

Optimal all-pay auction when signals are correlated.

Paulo Klinger Monteiro*
CORE†
34 voie du Roman Pays
1348 Louvain-la-Neuve
Belgium

January 1999

Abstract

In this note I prove the existence of the optimal all-pay auction when signals are correlated.

JEL: D44

1 Introduction.

An all-pay auction is an auction in which every bidder pays his bid and the highest bidder receives the object. Although not the most common auction, the all-pay auction also happens in the practice. For example in a tournament, the effort of each participant can be seen as his “bid”. A raffle is an all-pay auction in which every participant buys one or more of a fixed number of tickets and the winner is chosen randomly among the sold tickets. Alternatively we can look at the all-pay auction as a generalized raffle. The

*I acknowledge the suggestions of Fernando Branco. The financial support of a Guggenheim fellowship is gratefully acknowledged.

†Present affiliation: IMPA, Estrada Dona Castorina 110, 22460-320, Rio de Janeiro RJ, Brazil; email: pklm@impa.br

theoretical studies of all-pay auctions are not very numerous. For example, Bayer et al.(1996) study the all-pay auction with complete information. Krishna and Morgan(1997) study the war of attrition and the all-pay auction. In this paper I study the optimal all-pay auction when signals are correlated.

The study of optimal auctions begun with Myerson(1981). There the optimal auction of one object when the bidders signals are stochastically independent is determined. In the symmetric case the optimal auction is a second price sealed bid auction with a reserve price. However not much is known about optimal auctions if signals are not stochastically independent. Page(1998) proves the existence of optimal auctions in a general setting. An explicit characterization of the optimal auction is, naturally, not possible in the very general model of Page. However suppose we want the optimal auction of one object when signals are correlated. Is there a characterization like Myerson's in this case? This is not known in general. I will show that if we restrict ourselves to all-pay auctions then it is possible to obtain the optimal all-pay auction and its format is analogous to the independent signals optimal auction format.

2 The model.

The set of bidders is $\mathbb{I} = \{1, 2, \dots, I\}$. There is one object to be sold at auction. Bidder i has a signal $X_i : \Omega \rightarrow \mathbb{R}_+$, a random variable defined on the probability space (Ω, \mathcal{A}, P) . The random vector of all signals is $X = (X_1, \dots, X_I)$. It is possible to consider an unobserved signal $\mathbb{S} : \Omega \rightarrow \mathbb{R}_+$ as in Milgrom and Weber(1988). This would permit the inclusion of common value models. However this generalization doesn't present special difficulties but is notationally uncomfortable. I therefore decided for the more concise model. Finally Bidder's i evaluation of the object is the random variable $V_i = u_i(X)$, where $u_i : \mathbb{R}_+^I \rightarrow \mathbb{R}$ is a continuously differentiable function. Define $Z = \prod_{i \in \mathbb{I}} [m_i, n_i]$. The distribution, F , of the random vector X has a continuously differentiable density $f : Z \rightarrow \mathbb{R}_{++}$, $\int_Z f(x) dx = 1$. For each vector $x = (x_1, \dots, x_I)$ define the vector $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_I)$ obtained by the removal of the i^{th} coordinate of x . I also write $x = (x_i, x_{-i})$. The distribution function of X^i , F^i , has density f^i , $f^i(x_i) = \int_{Z_{-i}} f(x_i, x_{-i}) dx_{-i}$. Here $Z_{-i} = \{z_{-i}; z \in Z\}$. It is convenient to include the auctioneer as bidder 0. Thus $\mathbb{B} = \mathbb{I} \cup \{0\}$ is the set of bidders. If $A \neq \emptyset$ is finite define

$S(A) = \{(q_a)_{a \in A}; \sum_{a \in A} q_a = 1 \text{ and for every } a \in A, q_a \geq 0\}$.

