Pricing in Economies with a Variable Number of Commodities*

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Abstract

We present a general equilibrium model that encompasses the endogenous selection of a set of tradeable commodities. At its foundation we introduce the notion of a *trade infrastructure* as a set of social institutions describing the trade and production technologies available to the agents in the economy. Our model bridges the analyses of economies with a finite number of commodities and those with an infinite number, and it provides a general framework for investigating a very large class of possible applications. We discuss in detail a simple example on the development of a guild economy into a market based economy.

We introduce an equilibrium concept that describes the pricing of trade infrastructures, based on the notion of valuation equilibrium for economies with abstract public goods, as in Diamantaras and Gilles (1996, *International Economic Review,* 37, 851–860). Through this concept we are able to price the tradeability of a commodity by itself. As our main results we obtain the existence as well as the decentralization of Pareto efficient allocations using the concept of valuation equilibrium.

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1 Introduction

In this paper we present a model of determinants of the tradeability of a carrier of consumption properties. Whether a carrier of consumption properties is tradeable or not as a commodity in a market system has been a rather neglected issue in general equilibrium theory. Nevertheless it is clear that costs related to the use of the price mechanism make the selection of which commodities a society is able to trade, and which not, a non-trivial problem. We develop an *institutional* approach to address whether or not a certain carrier of consumption properties is tradeable in the market system. Our motivation is that institutional economic development is based on the marketing of new, innovative commodities.[1] Economic theory has not yet provided a comprehensive model that incorporates this type of institutional development. This paper is intended to contribute to a further understanding of the issues involved.

Our model is an extension of the classical Arrow-Debreu-McKenzie general equilibrium model to incorporate certain institutional elements that make up a costly market system. We call a configuration of trade institutions a *trade infrastructure*. A trade infrastructure consists of the collection of tradeable commodities, the feasible home production technology available to individual agents, the social production technology available, and the costs related to using the trade institutions represented in this particular trade infrastructure. By making home production technology part of a trade infrastructure we incorporate the division of labor as an institutional element in the economy. This approach seems to be in line with the classical institutional insights as developed in, e.g., Coase (1937) and North (1990 and 1994): it incorporates these insights in a rather natural fashion into a general equilibrium framework.

Our analysis revolves around the *endogenous* selection of a trade infrastructure. The literature usually assumes that market structures are given exogenously. Here we are referring to the theory of taxation as developed in Dierker and Haller (1990) and Guesnerie (1995), where the set of commodities is divided in a subset of tradeables (which the government can observe) and a subset of unobservable home produced

[1] Here we refer to the theory of institutional development by Douglass North, e.g., North (1990), North (1994), and Lombardini (1989). Other authors have explored related issues in a more formal fashion. Allen and Gale (1994) give an extended account of their work on the general equilibrium analysis of financial innovation (chapters 4 and 6–11) and a summary of the work of others (chapter 5). We discuss how our work relates to theirs in our concluding section, where we also comment on some other recent related work.

commodities.[2] In the literature on incomplete markets the endogenous determination of the market structure has not been addressed to the fullest extent (Magill and Quinzii, 1996, provide an overview). An exception is the seminal contribution by Hart (1975, Section 6), who found the "paradoxical" conclusion that opening up new markets could lead to a Pareto inferior situation. Hart (1975) notes that the reason for this paradox is that future consumption is weighted too heavily. Hence, the costs to the economy of allowing more commodities to be traded is higher than the resulting social benefits. If one applies our purely static framework to this type of situation, one could obtain similar insights. This is, however, not pursued in the current paper.

In our static approach a trade infrastructure has a collective nature. This calls for the modelling of a trade infrastructure as a (multifaceted) *public project*. We have demonstrated in previous work, e.g., Diamantaras and Gilles (1996) and Diamantaras, Gilles and Scotchmer (1996), that this approach offers many important insights. Gilles, Diamantaras and Ruys (1996) have also addressed the issue of costly trade from this perspective. Following this approach, a trade infrastructure is chosen collectively in view of its benefits and costs to the society. When the society moves to a trade infrastructure which allows for more commodities to be traded, the resulting benefits are identified as utility gains to the members of the society. In addition to those utility gains, there are possible cost savings and increments. Here, costs are divided in two categories, namely collectively borne costs of enhancing the market system to handle more commodities and individually borne costs related to accessing a more complex market system. Whether such a development is beneficial depends on whether the gains exceed the costs of its implementation.

The description of production as an integral part of the trade infrastructure is justified by the selection of which commodities are tradeable. A trade infrastructure in which certain commodities are not tradeable, has an impact on what technologies and inputs are available for use in production activities. Individual economic agents can own and consume non-tradeable commodities, but they are restricted to consuming their home-produced quantities of these non-tradeables. This implies the endogenous, social determination of a *division of labor*.

Finally, our model can also be interpreted as a bridge between general equilibrium

[2] In our model such a division is achieved endogenously through the concept of home production. First, the society as a whole selects the set of tradeables. Subsequently, individual agents select home production plans from the emerging set of feasible plans.

analysis with a finite number of commodities and an infinite number of commodities. In our model, agents decide collectively on what subset of the potentially available commodities is actually traded.

Our main results concern the existence of (Pareto) efficient allocations and their decentralization by means of appropriately chosen price systems. The equilibrium concept we choose for this purpose is that of *valuation equilibrium*, a notion originally proposed by Mas-Colell (1980) for partial equilibrium models with public projects and extended to general equilibrium models with collective goods by Diamantaras and Gilles (1996) and to economies with costly trade by Gilles, Diamantaras, and Ruys (1996). In a valuation equilibrium there is a price system for the private goods and a tax/subsidy system for the collectively borne costs of the trade infrastructure. The price system is *conjectural* in the sense that it specifies a vector of prices for the tradeable commodities for each possible trade infrastructure. This implies that agents are highly rational and contemplate price changes due to changes in the set of tradeable commodities, the institutions governing their trade, and the production technologies available.

