)-b(G(v))) f(v)}{1-F(v)}.$$

Differentiating the first equation above we obtain $b'(G(v)) + b'(v) = 0$. The following theorem describes the system of differential equations that an increasing equilibrium strategy—if there is one— to the subscription game must satisfy.

Theorem 3 *The solution of the system of differential equations below, with initial conditions defined by the primitives of the problem, characterizes a symmetric equilibrium involving strictly increasing strategies for the subscription game with two players with values determined by independent draws from a distribution $F : [0,1] \to \mathbb{R}$, where F is continuously differentiable and $f = F' > 0$ everywhere.*

$$\begin{aligned} b'(v) &= \frac{(G(v) + b(v) - c) f(v)}{1 - F(v)}, \\ G'(v) &= \frac{(1 - F(G(v)))(G(v) - c + b(v)) f(v)}{(1 - F(v))(v - b(v)) f(G(v))}. \end{aligned}$$

The above nonlinear system of ordinary differential equations that governs the subscription game will have to be solved for specific distribution functions to get a closed form solution. We are able to solve this system in the case of a uniform $[0,1]$ distribution.

In what follows, we provide some intuition how to solve this problem when we take into account the boundary conditions before we present the formal result in the next proposition.

For the uniform distribution on the interval $[0,1]$, the unconstrained solution to the above system of differential equations is given by the following pledging function:

$$b(v) = \frac{2c-1}{6} + \frac{v}{2} \tag{2}$$

Note, however, that in equilibrium a player may not follow $b(v)$ for any v in $[0,1]$ as this may lead to pledging more than his value or more than the cost of the public good. Hence we need to impose the following boundary conditions

$$b(v) \leq c \tag{3}$$

$$b(v) \leq v \tag{4}$$

$$b(v) \geq 0 \tag{5}$$

It turns out that (3) and (5) are not binding since $c > 1$. Condition (4) is binding as $b(v) > v$ for $v < \frac{2c-1}{3}$.

The next proposition characterizes a symmetric Bayesian equilibrium of the subscription game.

Proposition 1 *The following is a symmetric equilibrium pledging strategy for the subscription game with two players whose values are uniformly distributed* *on* $[0,1]$ *and* $1 < c < 2$*:*

$$b^*(v) = \begin{cases} \frac{2c-1}{6} + \frac{v}{2}, \ \textit{if} \ \frac{2c-1}{3} \leq v \leq 1 \\ 0, \ \textit{otherwise} \end{cases}$$

Proof. Given that Player 2 is following the proposed equilibrium pledging strategy, we have to find the best response of player 1. We first show that for $\frac{2c-1}{3} \leq v \leq 1$, $b_1(v) = \frac{2c-1}{6} + \frac{v}{2}$ is a best response to $b^*(v_2)$. Player 1's expected surplus, if he pledges b, is given by

$$\phi\left(b\right) = \left(v - b\right) \Pr\left(b + b^*(v_2) \geq c\right).$$

To find the maximum of ϕ first note that if $0 < c - b < b^*\left(\frac{2c-1}{3}\right) = \frac{2c-1}{3}$ then $\phi\left(b\right) = \left(v - b\right)\left(1 - \frac{2c-1}{3}\right) \leq \left(v - \frac{c+1}{3}\right)\left(1 - \frac{2c-1}{3}\right) = \phi\left(\ \frac{c+1}{3}\right)$. Note that if $c - b \geq b^*\left(1\right)$ then $\phi\left(b\right) = 0$. Let us consider now $c - b^*\left(1\right) < b < c - b^*\left(\frac{2c-1}{3}\right)$. Then we have

$$\begin{aligned} \phi\left(b\right) &= \left(v - b\right) \Pr\left(\left\{v_2 \geq \frac{2c-1}{3}; v_2 \geq \frac{4c+1}{3} - 2b\right\}\right) = \\ & \left(v - b\right)\left(1 - \max\left\{\frac{2c-1}{3}, \frac{4c+1}{3} - 2b\right\}\right). \end{aligned}$$

There are two cases to consider:

1) $\frac{2c-1}{3} > \frac{4c+1}{3} - 2b$

In this case $\phi\left(b\right) = \left(v - b\right)\left(1 - \frac{2c-1}{3}\right) < \phi\left(\frac{c+1}{3}\right)$.

