Synergies and Price Trends in Sequential Auctions*

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Abstract

In this paper we consider sequential auctions where an individual's value for a bundle of objects is either greater than the sum of the values for the objects separately (positive synergy) or less than the sum (negative synergy). We show that the existence of positive synergies implies in declining expected prices.

There are several corollaries. First, the seller is indifferent between selling the objects simultaneously as a bundle or sequentially when synergies are positive. When synergies are negative, the expected revenue generated by the simultaneous auction can be larger or smaller than the expected revenue generated by the sequential auction.

In addition, in the presence of positive synergies, an option to buy the additional object at the price of the first object is never exercised in the symmetric equilibrium and the seller's revenue in unchanged.

Finally, we examine two special cases with asymmetric players. In the first case, players have distinct synergies. In this example, even if

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one player has positive synergies and the other has negative synergies, it is still possible for expected prices to decline. In the second case, one player wants two objects and the remaining players want one object each. For this example, we show that expected prices may not necessarily decrease as predicted by Branco (1997). The reason is that players with single-unit demand will generally bid less than their true valuations in the first period. Therefore, there are two opposing forces; the reduction in the bid of the player with multiple-demand in the last auction and less aggressive bidding in the first auction by the players with single-unit demand.

1 Introduction

Weber (1983) considers a sequential auction of identical objects and shows that expected prices follow a martingale i.e., bidders expect prices will remain constant on average throughout the sequence of auctions within a sale. In Weber's model, bidders only purchase one of a fixed number of objects. That is, the marginal value for a bidder of a second object is zero.

The essence of Weber's result is that there are two opposite and exactly offsetting effects on price as the auction proceeds; a reduction in competition with fewer buyers puts downward pressure on price, while increased competition with fewer objects put upward pressure on price.

There is, however, empirical evidence that prices are not constant throughout sequential auction sales. Ashenfelter (1989) reports that identical cases of wine fetch different prices at sequential auctions in three auction houses from 1985 to 1987. Although the most common pattern was for prices to remain constant, prices were at least twice as likely to decline as to increase. Ashenfelter refers to this phenomenon as the "price decline anomaly."

McAfee and Vincent (1993) adopted a similar approach to Ashenfelter and examined data from Christie's wine auctions at Chicago in 1987. In addition to pairwise comparisons, they examined triples of identical wine sold in the same auction sale. Their results are very similar to those of Ashenfelter.

Similar empirical findings were identified in a number of other markets; cable television licenses (Gandal (1995); condominiums (Ashenfelter and Genesove (1992), and Vanderporten (1992-a,b); dairy cattle (Engelbrecht-Wiggans and Kahn (1992)); stamps (Taylor (1991) and Thiel and Petry (1990)) and wool (Jones, Menezes and Vella (1998)). Gandal provides evidence that prices increased in the sale of cable-TV licences in Israel. The price increases are attributed by Gandal to the interdependencies among licenses which may increase competition in the later rounds of the sale. Jones, Menezes and Vella indicate that prices may increase or decrease in sequential auctions of wool (adjusting prices to estimate wool of homogeneous quality).

Most theoretical explanations for price variation in sequential auctions have concentrated on explaining the price decline anomaly. In a two-object model, Black and de Meza (1992) explain the price decline anomaly by the existence of an option that gives the winner of the first auction the rights to purchase the second object at the same price. In particular, for the case where the value of a second object for a player is equal to a fraction of the value of a first object, they show the existence of an equilibrium in which expected prices increase in the absence of an option to buy and may decrease when the option is present. We will characterize price trends in a more general setting and determine the effect of the option on the seller's revenue. McAfee and Vincent explain the anomaly by considering the effects of risk aversion on bidding strategies. For identical objects they show how bids in the first round are equal to the expected prices in the second round plus a risk premium associated with the risky future price. They assume buyers have nondecreasing risk aversion and can only buy one object.

Von der Fehr (1994) uses participation costs to obtain different net valuations for identical objects. When bidders face a cost of participating in each auction of two identical objects sold sequentially, price is lower in the second auction than it is in the first. This follows because the number of buyers who stay for the second auction falls by more than the successful bidder in the first auction. Once again, buyers only buy one object.

Engelbrecht-Wiggans (1994) and Bernhardt and Scoones (1994) show how expected prices decline when the objects are statistically identical (i.e., where bidders' valuations for the objects are independent draws from a fixed distribution) and the distribution of values is bounded. Finally, Menezes and Monteiro (1997) replicate these results for the case when buyers are allowed to buy more than one object but participation is endogenous as bidders face participation costs.

In this paper we examine sequential auctions of identical objects where individuals demand more than one object. An individual's value for a bundle of objects is either greater than the sum of the values attributed to the separate objects (positive synergy) or less than the sum (negative synergy) — Black and de Meza consider a special case of negative synergies. Thus, in this paper we explore the type of interdependencies described, for example, by Gandal (1997) in reference to the cable-TV auctions in Israel.