We prove that under standard continuity conditions there exist Pareto efficient allocations, that every valuation equilibrium is Pareto efficient. Furthermore, under standard convexity and monotonicity assumptions every Pareto efficient allocation can be supported as a valuation equilibrium. It should be clear that our analysis has primarily a normative character; the decentralization results show that through the extension of the price mechanism we can appropriate "price" the tradeability of commodities. This is an important first step in a further extensive and more applied study of evaluating whether certain commodities should be tradeable or not. In the light of contemporary issues such as the drug problem and the privatization of public facilities, this type of economic analysis is highly relevant.

In Section 4 we develop a stylized model of an economy that can operate under a guild system, where all production is carried out at home, and under a market system, where all commodities are traded and production happens in public firms. We find that the market infrastructure Pareto dominates the guild one, but it requires transfers between agents to be supported as a valuation equilibrium.

We conclude the paper with a discussion of possible future extensions of the model and of questions raised in this paper. Furthermore, we discuss some related approaches to marketability of commodities developed in the literature.

2 Economies with tradeables and non-tradeables

A trade infrastructure is a set of social institutions that govern the terms of trade. Below we specify these institutions to be the commodities traded, the costs related to trade, the home production possibilities open to each individual agent, and the social production set representing the societal production technology.

We base our discussion on the notion of a "carrier of consumption properties" in the sense of Lancaster (1966). Let $\mathcal{L} := \{1, \dots, \ell\}$ denote a finite collection of *carriers* of consumption properties.[3] We distinguish these carriers from actual commodities that are traded in the economy. Whether a carrier of consumption properties is traded, is institutionally determined in the sense that in order to be tradeable a carrier has to satisfy three fundamental social economic properties:

- First, there has to exist a notional provision of the commodity in question. This implies that agents offer the carrier within the trade infrastructure in sufficient quantities to be socially recognizable as a "commodity." It is clear that agents submit offers through the trade infrastructure when they are relatively confident that these offers will be met by demands for the carrier in question.

- Second, the commodity should have a notional demand. Agents are assumed to submit demands through the trade infrastructure regarding a certain carrier if they can be relatively confident that their demands will be met.

These requirements of notional demand and notional supply with regard to a certain commodity are equally important. They cannot exist independently from each other.

- Third, there should exist social institutions that govern all trade related to that particular commodity. In the setting of a market mechanism this is usually called the *marketability* of the commodity. In the NYSE, for example, this is regulated by formal admission of the stock to the floor.

[3] Alternatively the set $\mathcal{L}$ can be set equal to the set of natural numbers, thus admitting a countably infinite number of carriers. In such a context we can consider trade infrastructures that support finite subsets $L \subset \mathcal{L}$ of tradeables. The carrier space would then be an ℓ_p space, with every commodity space $\mathbb{R}^L$ a subset of it. Our welfare theorems would not be affected by such a change, although the requirement that utility functions are real-valued would then impose certain implicit conditions. Our existence theorem does depend on the finiteness of the set $\mathcal{L}$.

Let A stand for the finite set of economic agents in the society. Each agent $a \in A$ derives satisfaction from the consumption of **all** potential carriers of consumption properties present in the economy as described by $\mathcal{L}$. This is embodied in a *utility function* $U : A \times \mathbb{R}_+^{\mathcal{L}} \to \mathbb{R}$, where $\mathbb{R}_+^{\mathcal{L}}$ is the *consumption space*.

We call the utility function $U : A \times \mathbb{R}_+^{\mathcal{L}} \to \mathbb{R}$ *monotone* if for every $a \in A$ and for all bundles $f, g \in \mathbb{R}_+^{\mathcal{L}}$: $f \gg g$ implies $U(a, f) > U(a, g)$ and *strictly monotone* if for every $a \in A$ and for all bundles $f, g \in \mathbb{R}_+^{\mathcal{L}}$: $f > g$ implies $U(a, f) > U(a, g)$, where we use the vector inequalities $\gg$, $>$, and $\geqq$. Strict monotonicity translates into the property that there exists a notional demand for each carrier.

A *trade infrastructure* embodies the social institutions that govern the terms under which a certain set of commodities is traded.[4] In the sequel we use Γ as an abstract index set representing the collection of all potential trade infrastructures. Generic potential trade infrastructures are denoted by $\gamma, \delta \in \Gamma$. Subsequently we attach to every $\gamma \in \Gamma$ elements completely describing what we envision a trade infrastructure to be.

Whether a carrier is tradeable is, thus, determined by the property whether it is recognized as such within the trade infrastructure: for each potential trade infrastructure $\gamma \in \Gamma$ we define $L(\gamma) \subset \mathcal{L}$ as the *collection of tradeable commodities* in that particular trade infrastructure. We call $\mathbb{R}^{L(\gamma)}$ the *commodity space* within trade infrastructure γ.