2) $\frac{2c-1}{3} \leq \frac{4c+1}{3} - 2b$

In this case $\phi\left(b\right) = \left(v - b\right)\left(1 - \frac{4c+1}{3} + 2b\right)$. This quadratic function has a unique maximum at $b^* = \frac{2c-1}{6} + \frac{v}{2}$. Thus b^* is the optimal pledge if $b^* \in \left[c - b\left(1\right), c - b\left(\frac{2c-1}{3}\right)\right]$ and $\frac{2c-1}{3} \leq \frac{4c+1}{3} - 2b^*$. The last inequality is valid for all $v \in [0,1]$. The first inequality is valid if $v \in [\frac{2c-1}{3}, 1]$. Thus $b(v) = \frac{2c-1}{6} + \frac{v}{2}$ is the best response to $b^*(v_2)$ if $v \in [\frac{2c-1}{3},\ 1]$. To finish let us find the best response for $v \in [0, \frac{2c-1}{3})$. It is clear from the reasoning in (2) above that the maximum of ϕ is not interior. Thus we need only to compare $\phi\left(c - b^*\left(1\right)\right) = \phi\left(\frac{2c-1}{3}\right) = \left(v - \frac{2c-1}{3}\right)\left(1 - \frac{4c+1}{3} + 2\frac{2c-1}{3}\right) = 0$ and $\phi\left(c - b^*\left(\frac{2c-1}{3}\right)\right) = \phi\left(\frac{c+1}{3}\right) = \left(v - \frac{c+1}{3}\right)\left(1 - \frac{2c-1}{3}\right) < 0$. Thus if $v \in [0, \frac{2c-1}{3})$ the maximum expected profit is zero. Hence since pledging zero and pledging $\frac{2c-1}{3}$ give the same expected surplus, we finished the proof that $b\left(\cdot\right)$ is an equilibrium.

The above proposition shows an equilibrium pledging strategy that is strictly increasing and differentiable by parts in the relevant range and, therefore, our previous analysis is justified.

Remark 5 *There are other non decreasing equilibrium pledging functions, both symmetric and asymmetric. The two other symmetric equilibria are "always pledge zero" and "pledge* $c/2$ *if one's value is greater than* $c/2$ *and pledge zero* *otherwise." There is a continuum of asymmetric equilibria where player 1 pledges αc if 1's value is above αc and zero otherwise and player 2 pledges $(1-\alpha)c$ if 2's value is above $(1-\alpha)c$ and zero otherwise, just as in the case of complete information.*

We now provide the intuition for the solution of the subscription game and compare it to the equilibrium in a first price sealed-bid auction. Recall that in a symmetric equilibrium of a first-price sealed-bid auction — where the object is awarded to the individual with the highest bid — an individual bids in such a way to outbid the opponent with the highest value. That is, conditional on his value being the highest, his bid is equal to the expected value of the first-order statistics of his opponents.

In the subscription game, however, the good is provided to both players if their contributions add up to the cost of provision c. Thus, the problem becomes one of forecasting the lowest pledge one can make, given that it is below one's value, and still have the good being provided. Thus, the equilibrium pledging strategy implies that a player pledges the equivalent to the expected value of the other player being lower than his own, conditional on the interval $[\frac{2c-1}{3}, 1]$, that is, on the relevant interval where pledges are less than or equal to the valuations for the public good, i.e. $b(v) \leq v$.

Notice the distinction between the solution of the subscription game and the solution of the first-price auction. In the latter, if a bidder's value is not the highest, then in any symmetric equilibrium with increasing bids he will lose the object and therefore, in equilibrium, he does not have to consider what he would do if his value is not the highest one. In the subscription game this is not the case. If his value is the lowest of the two, he may still obtain the object and, thus, following (*) guarantees that this is the minimum pledge so that the object is provided and the players are sharing the cost in such way as to equalize their marginal contributions. This property of the equilibrium pledging strategies is very distinct from the result for first-price auctions and it captures the nature of the trade-off between the probability of the public good being provided and the free-riding behavior.