Rosenthal and Krishna (1996) also consider the effects of synergies on bidding behavior. However, they concentrate on simultaneous auctions and consider only a very special type of positive synergy; where a bidder's value for two objects is simply equal to twice his value for an individual object plus a positive constant. (For example, for a player with a value close to zero, the marginal synergy is infinite). Branco (1997) provides an example of sequential auctions with positive synergy of the same type of Rosenthal and Krishna. In his example, equilibrium behavior implies in a decline in expected prices.

In contrast, we consider synergies of a general form, allowing for positive and negative synergies. We show that the existence of positive synergies implies in declining expected prices. When two objects are worth more as a bundle than as separate objects, whoever buys the first object has the opportunity to realize the synergy. Therefore, the price in the first period includes a premium to reflect such opportunity.

There are several corollaries. First, the seller is indifferent between selling the objects simultaneously as a bundle or sequentially when synergies are positive. Second, when synergies are negative, the simultaneous auction may yields a higher or smaller expected revenue than the sequential auction. Third, when the synergy is positive an option to buy the additional object at the price of the first object is never exercised in the symmetric equilibrium.

Finally, we present two special cases with asymmetric players. In the first example, players have distinct synergies. In this example, even if one player has positive synergies and the other has negative synergies, it is still possible for expected prices to decline. In the second example, one player wants two objects and the remaining players want one object each. For this example, we show that expected prices may not necessarily decrease as predicted by Branco (1997). The reason is that players with single-unit demand will generally bid less than their true valuations in the first period. Therefore, there are two opposing forces; the reduction in the bid of the player with multiple-demand in the last auction and less aggressive bidding in the first auction by the players with single-unit demand.

2 Price trends

We consider the sale of two identical objects sequentially through second-price sealed-bid auctions. Buyer i's utility from one object is given by $v_i, i = 1, ..., n$. The v_i's are drawn independently from a fixed distribution $F(\cdot)$ with $F(0) = 0$ and density $f > 0$. Buyer i's utility from owning the two objects is given by the continuous function $\delta(v_i)$. Define $Y^1 = \max\{v_j; j \geq 2\}$ and Y^2 as the second highest of $\{v_j; j \geq 2\}$.

If $\delta(x) > 2x$, we say that there are positive synergies. If $\delta(x) < 2x$, we say that there are negative synergies. Otherwise, there are no synergies. The next theorem characterizes bidding strategies in the symmetric equilibrium . As a corollary, we can predict whether prices are likely to increase, decrease or remain the same as a function of the existing synergies. We need the lemma

Lemma 1 *Suppose $n > 2$ and that $\delta(x) - x$ is strictly increasing. Then the function $g(x) = \frac{n-2}{F(x)^{n-2}} \int_0^x \max\{\delta(x) - x, y\} F(y)^{n-3} f(y)\, dy$ is strictly increasing.*

Proof. If $\delta(x) - x \geq x$ then $g(x) = \delta(x) - x$. If $\delta(x) - x < x$ first note that $\int_0^x \max\{\delta(x) - x, y\} F(y)^{n-3} f(y)\, dy = \int_0^{\delta(x)-x} (\delta(x) - x) F(y)^{n-3} f(y)\, dy + \int_{\delta(x)-x}^x yF(y)^{n-3} f(y)\, dy = \frac{(\delta(x)-x)}{n-2} F(\delta(x) - x)^{n-2} + \int_{\delta(x)-x}^x yF(y)^{n-3} f(y) dy$. Thus $F(x)^{n-2} g(x) = (\delta(x) - x) F(\delta(x) - x)^{n-2} + (n-2) \int_{\delta(x)-x}^x yF(y)^{n-3} f(y) dy$. Therefore we have $(n-2) F(x)^{n-3} f(x) g(x) + F(x)^{n-2} g'(x) = (\delta(x) - x)' F(\delta(x) - x)^{n-2} + (n-2) x F(x)^{n-3} f(x)$. Hence

$$F(x)^{n-2} g'(x) = (\delta(x) - x)' F(\delta(x) - x)^{n-2} + (n-2) x F(x)^{n-3} f(x)(x - g(x)).$$

Finally since $\delta(x) - x < x$, $g(x) < x$ and therefore $g'(x) > 0$. Thus g is strictly increasing.

Theorem 1 *Assume that $\delta(0) \geq 0$ and that $\delta(x) - x$ is strictly increasing. Then in the symmetric equilibrium, a player with value x for one object bids in the first auction $b(x) = \delta(x) - x$ if $n = 2$ and*

$$b(x) = \frac{n-2}{F(x)^{n-2}} \int_0^x \max\{\delta(x) - x, y\} F(y)^{n-3} f(y)\, dy, \text{ if } n > 2.$$

His bid in the second auction equals x in case he does not win the first object and equals $\delta(x) - x$ if he wins the first object.