Second, depending upon the infrastructure established in the economy, agents are endowed with a set that describes their opportunities to home-produce certain quantities of carriers of consumption properties; this extends the classical notion of an endowment. We denote by $W : A \times \Gamma \to\!\!\to \mathbb{R}_+^{\mathcal{L}}$ the *home production correspondence* assigning to each agent $a \in A$ within each potential trade infrastructure $\gamma \in \Gamma$ a set of bundles of carriers of consumption properties $W(a, \gamma) \subset \mathbb{R}_+^{\mathcal{L}}$. In case the correspondence W is a function, we recover the notion of an "endowment" of commodities as traditionally introduced within a pure exchange economy. The reason for assigning different home production sets to agents for different trade infrastructures is twofold:

1. We recognize that home production is a largely socially determined activity. In an economy with very few tradeables, economic agents are forced to provide

[4] This is adapted from Gilles, Diamantaras and Ruys (1996), where the definition given is firmly based on Knight (1992): a social institution consists of a set of rules, social norms, and desiderata that describe and guide certain behavior, in the case of trade infrastructure certain behavior regarding economic interaction and trade.

themselves with necessities through home production and to produce relatively little of the tradeable commodities. For example, we mention the resulting trade as observed for the medieval Anglo-Saxon and Angevin economies discussed by, e.g., Stenton (1947) and Mortimer (1994). In comparison, in a fully developed market economy home production is relatively unimportant since most of the carriers are tradeable in such a well developed economy.

2. Closely related to the point mentioned above, home production also embodies the division of labor in the economy. Hence, the division of labor is socially determined by the trade institutions implemented in the economy. This is captured in our framework by the correspondence W.

Let $\gamma \in \Gamma$ be some potential trade infrastructure. Then we denote by $W_\gamma : A \to\to \mathbb{R}_+^{L(\gamma)}$ the home production of tradeable commodities defined as the restriction of $W(\cdot, \gamma)$ to the commodity space $\mathbb{R}_+^{L(\gamma)}$, $W_\gamma(a) \equiv W(a, \gamma) \cap \mathbb{R}_+^{L(\gamma)}$.

Third,we assume that the marketability of a carrier of consumption properties is costly. Agents face individual costs regarding learning about the marketed commodities, i.e., there are individually borne access costs to the market. Also, there are collectively borne costs concerning the provision of an infrastructure of laws and physical facilities to accommodate the trade of these commodities. In particular the community has to provide physical market places, road systems, property law, and law enforcement to make trade possible.[5] The access costs related to a trade infrastructure are formally introduced for each trade infrastructure $\gamma \in \Gamma$ by a function $r_\gamma : A \to \mathbb{R}_+^{L(\gamma)}$, called the *access cost function* related to $\gamma \in \Gamma$. This function assigns to every agent a vector of tradeable commodities used in the agent's efforts to travel to and from the market place, to learn about the commodities traded in the market, and to access information about trade partners in the market. These costs are assumed to be independent of the amount traded in the trade infrastructure. Second, we introduce $c : \Gamma \to \mathbb{R}_+^{L(\gamma)}$ as the *setup cost function*, assigning to each potential trade infrastructure $\gamma \in \Gamma$ a vector of quantities of tradeables $c(\gamma) \in \mathbb{R}_+^{L(\gamma)}$ used in its establishment or maintenance.[6]

[5] Gilles, Diamantaras and Ruys (1996) also include individually borne transactions costs in their analysis as a third type of costs. These costs depend on the quantity traded within a trade infrastructure. Transaction costs are highly distortionary and to avoid unnecessary mathematical complications we do not incorporate these costs within the current framework. However, we mention that it is possible to extend the model developed above to incorporate transaction costs.

[6] This formulation may appear to rule out substitutability of inputs. However, we could have

Finally, a trade infrastructure embodies the (social) production possibilities available in the economy. As mentioned above, a trade infrastructure incorporates through the home production correspondence W a particular division of labor. This is complemented by the social production technology available to agents in the society. We introduce for every potential trade infrastructure $\gamma \in \Gamma$ a *social production set* $Y_\gamma \subset \mathbb{R}^{L(\gamma)}$ describing production possibilities in terms of tradeables only. Following Arrow and Debreu (1954) a production plan $y \in Y_\gamma$ consists of inputs (the negative coordinates of y) and outputs (the positive coordinates of y). We emphasize that the production set Y_γ is an abstraction; we do not explicitly rule out that there are multiple privately and/or collectively operated firms in the economy.

The listing of elements introduced above describes a trade infrastructure, i.e., $\gamma \in \Gamma$ represents a tuple $\langle L(\gamma), W(\cdot,\gamma), c(\gamma), r_\gamma, Y_\gamma \rangle$ consisting of the collection of tradeable commodities, the endowed home production technology, its setup costs, the induced access costs, and the social production technology available.

Definition 1 *An* ***economy*** *is a tuple* $\mathbb{E} = \langle A, \mathcal{L}, U, \Gamma, (L, W, c, r, Y) \rangle$ *such that the following conditions are satisfied:*

- *for every* $a \in A$ *the utility function* $U(a,\cdot)$ *continuous,*
- *for every potential trade infrastructure* $\gamma \in \Gamma : \ L(\gamma) \neq \emptyset$,
- *for every potential trade infrastructure* $\gamma \in \Gamma$ *the social production set* Y_γ *is closed, convex, comprehensive, i.e.,* $Y_\gamma + \mathbb{R}^{L(\gamma)}_- \subset Y_\gamma$, *and* $Y_\gamma \cap \mathbb{R}^{L(\gamma)}_+ = \{0\}$,
- *for every potential trade infrastructure* $\gamma \in \Gamma$ *and for every* $a \in A$, $W(a,\gamma)$ *is a non-empty, convex, and compact set in* $\mathbb{R}^{\mathcal{L}}_+$, *and*
- *each potential trade infrastructure* $\gamma \in \Gamma$ *can be established, i.e., for every* $\gamma \in \Gamma$ *and every* $w_\gamma : A \to \mathbb{R}^{L(\gamma)}_+$ *with* $w_\gamma(a) \in W_\gamma(a)$ *for all* $a \in A$ *it holds that*

$$\sum_{a\in A} r_\gamma(a) + c(\gamma) \ll \sum_{a\in A} w_\gamma(a) \tag{1}$$

The feasibility condition (1) requires that each potential trade infrastructure $\gamma \in \Gamma$ can in fact be established by *any* home production plan attainable in the economy.

specified c as well as r as correspondences to deal with this issue without any substantial impact on the analysis except for a notational complication — see Diamantaras, Gilles and Scotchmer (1996).