The following example illustrates that the equilibrium of the subscription game is inefficient.

Example 1 *For expositional ease, we assume that the realizations of v_1 and v_2 are both equal to $\frac{1}{2}$ and that $c = 1$. The predicted symmetric equilibrium according to Proposition 1 is for both players to pledge $b^*(v) = \frac{1}{6} + \frac{v}{2} = \frac{5}{12}$. Since $c = 1 \geq \frac{5}{12} + \frac{5}{12}$, the good is not provided in this equilibrium, although it is efficient to do so once we know individuals' valuations. Let us consider the game from Player 1's perspective and find what is his best response to Player 2 playing the proposed equilibrium strategy*

$$b^*(v_2) = \begin{cases} \frac{1}{6} + \frac{v_2}{2}, & \text{if } \frac{1}{3} \leq v_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Player 1's expected payoff is given by:

$$\pi_1\left(\frac{1}{2}, b_1, b^*(v_2)\right) = \left(\frac{1}{2} - b_1\right)\Pr\left(b_1 + b^*(v_2) \geq 1\right).$$

Maximizing with respect to b_1 *yields* $b_1 = \frac{5}{12}$ *as expected.*

Example 1 illustrates that, as one would expect, the subscription game may not be ex-post efficient. Ex-post efficiency is perhaps a very strong requirement. The next proposition provides an alternative measure of efficiency. It indicates the probability that the good will be provided whenever is efficient to do so.

Proposition 2 $Pr\left(b^*(v_1) + b^*(v_2) \geq c | v_1 + v_2 \geq c\right) = \frac{2}{3}$

Proof. Since $b^*(v_1) + b^*(v_2) \geq c$ implies that $v_1 + v_2 \geq c$ we need to compute

$$\frac{\Pr\left(\{(v_1, v_2); b^*(v_1) + b^*(v_2) \geq c\}\right)}{\Pr\left(\{(v_1, v_2); v_1 + v_2 \geq c\}\right)}.$$

Note that

$$\Pr\left(\{(v_1, v_2); b^*(v_1) + b^*(v_2) \geq c\ \}\right) =$$
$$\frac{1}{2}\left(\frac{4-2c}{3}\right)^2 + \left(\frac{2c-1}{3} - (c-1)\right)\left(1 - \frac{c+1}{3}\right) = \frac{4}{3} - \frac{4}{3}c + \frac{1}{3}c^2 = \frac{1}{3}(c-2)^2 .$$

As $\Pr\left(\{(v_1, v_2); v_1 + v_2 \geq c\}\right) = \frac{1}{2}(2-c)^2$ we finish the proof.

Notice that an increase in c affects this probability by two opposing effects. An increase in c causes $b(\cdot)$ to increase but the interval for which $b(\cdot)$ is different from zero shrinks. The random variable $v_1 + v_2$ has a triangular distribution (as the sum of two random variables that are uniformly distributed). Its density peaks at $c = 1$. Beyond this point, these two opposing effects completely offset each other as shown above. Therefore, a grant towards reducing the cost of provision has no effect on the probability of provision as it is perfectly offset by individual's behavior in equilibrium.

5 The Contribution Game with Two Players

In this section we analyze the contribution game where players make contributions that are not refunded if the threshold is not met. Players' values v_i are determined by independent draws from a distribution $F : [0,1] \to \mathbb{R}$, where F is continuously differentiable and $f = F' > 0$ everywhere.

Player 1's expected surplus given that he has a value v, contributes $x \geq 0$ and, given that $b : [0,1] \to \mathbb{R}$ is the strictly increasing contribution function of Player 2, is given by

$$\phi\left(x\right) = v \Pr\left(x + b\left(v_2\right) \geq c\right) - x.$$

where v_2 denotes 2's valuation (unknown to Player 1). That is, Player 1's expected payoff is equal to the expected value of completing the project minus his contribution. Note that player 1 pays x independently of whether the project is completed or not. Therefore

$$\phi\left(x\right) = v \Pr\left(v_2 \geq b^{-1}(c - x)\right) - x = v\left(1 - F\left(b^{-1}(c - x)\right)\right) - x.$$