Proof. We consider only the case $n > 2$. The case $n = 2$ is easier. Suppose bidders $i = 2, \ldots, n$ bid $b(x)$ in the first auction and $\delta(x) - x$ in case of winning the first auction and x in case of not winning the first auction. Let us find the best response of bidder 1. If he wins the first object he will bid $\delta(v) - v$ in the second auction. If he does not get the first object he will bid his signal v in the second auction. We need only to find his bid in the first auction. The expected utility of bidder one when his signal is v and he bids x is $H(x) =$
$E\left[\chi_{x \geq b(Y^1)} \left\{v - b(Y^1) + (\delta(v) - v - Y^1)^+\right\} + \chi_{x < b(Y^1)} (v - \max\{\delta(Y^1) - Y^1, Y^2\})^+\right]$.
Since $b(\cdot)$ is a continuous function its range is an interval. Therefore we may suppose without loss of generality that $x = b(\omega), \omega \geq 0$. Thus defining $h(\omega) = H(b(\omega))$ we have that $h(\omega) =$

$E\left[\chi_{\omega\geq Y^1}\left\{v-b\left(Y^1\right)+\left(\delta\left(v\right)-v-Y^1\right)^+\right\}+\chi_{\omega<Y^1}\left(v-\max\left\{\delta\left(Y^1\right)-Y^1,Y^2\right\}\right)^+\right].$
More explicitly we have that $h\left(\omega\right)=$

$$\int_0^{\omega}\left\{v-b\left(z\right)+\left(\delta\left(v\right)-v-z\right)^+\right\}\left(n-1\right)F\left(z\right)^{n-2}f\left(z\right)dz+\int_{\omega<z}\left(n-1\right)G\left(z\right)f\left(z\right)dz,$$

where $G\left(z\right)=\left(n-2\right)\int_0^z\left(v-\max\left\{\delta\left(z\right)-z,y\right\}\right)^+F\left(y\right)^{n-3}f\left(y\right)dy$. Therefore the derivative of h is $h'\left(\omega\right)=$
$\left\{v-b\left(\omega\right)+\left(\delta\left(v\right)-v-\omega\right)^+\right\}\left(n-1\right)F\left(\omega\right)^{n-2}f\left(\omega\right)-\left(n-1\right)f\left(\omega\right)G\left(\omega\right)$. To shorten the formulas below define $u(\omega,y)=\max\left\{\delta\left(\omega\right)-\omega,y\right\}$. Thus $\frac{h'(\omega)}{(n-1)f(\omega)}=$
$\left\{v-b\left(\omega\right)+\left(\delta\left(v\right)-v-\omega\right)^+\right\}F\left(\omega\right)^{n-2}-\left(n-2\right)\int_0^{\omega}\left(v-u\left(\omega,y\right)\right)^+F\left(y\right)^{n-3}f\left(y\right)dy=$
$\left\{v+\left(\delta\left(v\right)-v-\omega\right)^+\right\}F\left(\omega\right)^{n-2}-\left(n-2\right)\int_0^{\omega}\left(u\left(\omega,y\right)+\left(v-u\left(\omega,y\right)\right)^+\right)F\left(y\right)^{n-3}f\left(y\right)dy.$
Suppose first that $\omega>v$. Note that

$$k=\max\left\{\delta\left(\omega\right)-\omega,y\right\}+\left(v-\max\left\{\delta\left(\omega\right)-\omega,y\right\}\right)^+\geq v.$$

Also $k>v$ if $\omega>v$. If $\delta\left(v\right)-v\geq v$ we have that $\delta\left(\omega\right)-\omega>v$. Thus $G\left(\omega\right)=0$ and $h'\left(\omega\right)=\left(n-1\right)f\left(\omega\right)\left\{v-b\left(\omega\right)+\left(\delta\left(v\right)-v-\omega\right)^+\right\}F\left(\omega\right)^{n-2}<$
$\left(n-1\right)f\left(\omega\right)\left\{v-b\left(v\right)+\delta\left(v\right)-2v\right\}\leq 0$. If $\delta\left(v\right)-v<v$. Then

$$h'\left(\omega\right)<\left(n-1\right)f\left(\omega\right)\left[\left\{v+\left(\delta\left(v\right)-v-\omega\right)^+\right\}F\left(\omega\right)^{n-2}-vF\left(\omega\right)^{n-2}\right]=0.$$