This is a rather strong condition, but on the other hand a rather straightforward extension of similar feasibility conditions employed in the literature. We remark that in the current context condition (1) can be viewed as a notional supply condition.

Definition 2 *An* ***allocation*** *in $\mathbb{E}$ is a quadruple $(\gamma, f, y, \hat{w}_\gamma)$, where $\gamma \in \Gamma$ is a trade infrastructure, $f : A \to \mathbb{R}_+^{L(\gamma)}$ is a distribution of tradeables for final consumption within γ, $y \in Y_\gamma$ is an adopted social production plan, and $\hat{w}_\gamma : A \to \mathbb{R}_+^{\mathcal{L}}$ with $\hat{w}_\gamma(a) \in W(a, \gamma)$ for all $a \in A$ is a (feasible) home production plan. For an allocation $(\gamma, f, y, \hat{w}_\gamma)$ the pair (γ, y) is called the* ***social plan*** *within that allocation. An allocation $(\gamma, f, y, \hat{w}_\gamma)$ is* ***feasible*** *in $\mathbb{E}$ if*

$$\sum_{a \in A} f(a) + \sum_{a \in A} r_\gamma(a) + c(\gamma) \leqq \sum_{a \in A} w_\gamma(a) + y \tag{2}$$

where w_γ is the restriction of $\hat{w}_\gamma$ to the commodity space $\mathbb{R}_+^{L(\gamma)}$.

The definition of an allocation embodies several social elements. First, it describes the trade infrastructure $\gamma \in \Gamma$ that is established. Second, through $\hat{w}_\gamma$ it prescribes a home production plan, to be interpreted as a socially adopted division of labor. In this respect an allocation thus embodies the actual division of labor in the economy. Based on this are the social production plan y and, finally, the resulting allocation of traded commodities f. From condition (1) it is clear that there exist non-trivial feasible allocations.

Given the commodity space $\mathbb{R}_+^{L(\gamma)}$ related to the trade infrastructure $\gamma \in \Gamma$ and a home production plan $\hat{w}_\gamma : A \to \mathbb{R}_+^{\mathcal{L}}$ with $\hat{w}_\gamma(a) \in W(a, \gamma)$ for all $a \in A$, we introduce the *restricted* utility function $U_{\gamma,w} : A \times \mathbb{R}_+^{L(\gamma)} \to \mathbb{R}_+$ defined by $U_{\gamma,w}(a, f) := U\left(a, \widehat{f}_w\right)$, where for every $f \in \mathbb{R}_+^{L(\gamma)}$ and every $i \in \mathcal{L}$:

$$\widehat{f}_w^i(a) = \begin{cases} f^i(a) & \text{if} \quad i \in L(\gamma) \\ \hat{w}_\gamma^i(a) & \text{if} \quad i \notin L(\gamma) \end{cases}$$

The restricted utility function assigns to every bundle of commodities obtained in the trade infrastructure the level of utility resulting from consuming these obtained commodities and a selection from the home-produced quantities of the non-tradeable carriers of consumption properties. The next example shows that restriction of tradeable commodities in different trade infrastructures might lead to widespread externalities.

Example 1 In this example we consider a simple setting in which in different trade infrastructures, strictly different sets of carriers of consumption properties can be traded. To illustrate that in these circumstances for some agent one trade infrastructure might be uniformly better than another we only partially develop the description of two different trade infrastructures. Let $\mathcal{L} = \{1,2,3,4\}$ be the set of carriers of consumption properties and let $\Gamma = \{\alpha, \beta\}$ be the set of potential trade infrastructural configurations. Now the consumption space is $\mathbb{R}^4_+$. Next we limit ourselves to the sets of tradeable commodities in each trade infrastructure: $L(\alpha) = \{1,2\}$ and $L(\beta) = \{1,3\}$.[7] From these definitions it is clear that the commodity space is represented by $\mathbb{R}^2_+$. Next we consider a single agent a endowed with $W(a,\alpha) = \{(10,16,1,1)\}$ and $W(a,\beta) = \{(10,1,1,16)\}$. Furthermore this agent has a strictly monotone, continuous, and strictly quasi-concave utility function representing his preferences on $\mathbb{R}^4_+$ given by

$$U(a,f) = 4 - e^{-f_1} - e^{-f_2} + \sqrt{f_3} + \sqrt{f_4},$$

where $f = (f_1, f_2, f_3, f_4) \in \mathbb{R}^4_+$. From this we derive for every $f \in \mathbb{R}^{\{1,2\}}_+$ and for every $g \in \mathbb{R}^{\{1,3\}}_+$ that

$$\begin{aligned} U_\alpha(a,f) &= 6 - e^{-f_1} - e^{-f_2} < 6 \qquad \text{and} \\ U_\beta(a,g) &= 8 - \frac{1}{e} - e^{-g_1} + \sqrt{g_3} \geqq 7 - \frac{1}{e} > 6 \end{aligned}$$

Hence, trade infrastructure β is *uniformly* better for agent a than trade infrastructure α, since it always delivers a higher utility level for that agent. Hence, this agent cannot be compensated with any amount of tradeables to overcome the disutility generated by the implementation of trade infrastructure α instead of β. ◇◇

3 Pricing with variable commodity spaces

We investigate Pareto efficiency in the setting developed above. In our setting all costs and externalities generated by the trade infrastructure are taken into account