Lemma 1 *The densities $f(x)$, and $f(x_{-i}|x_i)$ are (uniformly) lipschitzian in x_i .*

Proof. The density f is lipschitzian since its support is compact and its derivative is continuous. From

$$|f^i(a) - f^i(b)| \leq \int |f(a, x_{-i}) - f(b, x_{-i})| dx_{-i} \leq L'|a - b|$$

it follows that f^i is lipschitzian too. Then using that $\inf_{z \in Z} f(z) > 0$ and the inequality:

$$\left| \frac{f(a, x_{-i})}{f^i(a)} - \frac{f(b, x_{-i})}{f^i(b)} \right| \leq \frac{|f(a, x_{-i}) - f(b, x_{-i})|}{f^i(a)} + f(b, x_{-i}) \frac{|f^i(b) - f^i(a)|}{f^i(a)f^i(b)}$$

we conclude that $f(x_{-i}|x_i) = \frac{1}{f^i(x_i)}f(x)$ is lipschitzian. QED

The auction will proceed in the following way:

1. The auctioneer publicly announces the functions¹ $q : \mathbb{R}_+^{\mathbb{I}} \rightarrow S(\mathbb{B})$ and $P = (P^1, \dots, P^I), P^i : \mathbb{R}_+ \rightarrow \mathbb{R}, 1 \leq i \leq I$;
2. Each bidder knowing his signal, X_i , announces a number $x_i \geq 0$ privately to the auctioneer. The auctioneer forms the vector $x = (x_1, \dots, x_I)$;
3. Bidder $i \in \mathbb{I}$ pays $P^i(x_i)$;
4. The objects are delivered accordingly to $u \in \mathbb{B}$ drawn with probability $q_u(x) = q(x)(u), u \in \mathbb{B}$.

As in Myerson(1981), pages 62-63, the direct mechanisms (q, P) will be chosen among the direct mechanisms that satisfy individual rationality and incentive compatibility constraints. Define the function $T_i(x_i)$ as the expected utility of bidder i if his signal is x_i . Thus we have that

$$T_i(x_i) = \int q_i(x) u_i(x) f(x_{-i}|x_i) dx_{-i} - P^i(x_i). \quad (1)$$

¹Henceforward called direct mechanisms.

If i has valuation x_i and declares valuation x'_i then we have the incentive compatibility constraints:

$$T_i(x_i) \geq \int q_i(x'_i, x_{-i}) u_i(x) f(x_{-i}|x_i) dx_{-i} - P^i(x'_i). \quad (2)$$

The individual rationality constraints are:

$$T_i(x_i) \geq 0. \quad (3)$$

Lemma 2 *The following is true:*

1. T_i is a primitive: $T_i(z) = T_i(0) + \int_0^z T'_i(u) du$ for every $z \geq 0$;
2. For almost every $x_i \in (m_i, n_i)$,

$$T'_i(x_i) = \int q_i(x) D_{x_i} (u_i(x) f(x_{-i}|x_i)) dx_{-i}. \quad (4)$$

Before proving the lemma we define $\hat{u}_i(x) = u_i(x) f(x_{-i}|x_i)$.

Proof. Suppose $m_i < a < b < n_i$. From the incentive compatibility constraints we have

$$T_i(b) - T_i(a) \geq \int q_i(a, x_{-i}) [\hat{u}_i(b, x_{-i}) - \hat{u}_i(a, x_{-i})] dx_{-i}.$$

There exists K such that $|\hat{u}_i(b, x_{-i}) - \hat{u}_i(a, x_{-i})| \leq K|b - a|$ and $|f(x_{-i}|b) - f(x_{-i}|a)| \leq K|b - a|$ for every x_{-i} . Since $\sum_{i=0}^I q_i(x) = 1$ we have that

$$T_i(b) - T_i(a) \geq -K|b - a|.$$

Changing places between a and b we obtain $T_i(a) - T_i(b) \geq -K|b - a|$. So $|T_i(b) - T_i(a)| \leq K|b - a|$. Therefore T_i is absolutely continuous since it is lipschitz. Hence it is almost everywhere differentiable and a primitive. This proves item (1). Suppose $a \in (m_i, n_i)$ is a point of differentiability of T_i . If $b > a$ then

$$\frac{T_i(b) - T_i(a)}{b - a} \geq \int q_i(a, x_{-i}) \frac{\hat{u}_i(b, x_{-i}) - \hat{u}_i(a, x_{-i})}{b - a} dx_{-i}. \quad (5)$$