Thus $\omega>v$ implies $h'\left(\omega\right)<0$. Suppose now that $\omega<v$. If $\delta\left(v\right)-v\leq v$ then $\delta\left(\omega\right)-\omega\leq v$ and so

$$\begin{aligned}&h'\left(\omega\right)>\left(n-1\right)f\left(\omega\right)\left[\left\{v+\left(\delta\left(v\right)-v-\omega\right)^+\right\}F\left(\omega\right)^{n-2}-vF\left(\omega\right)^{n-2}\right]=\\&\left(\delta\left(v\right)-v-\omega\right)^+F\left(\omega\right)^{n-2}\geq 0.\end{aligned}$$

Now if $\delta\left(v\right)-v>v$ then $\delta\left(\omega\right)-\omega\leq\delta\left(v\right)-v$ and

$$h'\left(\omega\right)>\left(n-1\right)f\left(\omega\right)\left[\left\{v+\left(\delta\left(v\right)-v-\omega\right)^+\right\}F\left(\omega\right)^{n-2}-\left(\delta\left(v\right)-v\right)F\left(\omega\right)^{n-2}\right]=$$

$$\left(v-\omega\right)F\left(\omega\right)^{n-2}>0.$$

To conclude we have shown that $\omega>v$ implies $h'\left(\omega\right)<0$ and that $\omega<v$ implies that $h'\left(\omega\right)>0$. Therefore the maximum of h is achieved at $\omega=v$.
QED
Defining X^1 as the largest of the signals $\{v_j;j\geq 1\}$ and X^2 as the second largest of the signals $\{v_j;j\geq 1\}$, X^3 as the third largest of the signals and $\{X^2,X^3,\delta(X^1)-X^1\}^2$ as the second highest among $\{X^2,X^3,\delta(X^1)-X^1\}$ the equilibrium prices in each auction are given by, respectively:

$$\begin{aligned}P^1&=b\left(X^2\right),\\P^2&=\{X^2,X^3,\delta(X^1)-X^1\}^2.\end{aligned}$$

If synergies are positive then $P^2 = X^2 \leq \delta(X^2) - X^2 = P^1$. That is the price in the second auction is not greater than the price in the first auction and is in general smaller. Thus, in equilibrium, it follows that prices decrease if the synergy is positive. The price remains the same in the absence of synergies. If $n = 2$ and the synergy is negative then $P^1 \leq P^2$. In general if the synergy is negative the price can go up or down.

The reason for prices to increase is rather intuitive. In each period, bidder's bid their true marginal valuations. In the first period, a player bids the difference between his value for the bundle ($\delta(v)$) and his value from owning the first object only (v). In the second period, he bids either his value for one object or again his marginal value, depending whether or not he won the first object. Thus, in the presence of positive synergies, we have $\delta(v) \geq 2v$ (that is $\delta(v) - v \geq v$), which results in decreasing prices; the price in the first period includes a premium for the opportunity to realize the synergies.

With positive synergy, the expected revenue of the sequential auction is $E[\delta(X^2)]$, which coincides with the expected revenue when the two objects are sold simultaneously as a bundle. In this case, the revenue equivalence theorem holds because the individual with the highest signal receives both objects in either type of auction.

Let us consider an example with negative synergy.

Example 1 *We suppose $\delta(x) = x$. It is not strictly in our hypotheses but it is the "limit" of $\delta(x) = (1+\lambda)x$ when $\lambda \to 0$. Then $b(x) = \frac{n-2}{F(x)^{n-2}} \int_0^x yF(y)^{n-3} f(y)dy$. Thus $E[P^1] = E[b(X^2)] = E[X^3]$. And $P^2 = X^3$. The revenue from selling in bundles is $E[X^2]$. It is clear that depending on the distribution $E[X^2]$ can be greater or smaller than $2E[X^3]$.*

3 The value of an option to buy

Black and de Meza (1992) consider the case where the value of a second object for a player is equal to a fraction of the value of a first object. They characterize an equilibrium in which expected prices increase in the absence of an option to buy and may decrease when the option is present. Moreover, they show that the option may increase the seller's expected revenue. Here, however, in the presence of positive synergies, the option to buy is never exercised and the seller's expected revenue is not affected by the introduction of an option.

3.1 The option to buy if the synergy is positive.

The model is the same as in the previous section and so is the notation. The distinction is that now the winner of the first auction has the right to buy the second object at the same price paid for the first object. The timing is as follows. Each bidder submits a bid in the first auction. The winner of the first auction is given an option to buy the second object at the price paid in the first auction. If this option is exercised, there is no second auction. Otherwise, bidders submit bids for the second object and the winner is determined. The next theorem characterizes equilibrium behavior in the presence of positive synergies.