[7] We might interpret commodity 1 as money, commodities 2 and 3 as differentiated luxury commodities with legal limitations on free trade, and commodity 4 as a nontradeable home produced commodity. In particular we might interpret Commodities 2, 3 and 4 as carriers of addictive properties such as tobacco, alcoholic beverages, and marijuana. The interpretation here is that the government only allows one of these addictive commodities to be traded in the economy. Which commodity is traded thus becomes an institutional element of the trade infrastructure.

through the defined notion of feasibility. The concept of Pareto efficiency is, therefore, relatively easy to generalize to this setting:

Definition 3 *A feasible allocation $(\gamma, f, y, \hat{w}_\gamma)$ is **Pareto efficient** in the economy $\mathbb{E}$ if there is no alternative feasible allocation $(\delta, g, z, \hat{v}_\delta)$ such that $U_{\delta,v}(a, g(a)) \geqq U_{\gamma,w}(a, f(a))$ for all agents $a \in A$ and $U_{\delta,v}(b, g(b)) > U_{\gamma,w}(b, f(b))$ for some agent $b \in A$.*

The first question addressed is the one regarding the existence of Pareto efficient allocations. For that purpose we have to introduce some auxiliary notation. For any $L \subset \mathcal{L}$ we define

$$\Gamma_L := \{\gamma \in \Gamma \mid L(\gamma) = L\} \tag{3}$$

as the set of trade infrastructures with the same collection of tradeables. Now $\{\Gamma_L\}_{L \subset \mathcal{L}}$ forms a (finite) partition of Γ. For $L \subset \mathcal{L}$ we define c_L as the function $\gamma \mapsto c(\gamma)$, Y_L as the correspondence $\gamma \mapsto Y_\gamma$, W_L as the correspondence $(a, \gamma) \mapsto W(a, \gamma)$, and finally r_L as the function $(a, \gamma) \mapsto r_\gamma(a)$ restricted to Γ_L. We are now able to formulate our existence result.

Theorem 1 *Let for every $L \subset \mathcal{L}$ the subset Γ_L be a compact metric space. Furthermore, suppose that for every $L \subset \mathcal{L}$ the functions r_L and c_L as well as the correspondences Y_L and W_L are continuous on Γ_L. Then there exists at least one Pareto efficient allocation in $\mathbb{E}$.*

For the proof of Theorem 1 we refer to the appendix.

The continuity conditions that guarantee existence of efficient allocations are easily interpretable. Namely, the mappings describing the economy $\mathbb{E}$ are piecewise continuous in the sense that a discontinuity might occur when there is a (discrete) change in the set of tradeable commodities. If the set of tradeables is constant, however, and other changes in the trade infrastructure occur, then these changes have a continuous character. Intuitively this is quite acceptable, since changes in the set of tradeables have to be interpreted as "disruptive" and other changes are mainly related to production technologies.

Next we address the question whether Pareto efficient allocations can be decentralized through appropriate price systems. To this aim we use an extension of the concept of a *valuation equilibrium*, seminally introduced by Mas-Colell (1980). In a valuation equilibrium agents take as given a conjectural price system for the tradeable commodities and a valuation system that assigns to every potential trade infrastructure some personalized admission price. Prices being conjectural means that agents are assumed to anticipate price changes resulting from the changes in the trade infrastructure. In our model it is obvious that prices have to be conjectural, since the collection of tradeable commodities depends on the trade infrastructure established.

A valuation system can be viewed in two different lights. First, it can be considered a Lindahl price for the trade infrastructure. In this view a valuation equilibrium is considered a generalization of the Lindahl equilibrium concept as originally formulated for economies with pure public goods. This interpretation emphasizes the public nature of a trade infrastructure. In this respect the marketability of a potential commodity itself is subject to valuation, i.e., the valuation system assigns a value to the possibility to trade a commodity on the market. Second, valuations can be interpreted as income redistribution and taxation devices. An authority is assumed to select the trade infrastructure and to levy taxes to cover its establishment costs. In accordance with the traditional second welfare theorem, moreover, the authority has to redistribute income to support its decision in the decentralized decision making process by the individual agents in the economy.

The formal representation requires the definition of a *price system* as a function $p : \Gamma \to \Delta$, where Δ is the unit simplex in $\mathbb{R}^{\mathcal{L}}$, such that for every $\gamma \in \Gamma$, $p(\gamma) \in \Delta \cap \mathbb{R}^{L(\gamma)}$. We denote by $\mathcal{P}$ the class of all price systems.

Definition 4 *A feasible allocation* $(\gamma, f, y, \hat{w}_\gamma)$ *in* $\mathbb{E}$ *is a* ***valuation equilibrium*** *if there exist a price system* $p \in \mathcal{P}$ *and a valuation function* $V : A \times \Gamma \to \mathbb{R}$ *such that*

(i) *there is budget balance in the sense that the total collected valuations cover the establishment cost of the trade infrastructure and distribute the profits over the agents in the economy, i.e.,* $\sum_{a\in A} V(a, \gamma) = p(\gamma) \cdot c(\gamma) - p(\gamma) \cdot y$*,*

(ii) *the chosen social plan* (γ, y) *maximizes the joint profit given by* $p(\delta)\cdot[z - c(\delta)] + \sum_{a\in A} V(a, \delta)$ *over all social plans* $\delta \in \Gamma$ *and* $z \in Y_\delta$*, and*

(iii) *for every agent* $a \in A$*, the final consumption bundle and home production plan* $(f(a), \hat{w}_\gamma(a))$ *maximize the utility function* $U(a, \cdot)$ *on the union of the* *budget sets* $\cup_{\delta\in\Gamma}B_\delta(a)$*, where*

$$B_\delta(a) = \left\{ (\widehat{g}_w, w) \in \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{L}} \;\middle|\; \begin{array}{c} w \in W(a,\delta), \\ p(\delta)\cdot[g + r_\delta(a)] + V(a,\delta) \\ \leqq p(\delta)\cdot \widetilde{w}_\delta, \quad g \in \mathbb{R}_+^{L(\delta)} \end{array} \right\}$$

and $\widetilde{w}_\delta$ *is the restriction of* $w \in \mathbb{R}_+^{\mathcal{L}}$ *to the commodity space* $\mathbb{R}_+^{L(\delta)}$*.*

The main decentralization result of our model of a trade economy is given by the following theorem.