Hence using the dominated convergence theorem,

$$T'_i(a) \geq \int q_i(a, x_{-i}) D_a \widehat{u}_i(a, x_{-i}) dx_{-i}. \quad (6)$$

Analogously if $b < a$ we obtain the inequality (6) reversed and therefore (4).
QED

Lemma 3 *If $D_z(u_i(z, x_{-i})f(x_{-i}|z)) \geq 0$ for every $(z, x_{-i}) \in Z$ and $q_i(z, x_{-i})$ is increasing in z then the mechanism $(q_i, \widehat{P}^i)_{i \in \mathbb{I}}$ satisfy the incentive compatibility constraints (2).*

The proof is easy and will be omitted.

2.1 The auctioneer revenue.

From the definition of T_i we obtain the payment function, \widehat{P}^i :

$$\widehat{P}^i(z) = \int q_i(z, x_{-i}) \widehat{u}_i(z, x_{-i}) dx_{-i} - T_i(0) - \int_0^z T'_i(w) dw.$$

Since the individual rationality constraint requires that $T_i(0) \geq 0$ the auctioneer will choose $T_i(0) = 0$. Therefore the auctioneer expected revenue from Bidder i is

$$\begin{aligned} \int P^i(x_i) f(x) dx &= \int P^i(x_i) f^i(x_i) dx_i = \\ &= \int q_i(x) u_i(x) f(x_{-i}|x_i) f^i(x_i) dx - \int \chi_{(0, x_i)}(w) T'_i(w) f^i(x_i) dw dx_i = \\ &= \int q_i(x) u_i(x) f(x) dx - \int q_i(w, x_{-i}) (1 - F^i(w)) D_w(\widehat{u}_i(w, x_{-i})) dx_{-i} dw. \end{aligned}$$

Above I used (4) to eliminate T'_i . Define

$$h^i(x) = u_i(x) - \frac{(1 - F^i(x_i)) D_{x_i}(u_i(x) f(x_{-i}|x_i))}{f(x)} \quad (7)$$

if $i \in \mathbb{I}$ and $h^0(x) = 0$. The auctioneer revenue is

$$\int \sum_{i=1}^I P^i(x) f(x) dx = \int \sum_{i=1}^I q_i(x) h^i(x) f(x) dx = \int \sum_{i=0}^I q_i(x) h^i(x) f(x) dx.$$

The following hypothesis permits to obtain the optimal auction in analogy with Myerson(1981):

$$h^i(x) = h^j(x) \text{ and } i \neq j \Rightarrow \frac{\partial}{\partial x_i}(h^i(x'_i, x_{-i}) - h^j(x'_i, x_{-i})) > 0, \forall x'_i > x_i. \quad (8)$$

Theorem 1 *Suppose (8) and that $D_{x_i}(u_i(x)f(x_{-i}|x_i)) \geq 0$ for every x_i . Then the optimal auction $(q_i, \hat{P}^i)_{i \in \mathbb{I}}$ has the following form*

1. $q_i(x) = \begin{cases} 0 & \text{if } h^i(x) < \max_{j \in \mathbb{B}} h^j(x); \\ \frac{1}{\#\{i \in \mathbb{I}; h^i(x) = \max_{j \in \mathbb{B}} h^j(x)\}} & \text{if } h^i(x) = \max_{j \in \mathbb{B}} h^j(x). \end{cases}$
2. $\hat{P}^i(x_i) = \int q_i(x) \hat{u}_i(x) dx_{-i} - \int_0^{x_i} [q_i(w, x_{-i}) D_w \hat{u}_i(w, x_{-i}) dx_{-i}] dw.$