Theorem 2 *Suppose the synergy is always non-negative. Then the equilibrium strategy defined in theorem 1, $b(x) = \delta(x) - x$ is also an equilibrium strategy when there is an option to buy both objects in the first auction. In equilibrium the option is never exercised.*

Proof. We assume that players $2, ..., n$ bid in the first period according to the function $b(\cdot)$. We suppose also that they never exercise their option to buy. Their behavior in the second auction is the same as in theorem 1. We consider the game from Player 1's perspective. Let us find what is Player 1's best response. Let v denote 1's value and x his bid. We can write his expected profits as follows $H(x) =$

$E[\chi_{x\geq b(Y^1)} \max\left\{\delta(v) - 2b(Y^1), v - b(Y^1) + (\delta(v) - v - Y^1)^+\right\} +$
$\chi_{x<b(Y^1)} (v - \max(\delta(Y^1) - Y^1, Y^2))^+]$

If Player 1 has the highest bid in the first auction and he exercises the option, his profits are $\delta(v) - 2b(Y^1)$. If Player 1 wins the first object and does not exercise the option, his profits are $v - b(Y^1) + (\delta(v) - v - Y^1)^+$, where the expression in parentheses are his profits if he wins the second object. He will exercise his option if $k = \delta(v) - 2b(Y^1) - [v - b(Y^1) + (\delta(v) - v - Y^1)^+] > 0$. However if $Y^1 \geq \delta(v) - v$, $k = \delta(v) - v - (\delta(Y^1) - Y^1 \leq 0$ since from the positive synergy assumption we have that $Y^1 \geq v$. If $Y^1 < \delta(v) - v$, $k = -(\delta(Y^1) - Y^1) \leq 0$. Therefore Player 1 never exercises his option. Thus the maximizing problem of Player 1 is the same as in theorem (1). And the solution is therefore the same. QED

In the presence of positive synergies we have $\delta(v) - v \geq \dfrac{\delta(v)}{2}$. Therefore, a player who exercises the option to buy would pay too much for the object. (Recall that expected prices are decreasing in the equilibrium of the sequential auction when the option is not exercised).

3.2 The option to buy when the synergy is negative.

It happens sometimes in auctions that the option to buy both objects is exercised. We have seen that if the synergy is always positive this will not happen in equilibrium. However Black and Meza have shown that if the average synergy is negative and constant then the option may be exercised sometimes. In general when the synergy is negative we cannot find a closed form solution as we found before. In this section we expound the difficulties in the negative synergy case but go no farther.

We suppose that $\delta(x) - x \leq x$ for every x. We look for a symmetric equilibrium $b(\cdot)$ such that $\delta(x)-x \leq b(x) \leq x$. We follow the same procedure as in theorem (1) proof. Suppose Player 1 bids $x = b(\omega)$. If $x > b(Y^1)$ he will exercise his option[1] if

$$k = \delta(v) - 2b(Y^1) - [v - b(Y^1) + (\delta(v) - v - Y^1)^+] \geq 0.$$

If $Y^1 \leq \delta(v) - v$ Player 1 exercises his option since $k = Y^1 - b(Y^1) \geq 0$. If $Y^1 > \delta(v) - v$, $k = \delta(v) - v - b(Y^1)$. Since $b(\delta(v) - v) \leq \delta(v) - v$ the option will be exercised if and only if $\delta(v) - v \geq b(Y^1)$. Thus symetrically the Player with valuation Y^1 does not exercise his option if and only if $\delta(Y^1) - Y^1 < \max\{x, b(Y^2)\}$. Therefore Player 1 expected utility is $h(\omega) =$

$$\int_0^\omega \max\{\delta(v) - 2b(y), v - b(y) + (\delta(v) - v - y)^+\} f_Y(y)dy +$$

$$E\left[\chi_{\omega<Y^1<g(b(\max\{\omega,Y^2\}))}(v - \max\{\delta(Y^1) - Y^1, Y^2\})^+\right].$$

Where g is the inverse of $\delta(y) - y$. Given that $b(y) \geq \delta(y) - y$ we can show that no $\omega > v$ maximizes $h(\omega)$. Naturally it is difficult to go deaper in the maximization of h and we have nothing to offer in this regard. One thing we can be sure without solving the maximization problem: If there is a symmetric equilibrium with $b(x) \leq x$ then prices may go up or down. This is the content of the next lemma:

Lemma 2 *If there is a symmetric equilibrium with $b(x) \leq x$ for every x then in the negative synergy case equilibrium price may go up or down.*

Proof. The first period equilibrium price is $P^1 = b(X^2)$. If the option is not exercised then we have that $\delta(X^1) - X^1 < b(X^2)$. Since second period equilibrium price is $P^2 = \{X^2, X^3, \delta(X^1) - X^1\}$ then since necessarily $\delta(X^1) - X^1 < X^2$ we have that $P^2 = \max\{X^3, \delta(X^1) - X^1\}$. Thus if $X^3 \in (b(X^2), X^2)$, $P^2 > P^1$. And the price decreases otherwise. QED

[1]In this section we suppose that the option is exercised if the bidder is indiferrent between exercising it or not.