Hence the first-order condition is given by

$$\phi'\left(x\right) = vf\left(b^{-1}(c - x)\right)\left(b^{-1}\right)'(c - x) - 1 = 0.$$

In a symmetric equilibrium, $x = b\left(v\right)$ is the solution. Thus in a symmetric equilibrium we have

$$vf\left(b^{-1}\left(c - b\left(v\right)\right)\right)\left(b^{-1}\right)'\left(c - b\left(v\right)\right) = 1.$$

This leads to a differential equation that is not standard since the function itself appears in the argument. To solve it we first define the following change of variables $G\left(v\right) = b^{-1}\left(c - b\left(v\right)\right)$. It follows that

$$b\left(G(v)\right) + b\left(v\right) = c \tag{6}$$

$$vf(G(v))(b^{-1})'(b(G(v))) = 1 \tag{7}$$

The following theorem describes the system of differential equations that an increasing equilibrium strategy—if there is one—to the contribution game must satisfy.

Theorem 4 *The solution of the system of differential equations below with initial conditions defined by the primitives of the problem, characterizes a symmetric equilibrium involving strictly increasing strategies for the contribution game with two players with values determined by independent draws from a distribution $F : [0,1] \to \mathbb{R}$, where F is continuously differentiable and $f = F' > 0$ everywhere:*

$$\begin{aligned} b'\left(v\right) &= G\left(v\right) f\left(v\right), \\ G'\left(v\right) &= -\frac{G\left(v\right) f\left(v\right)}{vf\left(G\left(v\right)\right)}. \end{aligned}$$

Proof. First note that since $G\left(v\right) = b^{-1}\left(c - b\left(v\right)\right)$, it follows that

$$b\left(G\left(G(v)\right)\right) + b\left(G(v)\right) = c = b\left(G(v)\right) + b\left(v\right)$$

This implies that

$$b\left(G\left(G(v)\right)\right) = b\left(v\right)$$

Hence $G\left(G(v)\right) = v$. Therefore we can rewrite equation (2) using the fact that

$$\left(b^{-1}\right)'\left(b(G(v))\right) = \left(b'\left(b^{-1}(b\left(G\left(v\right)\right))\right)\right)^{-1} = \left(b'\left(G\left(v\right)\right)\right)^{-1}$$

as

$$b'\left(G\left(v\right)\right) = vf\left(G\left(v\right)\right) \ \Rightarrow b'\left(v\right) = G\left(v\right)f\left(v\right)$$

Differentiating equation (1) we obtain

$$b'\left(G\left(v\right)\right)G'\left(v\right) + b'\left(v\right) = 0.$$

Finally we get the following system of ordinary differential equations as desired.

$$\begin{aligned} b'\left(v\right) &= G\left(v\right)f\left(v\right), \\ G'\left(v\right) &= -\frac{G\left(v\right)f\left(v\right)}{vf\left(G\left(v\right)\right)}. \end{aligned}$$

Example 2 *Suppose* $f\left(v\right) = 1$ *then* $G\left(v\right) = \frac{k}{v}$ *and* $b\left(v\right) = a + k\log\left(v\right)$ *is a solution of the system. It shall be clear that* $b\left(\cdot\right)$ *cannot be a solution on the whole interval* $[0,1]$ *since it is eventually negative for small* v*. If* $0 < c < 1/e$ *an equilibrium strategy is* $b\left(x\right) = \max\left\{c + k\log\left(x\right), 0\right\}$ *where* k *is such that* $k^k = e^{-c}$.

The above nonlinear system of ordinary differential equations will have to be solved for specific distribution functions to get a closed form solution. However, we are able to show directly that the strategy "contributing zero" for both players is the only equilibrium for a general family of distributions.

Proposition 3 *Suppose* $1 < c < 2$*. If the distribution function satisfies* $F(z) \geq z$, $z \in (0,1)$ *then the best response to* $b_2(\cdot)$ *such that* $b_2\left(v_2\right) \leq v_2$ *is* $b_1\left(v_1\right) = 0$.