Theorem 2 *Let* $\mathbb{E}$ *be an economy such that all preferences are monotone. Then the following statements hold:*

(a) *Each valuation equilibrium in* $\mathbb{E}$ *is Pareto efficient.*

(b) *Let for every* $w \in W_\delta(a)$ *hold that* $w \gg r_\delta(a)$ *for every potential trade infrastructure* $\delta \in \Gamma$ *and every agent* $a \in A$*. If for every agent* $a \in A$ *the utility function* $U(a,\cdot)$ *is quasi-concave and strictly monotone, then each Pareto efficient allocation can be supported as a valuation equilibrium with a strictly positive price system.*

For a proof of Theorem 2 we refer to the appendix.

We remark that the assumptions made in assertion (b) of Theorem 2 imply that two of the three requirements for the marketability of a carrier of consumption properties are satisfied. First, strict monotonicity of the utility function implies that there is a non-trivial notional demand for any $i \in \mathcal{L}$. Second, the requirement that for every $w \in W_\delta(a)$ it holds that $w \gg r_\delta(a)$ for any $a \in A$ and $\delta \in \Gamma$ implies that potentially each carrier $i \in \mathcal{L}$ has a non-trivial supply. Hence, the marketability of a potential commodity has been reduced completely to a *social* decision whether to "admit" it to the market system or not.

The following consequence of Theorem 2 states conditions under which Pareto efficient allocations can be supported by constant price systems. These conditions are exactly the properties that reduce the choice of a trade infrastructure to the choice of different trade technologies. Thus, under these conditions there is a fixed production technology as well as a fixed set of tradeables.

Corollary 1 *Let $\mathbb{E}$ be an economy such that the utility function U is continuous, quasi-concave and strictly monotone. If for all $\gamma, \delta \in \Gamma : L(\gamma) = L(\delta)$, $\sum_{a \in A} r_\gamma(a) + c(\gamma) = \sum_{a \in A} r_\delta(a) + c(\delta)$, $W(a, \gamma) = W(a, \delta)$ for all $a \in A$, and $Y_\gamma = Y_\delta$, then every Pareto efficient allocation can be supported as a valuation equilibrium with a constant price system, i.e., the conjectural price system for tradeables is constant over all potential trade infrastructures.*

Proof. The proof of this corollary is a direct adaptation of the proof of Theorem 2(b), noting that if the conditions of the corollary are satisfied, $F(\gamma) = F(\delta)$ for all $\gamma, \delta \in \Gamma$. Hence, the price constructed in that proof can be chosen uniquely, irrespective of the trade center $\gamma \in \Gamma$. ■

4 An application: guilds versus complete markets

An important feature of our model involves the consideration of home production and the endogenous division of labor. The following example illustrates the explanatory gain that this approach affords us.

In a stylized medieval economy there are m bakers and n carpenters. There are three commodities, bread, housing, and leisure, numbered in order of appearance, so $\mathcal{L} = \{1, 2, 3\}$. Of these commodities, the first two are traded under a guild trade infrastructure; calling this infrastructure g, we have $L(g) = \{1, 2\}$. The idea behind the guild system is that there are baker and carpenter guilds that completely regulate the admission of bakers and carpenters to the economy. Production activities are limited to home production only. Thus, the division of labor is not determined endogenously by a labor market, but rather given exogenously as a static social institution in the economy.

The other trade infrastructure considered is that of a fully developed market economy, called l. Under it, all three commodities are tradeable, so $L(l) = \{1, 2, 3\}$. Consequently all productive activities are socialized. Thus, the division of labor is determined endogenously through the labor market.

In this economy all agents have the same Cobb-Douglas utility function on the consumption space $\mathbb{R}^3_+$, $U(x_1, x_2, x_3) = x_1 x_2 x_3$, and the same initial 2 units of time. To simplify, we assume that all the costs related to the establishment of and access to the two infrastructures are zero.

Under the guild system, bakers produce bread with the home production technology described by the set

$$W_b(g) = \left\{ \left(w_1^b, 0, w_3^b\right) \,\middle|\, w_1^b \leqq \sqrt{k},\ w_3^b \leqq 2 - k,\ 0 \leqq k \leqq 2 \right\}.$$

This means that an individual baker has two units of time which he can devote to leisure, w_3^b, or to the home production of bread, w_1^b. Home production of leisure is directly consumed, so $x_3^b = w_3^b$, and some of the home production of bread is traded while the rest is consumed, x_1^b. Carpenters have exactly analogous home production possibilities:

$$W_c(g) = \left\{ (0, w_2^c, w_3^c) \,\middle|\, w_2^c \leqq \sqrt{k},\ w_3^c \leqq 2 - k,\ 0 \leqq k \leqq 2 \right\}.$$

There is no social production set under the guild system, emphasizing the shielded type of guild production in these types of economies.

Under the labor market system, there is only one collective production technology, and no home production possibilities:

$$\begin{aligned} Y(l) &= \left\{ \left(m\sqrt{\frac{k_1}{m}}, n\sqrt{\frac{k_2}{n}}, -k_1 - k_2 \right) \,\middle|\, k_1 \geqq 0,\ k_2 \geqq 0 \right\}, \\ W(l) &= \{(0, 0, 2)\} \text{ for all agents.} \end{aligned}$$

It is evident that this form of the production set is chosen to make the same allocation of inputs and outputs that arises in equilibrium under the guild system production efficient under the market system. Thus any differences in the equilibrium allocations under the two systems are entirely attributable to the difference in the institutional setups.