So far we examined price trends in sequential auctions with symmetric players. A feature of sequential auctions, however, is the existence of asymmetric players. In the sale of wine, for example, restaurant owners and collectors form two distinct groups of players. In the next two sections we provide examples with asymmetric players where again we are able to obtain expected prices that may be decreasing or increasing.

4 Asymmetric synergies

There are two objects and two bidders. The objects are sold sequentially through second-price auctions. Bidder i, $i = 1, 2$, values one object at v_i, and two objects at $\delta_i v_i$. We assume that $\delta_1 \geq \delta_2 \geq 2$. Suppose the v_i's are determined by independent draws from the uniform-[0,1] distribution and that each bidder knows only his value. As before, in the second auction it is a dominant strategy for player i to bid v_i if he did not win the first object and $\delta_i v_i - v_i$ if he won the first object. We now assume that player 2, who has a value y, follows a strategy $b_2(y)$ in the first auction and compute player 1's best response. Denote 1's bid by x and his value by v. His expected profits are given by:

$$H(x) = E\left[\chi_{x \geq b_2(y)} \left\{v - b_2(y) + (\delta_1 v - v - y)^+\right\} + \chi_{x < b_2(y)} \left\{(v - (\delta_2 - 1)y)^+\right\}\right]$$

Taking the expected value we obtain

$$H(x) = \int_0^{b_2^{-1}(x)} \left[v - b_2(y) + (\delta_1 v - v - y)^+\right] dy + \int_{b_2^{-1}(x)}^{1} (v - (\delta_2 - 1)y)^+ \, dy$$

Player 1 chooses x to maximize $H(x)$ yielding $H'(x) =$

$$\left(b_2^{-1}\right)'(x) \left[v - x + \left(\delta_1 v - v - b_2^{-1}(x)\right)^+ - \left(v - (\delta_2 - 1) b_2^{-1}(x)\right)^+\right] = 0.$$

We will assume for the moment that $b_2 = k_2 y$ and examine the expression

$$v - x + \left((\delta_1 - 1)v - \frac{x}{k_2}\right)^+ - \left(v - (\delta_2 - 1)\frac{x}{k_2}\right)^+ = 0.$$

It suffices to consider only three cases.

(i) If the two expressions between parentheses are less than zero, then $v = x$. This will occur if $k_2 \leq \min\{\frac{1}{\delta_1 - 1}, \delta_2 - 1\}$.

(ii) If the first expression between parentheses is greater than zero and the second less than zero, then $x = \dfrac{\delta_1 k_2 v}{k_2 + 1}$. This will occur if $k_2 + 1 \leq \delta_1(\delta_2 - 1)$.

(iii) If the two expressions between parentheses are positive, then $x = \frac{(\delta_1 - 1)\, k_2 v}{2 + k_2 - \delta_2}$. This holds whenever $k_2 + 1 \geq \delta_1(\delta_2 - 1)$.

Given that this is an asymmetric game, we have to computer 2's best response assuming that $b_1 = k_1 v$. Given that we are looking for an equilibrium, we will consider the case where both players are in case ii) above. Therefore, we obtain $k_1 = \frac{\delta_1 k_2}{k_2 + 1}$ and $k_2 = \frac{\delta_2 k_2}{k_1 + 1}$.

This holds if $\frac{\delta_2}{\delta_2 - 1} \leq \delta_1$. In this case, we have the following period 1 equilibrium bids:

$$b_1(v_1) = \frac{\delta_1\delta_2 - 1}{\delta_2 + 1} v_1 \text{ and } b_1(v_2) = \frac{\delta_1\delta_2 - 1}{\delta_1 + 1} v_2.$$

Equilibrium prices are given by

$$p^{(1)} = \min\left\{\frac{\delta_1\delta_2 - 1}{\delta_2 + 1} v_1, \frac{\delta_1\delta_2 - 1}{\delta_1 + 1} v_2\right\}$$

$$p^{(2)} = \begin{cases} \min\left\{(\delta_1 - 1)\, v_1, v_2\right\}, & \text{if Player 1 wins the first object} \\ \min\left\{(\delta_2 - 1)\, v_2, v_1\right\}, & \text{if Player 2 wins the first object} \end{cases}$$

Now if Player 1 wins the first object, we have

$$p^{(1)} = \frac{\delta_1\delta_2 - 1}{\delta_1 + 1} v_2 \text{ and } p^{(2)} = \min\{(\delta_1 - 1)\, v_1, v_2\}$$

Note that since $\delta_1 (\delta_2 - 1) > 2$, we have $\frac{\delta_1\delta_2 - 1}{\delta_1 + 1} > 1$. Therefore, if $(\delta_1 - 1)\, v_1 > v_2$, prices decrease. If $p^{(2)} = (\delta_1 - 1)\, v_1 < v_2 < \frac{\delta_1\delta_2 - 1}{\delta_1 + 1} v_2 = p^{(1)}$. That is, prices also fall.