Proof. Define $\phi\left(x\right) = vP(b_2(v_2) \geq c - x) - x$. If $x > v$ then $\phi\left(x\right) \leq v - x < 0$. Thus to maximize ϕ we must have $x \leq v$. Suppose $x > 0$. If $x < c - 1$ then $\phi(x) = v \cdot 0 - x = -x < 0$. Suppose now $c - 1 \leq x \leq v$. We have $\phi(c-1) = -x$ and $\phi(v) \leq 0$. Now:

$$\phi\left(x\right) \leq vP(v_2 \geq c - x) - x = v(1 - F(c - x)) - x =: g(x).$$

Now

$$g(x) \leq v(1 - (c - x)) - x = v - c + (v - 1)x < 0.$$

Note that the above family of distributions includes well known distributions such as the uniform and the exponential distributions. More generally any concave distribution satisfies it. This implies that the coordination problem is so severe in the contribution game that "contributing zero" is the only equilibrium outcome for the above class of distribution functions.

Of course, if the distributions are highly skewed towards high valuations, then we get an equilibrium with nonzero contributions as the following proposition shows.

Proposition 4 *Suppose the distribution F has support $[0,1]$, that $c > 1$ and that there exists an $\alpha \in (0,1)$ such that $F(\frac{c}{2\alpha}) = 1 - \alpha$. Then*

$$b(v) = \begin{cases} \frac{c}{2} & \text{if} \quad v \geq \frac{c}{2\alpha} \\ 0 & \text{if} \quad v < \frac{c}{2\alpha} \end{cases}$$

is an equilibrium strategy for the contribution game with two players.

Proof. Define $\phi(x) = v \Pr(b(v_2) \geq c - x) - x$, $x \geq 0$. If $c \leq x$, $\phi(x) = v - x \leq v - c < 0$. If $\frac{c}{2} \geq c - x > 0$, $\phi(x) = v\left(1 - F\left(\frac{c}{2\alpha}\right)\right) - x = v\alpha - x \leq \phi\left(\frac{c}{2}\right)$. If $c - x > \frac{c}{2}$, $\phi(x) = -x \leq \phi(0)$. Thus

$$\max\{\phi(x); x \geq 0\} = \max\left\{\phi(0), \phi\left(\frac{c}{2}\right)\right\} = \max\left\{0, v\alpha - \frac{c}{2}\right\}.$$

Thus $x = \frac{c}{2}$ maximizes ϕ if and only if $v \geq \frac{c}{2\alpha}$, and $x = 0$ otherwise.

Example 3 *If $F(x) = x^{\gamma}$, α is a solution of $(\frac{c}{2})^{\gamma} = \alpha^{\gamma}(1-\alpha)$. Since $\max\{\alpha^{\gamma}(1-\alpha); \alpha \in [0,1]\} = \left(\frac{\gamma}{\gamma+1}\right)^{\gamma}\left(\frac{1}{\gamma+1}\right)$ there is a solution if $c \leq 2\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma}}\frac{\gamma}{\gamma+1}$. For example if $\gamma = 13, c = \frac{3}{2}$ we have $\alpha = 0.88$.*

Remark 6 *If there is an α for the distribution F then there is an α for a distribution $G \leq F$. The proof is simple and will be omitted.*

6 Conclusion

We have developed a model of private provision of discrete public goods with incomplete information and continuous contributions. Within this framework, we first characterized the equilibrium to subscription games and contribution games when a single player can provide the good by himself. We then analyzed N-person contribution and subscription games.

Our analysis showed that, unlike the model with incomplete information, efficiency is no longer a possible outcome. This inefficiency stems from the difficulties arising in coordinating to overcome the free-rider problem in the presence of incomplete information. Although this type of inefficiency is well known in economic theory, we go beyond that by explicitly characterizing the trade-off involved when information is incomplete. More specifically, we explicitly analyzed the trade-off between free riding behavior and the probability of provision of the public good. We measure how inefficient the provision is and we are able to do that since we are able to explicitly solve for the contribution functions of the players.

Finally, we showed that there is a significant difference in the equilibrium outcome between subscription and contribution games for a general class of distribution functions. We showed that for the contribution game, for a class of distribution functions, the only equilibrium outcome is where both players contribute zero regardless of their valuations. This confirms the evidence from the experimental literature of the superiority of subscription games over contribution games.

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