We develop our analysis of this economy in two stages. In the first stage we consider straightforward competitive equilibria in each of the two infrastructural settings. This shows that it is certainly not trivial to establish the selection of infrastructure l over g. Next we show that there exists an appropriate compensation scheme that makes it possible to rationalize the choice of the Pareto optimal complete market system l over the guild system g.

Competitive equilibrium under the guild system.

In order to compute a straightforward competitive equilibrium under the guild system g, we start with the bakers' utility maximization problem:

$$\max_{x_1, x_2, x_3, k, w_1, w_3} x_1 x_2 x_3 \quad \text{subject to} \quad w_1 = \sqrt{k},\ w_3 = 2 - k,\ x_3 = w_3,\ p_1 w_1 = p_1 x_1 + p_2 x_2.$$

The reader can verify that the following is the solution to this maximization problem: $\left(x_1^b, x_2^b, x_3^b, k^b, w_1^b, w_3^b\right) = \left(\frac{1}{2}, \frac{p_1}{2p_2}, 1, 1, 1, 1\right)$. Analogously, the carpenters solve the problem

$$\max_{x_1, x_2, x_3, k, w_2, w_3} x_1 x_2 x_3 \quad \text{subject to} \quad w_2 = \sqrt{k},\ w_3 = 2-k,\ x_3 = w_3,\ p_2 w_2 = p_1 x_1 + p_2 x_2,$$

and obtain the solution $\left(x_1^c, x_2^c, x_3^c, k^c, w_1^c, w_3^c\right) = \left(\frac{p_2}{2p_1}, \frac{1}{2}, 1, 1, 1, 1\right)$. Hence, the market demand for bread is equal to

$$\frac{m}{2} + \frac{np_2}{2p_1},$$

while the market supply is $mw_1^b = m$. This yields the price ratio

$$\frac{p_2}{p_1} = \frac{m}{n},$$

and the consumptions $x_2^b = \frac{n}{2m}$, $x_1^c = \frac{m}{2n}$. For later comparison, each baker receives utility level $U_b^*(g) = \frac{n}{4m}$ and each carpenter receives $U_c^*(g) = \frac{m}{4n}$.

Competitive equilibrium under a market system.

We consider the trade infrastructure l and assume that all agents in the economy get an equal share of the profits made.[8] Hence, the maximal profits π^* are distributed equally over the agents via the valuation system $V(a, l) = -\pi^*/(m+n)$. Profits now equal

$$\pi\left(k_1, k_2\right) = p_1 m \sqrt{\frac{k_1}{m}} + p_2 n \sqrt{\frac{k_2}{n}} - p_3\left(k_1 + k_2\right).$$

Maximizing this expression by choice of k_1, k_2 yields

$$k_1 = \frac{mp_1^2}{4p_3^2}, \quad k_2 = \frac{np_2^2}{4p_3^2}.$$

Therefore, the maximized profit equals

$$\pi^* = \frac{mp_1^2 + np_2^2}{4p_3}.$$

[8] Concerning the use of the collective production technology we note that if all bakers were to decide to use their labor making bread, and all carpenters making housing, the outputs of bread and housing would be exactly m and n, respectively, as under the guild infrastructure. However, with a labor market each agent's labor is generic and can be hired to either purpose. In particular, we should expect that when $m \neq n$, this possibility will induce an equilibrium allocation different from the one under the guild infrastructure.

Every agent maximizes $x_1x_2x_3$ subject to

$$p_1x_1 + p_2x_2 + p_3x_3 = 2p_3 + \frac{mp_1^2 + np_2^2}{4p_3(m+n)}.$$

And thus it follows that

$$\begin{aligned} x_1 &= \frac{8p_3^2(m+n) + mp_1^2 + np_2^2}{12p_1p_3(m+n)}, \\ x_2 &= \frac{8p_3^2(m+n) + mp_1^2 + np_2^2}{12p_2p_3(m+n)}, \\ x_3 &= \frac{8p_3^2(m+n) + mp_1^2 + np_2^2}{12p_3^2(m+n)}. \end{aligned}$$

From these and the expressions for k_1, k_2 above, we obtain the market equilibrium condition for bread,

$$\frac{mp_1}{2p_3} = \frac{8p_3^2(m+n) + mp_1^2 + np_2^2}{12p_1p_3}.$$

Analogously, for housing we obtain

$$\frac{np_2}{2p_3} = \frac{8p_3^2(m+n) + mp_1^2 + np_2^2}{12p_2p_3}.$$

From these we conclude that

$$\frac{p_2}{p_1} = \sqrt{\frac{m}{n}},$$

unlike the situation under a guild (unless $m = n$). Using this, the normalization of prices, and the market equations, we can solve for the equilibrium prices. Unfortunately, this implies the computation of the root of a rather lengthy quadratic polynomial in one of the prices.

To illustrate the resulting equilibrium we consider the numerical results for $m = 30$ and $n = 10$. For these values we obtain the approximate prices: $p_1 = 0.299005$, $p_2 = 0.517892$, and $p_3 = 0.183103$. Under such prices, each agent consumes the vector $(0.612373, 0.353554, 1)$, and obtains utility $U_a^*(l) = 0.216506$. (By contrast, the utilities achieved by the bakers and the carpenters under the guild system, with the chosen values of m and n, are $U_b^*(g) = \frac{1}{12}$ and $U_c^*(g) = \frac{3}{4}$.)

Valuation equilibrium to support the market system.