If player 2 wins the first object we have

$$p^{(1)} = \frac{\delta_1\delta_2 - 1}{\delta_2 + 1} v_1 \text{ and } p^{(2)} = \min\{(\delta_2 - 1)\, v_2, v_1\}.$$

A similar analysis demonstrates that equilibrium prices will also fall as long as $\delta_2 (\delta_1 - 1) > 2$. It should not be difficult to provide an example where player 2 has negative synergy, player 1 has positive synergy and expected prices increase.

5 Asymmetric Demands

The example in Branco(1997) is such that Player 1 wants the first object only, Player 2 wants the second object only and Player 3 wants both objects and has positive synergy. Player $i, i = 1,2,3$, receives independently a signal x_i from a uniform distribution in the interval [1,2]. The value of the object for player i, $i = 1,2$, is simply $\frac{x_i}{2}$. The value of the two objects for player 3 is equal to $x_3 + \alpha$, where $\alpha > 0$ is a constant that is known by all players.

As a result of the assumption that Player 1 wants only the first object and that Player 2 wants only the second object, these two players behave as in a single-object second-price auction and bid their valuations. Prices then increase because player 3's bidding behavior in the first auction reflects the value of winning the first object for the realization of the positive synergies. Branco argues that this intuition should carry out to a more general model

However, this may not hold in general. When players can buy either the first or the second object, those players with single-unit demand do not bid their true valuations in the first period because winning in the first period precludes them from winning the second object for a price that may be inferior. This is demonstrated next.

Theorem 3 *Suppose that Player 1 wants both objects and there are synergies from owning the two identical objects. Assume that Players* $2, ...N$ *want only one object. If there is an equilibrium strategy* $(b(\cdot), c(\cdot), \ldots, c(\cdot))$ *such that* $b(v) \leq \delta(v) - v$ *for every* v, *then* $c(y) \leq y$ *for every* y.

Proof. Let us suppose bidders $i = 3, \ldots, N$ play $c(\cdot)$ and Bidder 1 plays $b(\cdot)$. Define $Z = \max\{y_3, \ldots, y_N\}$ and $Z^{(2)}$ as the second greatest among $\{y_3, \ldots, y_N\}$. If bidder 2 bid x his expected profit

$g(x) = E[\chi_{x \geq \max\{b(y_1), c(Z)\}} (v - \max\{b(y_1), c(Z)\}) +$

$\chi_{x < \max\{b(y_1), c(Z)\}} \{\chi_{b(y_1) > \max\{x, c(Z)\}} (v - \max\{\delta(y_1) - y_1, Z\})^+ +$
$\chi_{b(y_1) < \max\{x, c(Z)\}} \left(v - \max\left\{y_1, Z^{(2)}\right\}\right)^+\}$

Suppose now that $x > v$. Then
$\chi_{x < \max\{b(y_1), c(Z)\}} \chi_{b(y_1) > \max\{x, c(Z)\}} (v - \max\{\delta(y_1) - y_1, Z\})^+ = 0$
since $v < x < \max\{x, c(Z)\} < b(y_1) \leq \delta(y_1) - y_1$ implies that
$(v - \max\{\delta(y_1) - y_1, Z\})^+ = 0$.
Therefore

$$g(x) = E[\chi_{x \geq \max\{b(y_1), c(Z)\}} (v - \max\{b(y_1), c(Z)\}) +$$

$$\chi_{x<\max\{b(y_1),c(Z)\}}\chi_{b(y_1)<\max\{x,c(Z)\}}\left(v-\max\left\{y_1,Z^{(2)}\right\}\right)^+] =$$

$$E[\chi_{x\geq\max\{b(y_1),c(Z)\}}\left(v-\max\left\{b\left(y_1\right),c\left(Z\right)\right\}\right)+\chi_{x<c(Z)}\chi_{b(y_1)<c(Z)}\left(v-\max\left\{y_1,Z^{(2)}\right\}\right)^+].$$

Both summands increase if x decreases as long as $x>v$. Thus in equilibrium the optimum bid is $x=c(v)\leq v$. QED

One can also show that $c(y)=y$ cannot be an equilibrium bidding strategy when $b\left(v\right)=\delta\left(v\right)-v$. The proof is straightforward and will be omitted. Moreover, to show that the condition on $b\left(\cdot\right)$ in the theorem above is not vacuous let us calculate the best response of bidder 1 to $c(y)=y$, $N=3$ in the uniform-[0,1] distribution case. We suppose also that $\delta\left(v\right)\leq\frac{5v}{2}$. If bidder 1 bids x his expected profit is

$$\begin{aligned} h\left(x\right) &= E[\chi_{x\geq\max\{y_2,y_3\}}\left(v-\max\left\{y_2,y_3\right\}+\left(\delta\left(v\right)-v-\max\left\{y_2,y_3\right\}\right)^+\right)+ \\ &\quad \chi_{x<\max\{y_2,y_3\}}\left(v-\min\left\{y_2,y_3\right\}\right)^+] \end{aligned}$$