Moreover the auctioneer expected revenue is $\int \max_{i \in \mathbb{B}} h^i(x) f(x) dx.$

Corollary 1 *The auctioneer expected revenue is less than $\int \max_{i \in \mathbb{I}} u^i(x) f(x) dx$ if for every i , $D_{x_i} \hat{u}_i(x) > 0$ for every $x \in Z$.*

Proof. Immediate since $h^i(x) < u^i(x)$ for every $i \in \mathbb{I}$.

Remark 1 *Corollary 1 shows that in general the optimal all-pay auction doesn't extract all the surplus. It however sometimes do as an example in Bulow and Klemplerer(1994) shows.*

In the following, an example showing the calculations necessary to find the optimal auction is presented.

Example 1 *There are two bidders with valuations $u_i(x_1, x_2) = x_i$, $i = 1, 2$. The signals vector $X = (X_1, X_2)$ has a distribution with density $f(x, y) = \frac{4(1+xy)}{5}$, $(x, y) \in [0, 1]^2$. The first thing to calculate is the marginal $f^1(x) = \int_0^1 f(x, y) dy = \frac{4+2x}{5}$. The distribution of X_1 is therefore $F^1(x) = \int_0^x f^1(z) dz = \frac{4x+x^2}{5}$. Now we need the conditional density $f(y|x) = \frac{f(x,y)}{f^1(x)} = \frac{2(1+xy)}{2+x}$. Thus from (7) we have*

$$h^1(x, y) = x - \frac{(5 - 4x - x^2) D_x \left(x \frac{2(1+xy)}{2+x} \right)}{4(1+xy)}.$$

Since $D_x \left(x \frac{2(1+xy)}{2+x} \right) = \frac{2(2+4xy+x^2y)}{(2+x)^2}$ we obtain:

$$h^1(x, y) = x - \frac{(5 - 4x - x^2)(2 + 4xy + x^2y)}{2(2 + x)^2(1 + xy)}.$$

The auctioneer expected profits is therefore:

$$\int_0^1 \max\{h^1(x, y), h^1(y, x), 0\} \frac{4}{5}(1 + xy) dx dy \approx 0.457.$$

To compare for example with the war of attrition we define $F(y|x) = \int_0^y f(z|x) dz$ and $\lambda(y|x) = \frac{f(y|x)}{1-F(y|x)}$. The expected payment of each bidder, if his signal is x , is given by (see Krishna and Morgan (1997) page 352, equation (13).):

$$e^W(x) = \int_0^x y f(y|x) \frac{\lambda[y, y]}{\lambda[y, x]} dy.$$

Thus the auctioneer expected revenue is

$$2 \int_0^1 e^W(x) f^1(x) dx \approx 0.389.$$

References.

1. Baye, M. R., D. Kovenock, C. G. de Vries, 1996, The all-pay auction with complete information, *Economic Theory* 8, *n*^o2, 291-305;
2. Bulow, J. and P. Klemplerer, 1994, Auctions vs. negotiations, Discussion paper *n*^o 924, Center for Economic Policy and Research, London;
3. Krishna, V. and J. Morgan, 1997, An analysis of the war of attrition and the all-pay auction, *Journal of Economic Theory* v.72, *n*^o2, 343-362;
4. Milgrom, P. and R. Weber, 1988, *Econometrica* v.50, *n*^o5, 1089-1121;
5. Myerson, R. B., 1981, Optimal auction design, *Mathematics of operations research*, v.6 *n*^o13, 619-632;
6. Page, Fr., 1998, Existence of optimal auctions in general environments, *Journal Mathematical Economics* v.29, *n*^o4, 387-416;