Or

$$\begin{aligned} h\left(x\right) &= \int_0^x\left(v-z+\left(\delta\left(v\right)-v-z\right)^+\right)2zdz+2\int_{x<y_2}\int_{y_3<y_2}\left(v-y_3\right)^+dy_2dy_3 = \\ &\quad \int_0^x\left(v-z+\left(\delta\left(v\right)-v-z\right)^+\right)2zdz+2\int_x^1G\left(y_2\right)dy_2. \end{aligned}$$

Here $G\left(y_2\right)=\int_0^{y_2}\left(v-y_3\right)^+dy_3$. Thus

$$h'\left(x\right)=2x\left(v-x+\left(\delta\left(v\right)-v-x\right)^+\right)-2G\left(x\right)=0.$$

If $x\leq v$, $G\left(x\right)=vx-x^2/2$. Thus

$h'(x)=2x\left[v-x+\delta\left(v\right)-v-x-v+\frac{x}{2}\right]=2x\left[\delta\left(v\right)-v-\frac{3x}{2}\right]$.

Therefore, $x^*=\frac{2(\delta(v)-v)}{3}\leq v$ (since $\delta\left(v\right)\leq\frac{5v}{2}$) and x^* is the maximum of h on $\left[0,v\right]$.

If $x\geq\delta\left(v\right)-v$, then $h'\left(x\right)<0$. Suppose now that $x\in\left(v,\delta\left(v\right)-v\right)$. Then $h'\left(x\right)=2x\left(\delta\left(v\right)-2x\right)-v^2$. The solution is now $x'=\frac{\delta(v)+\sqrt{\delta(v)^2-4v^2}}{4}$. However $x'>v$ if and only if $\sqrt{\delta\left(v\right)^2-4v^2}>4v-\delta\left(v\right)$. This inequality is true if $\delta\left(v\right)\geq 4v$. When $\delta\left(v\right)<4v$, the inequality is true if and only if $\delta\left(v\right)^2-4v^2>16v^2-8v\delta\left(v\right)+\delta\left(v\right)^2\Leftrightarrow 8v\delta\left(v\right)>20v^2\Leftrightarrow\delta\left(v\right)>\frac{5v}{2}$.

Thus the optimal bid of player 1 is $b\left(v\right)=\frac{2(\delta(v)-v)}{3}\leq v$.

Branco's prediction that expected prices will decrease rely on the assumption that the bidder who only wants the first object and the bidder who only wants the second object — despite the objects being ex-ante identical — will bid their true valuations in the first and second auctions, respectively. At the same time, the bidder who wants the two objects will bid more aggressively in the first auction. However, the above theorem demonstrates that players with single-unit demand will generally not bid their true valuations in the first period when they are allowed to bid for any of the two identical objects. There are two opposing forces; the reduction in the bid of the player with multiple-demand in the last auction and less aggressive bidding in the first auction by the players with single-unit demand. That is, there is no clear tendency for a declining price.

6 Conclusion

In this paper we examine sequential auctions of identical objects where individuals demand more than one object and there are synergies. We show that the existence of positive synergies implies in declining expected prices. When two objects are worth more as a bundle than as separate objects, whoever buys the first object has the opportunity to realize the synergy. Therefore, the price in the first period includes a premium to reflect such opportunity. In addition, when the synergies are negative, we show that expected prices increase.

When synergies are positive, we show that 1) the seller's expected revenue is the same under both simultaneous or sequential auctions; 2) an option to buy the additional object at the price of the first object is never exercised in the symmetric equilibrium. When synergies are negative, such option is never exercised either.

Moreover, in the case of negative synergies, the revenue equivalence theorem does not hold as the individual with the highest signal, who wins the simultaneous auction when the two objects are sold as a bundle, always wins the first auction but may not win the second auction when the objects are sold sequentially. In this case, we show that selling in bundles generates more revenue than selling sequentially. Finally, we present two examples with asymmetric players. In the first example, players have distinct synergies. In this example, even if one player has positive synergies and the other has negative synergies, it is still possible for expected prices to decline. In the second example, one player wants two objects and the remaining players want one object each. For this example, we show that expected prices may not necessarily decrease as predicted by Branco (1997). The reason is that players with single-unit demand will generally bid less than their true valuations in the first period. Therefore, there are two opposing forces; the reduction in the bid of the player with multiple-demand in the last auction and less aggressive bidding in the first auction by the players with single-unit demand.

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