The above shows that there are cases that only one group of agents — in this case the bakers — benefit from the establishment of a complete market system. The other group (the carpenters) will only support the abolishment of their guild if they are compensated sufficiently, although clearly a complete market system is Pareto superior to a guild system. The appropriate compensations are expressed in a completely developed valuation equilibrium for this economy.

To compute the correct valuation system we let r denote the amount to be paid to each carpenter. Consider then the valuation system V' which is defined by:

$$V'(a,l) = \begin{cases} -\frac{\pi^*}{(m+n)} + \frac{n}{m}r, \text{ if } a = b, \\ -\frac{\pi^*}{(m+n)} - r, \text{ if } a = c. \end{cases}$$

As before, $\sum_a V'(a,l) = -\pi^*$. The changes in the agents' incomes will modify their consumptions in equal proportions, since they all share a Cobb-Douglas utility function. The changes in the demand functions for bakers and carpenters cancel each other out because they all have the same utility function; hence we do not need to recompute the price ratio.[9] Our task is to find a value or a range of values for r such that the maximized utility of a baker under the labor infrastructure is at least $1/12$ and that of a carpenter at least $3/4$. The reader can check that any r in the approximate range $(0.281842, 0.449199)$ will do. For instance, using $r = 0.3$ we find $U_b^*(l) = 0.118483$ and $U_c^*(l) = 0.800238$.

We conclude that a complete market system Pareto dominates the guild system, but that without appropriate transfers from the bakers to the carpenters, it would not be an equilibrium in an economy in which profits are distributed equally. This raises questions regarding the historic emergence of a complete market system: How did such a market system get established, even though certain groups of agents would have suffered without appropriate compensation? Second, it gives some guidance on the construction of explicit dynamic models aimed to explain this emergence.

5 Concluding remarks

The model of this paper attempts to provide a clearer view of the institutions of trade. Our standpoint is in between a completely individual-based analysis, as economic analysis often strives to be, and a cooperative analysis of society's choice of a trading system, which would downplay individual motives. We feel that this area

[9] The consumption of commodity i by each baker decreases by $r/9p_i$ and the consumption of commodity i by each carpenter increases by $r/3p_i$.

has remained unexplored for too long, given the rich results that it can yield when explored systematically.

An unanswered question that comes up with the example developed in Section 4 is that related to the influence of groups or coalitions on the determination of trade institutions. It has yet to be determined what exactly constitutes the power of such a coalition, in particular concerning the establishment of new markets. Until then a proper definition of a core allocation remains elusive. A preliminary analysis is given in Gilles, Diamantaras and Ruys (1996), although their model does not allow for variable commodity spaces.

The kind of analysis undertaken in this paper can be applied to more specifically targeted models of economic exchange. Allowing for a variable number of commodities, in particular, makes it possible to consider applications to models with geographical diversity (e.g., customs unions) or temporal diversity such as overlapping-generations models. Such extensions will be pursued in future papers.

We now turn to some comments on related literature. Yang and Ng (1993) develop a number of highly specified general equilibrium models in which the division of labor and the extent of markets are endogenous. The present paper can be seen as a general framework that affords the analyst a high enough degree of generality to make some underlying features of the economic situation stand out sharply encompassing seberal insights developed in Yang and Ng.

Allen and Gale (1994) work within the framework of financial economics. This means that they take as given many aspects of what we call a trade infrastructure. They emphasize the choice of financial instruments used by firms to finance their activities. With this aim, the authors further specialize their inquiry to the analysis of risk-sharing aspects of financial innovation, although they do mention other approaches which emphasize transaction costs. Their main model has two periods and one composite commodity. Firms have a choice of financial instruments, and this choice is nontrivial when markets are incomplete. Our model can be applied to this type of framework (though this belongs to another paper); we would then be able to encompass a much wider scope for financial innovation and to take into account the endogeneity of tradeable commodities. This last feature, in particular, is ideally suited to models with incomplete markets for future commodity deliveries, and it would provide new insights on market incompleteness.

Makowski and Ostroy (1995) study a model related to ours, and ask one of the same questions we address. In their model, prices as well as the set of tradeable commodities respond to individuals' choices of occupations, which determine the individuals' trading possibility sets, and so determine the social marketability of commodities. They show that the conditions of full appropriability (individuals' private benefits from their occupational choices coincide with the social benefits of these choices) and noncomplementarity (no centrally coordinated occupational switches achieve more return than the sum of their individual component switches) are sufficient to make occupational equilibria Pareto efficient. They also show that failure of efficiency may occur when these conditions do not hold. (In such cases, there are opportunites for coordination failures—see also Heller 1997.) Their model uses quasi-linear utility functions and, from our point of view, gives a clear argument that central coordination of some decisions in economies with endogenous marketability is desirable. In this paper we provided a characterization of such centralized decision-making in a framework more general than Makowski and Ostroy's. Further work to elaborate on the implementation of such decision-making appears highly desirable, but beyond the scope of this paper.

Finally, Romer (1994) makes a well-stated case for the extension of economic analysis to encompass genuinely new commodities. He emphasizes that economists do not know all the possible consumption characteristics; economic theories should recognize that new commodities could be introduced and developed. He thus calls for a rather different philosophy of economics. His approach suggests some intriguing questions beyond the scope of this paper, which show the importance of studying new goods and with which we are only partly concerned in this paper. For new goods, Romer's viewpoint would not agree with our assumptions that individuals have coherent preferences over all possible consumption characteristics, and that home production of all commodities is possible.

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Appendix: Proofs

Proof of Theorem 1

Let $L \subset \mathcal{L}$ be a subset of tradeables. Without loss of generality we may assume that $\Gamma_L \neq \varnothing$.

Since for any agent $a \in A$ the restricted correspondence $W_L(a, \cdot)$ is lower hemicontinuous on Γ_L and $W(a, \gamma)$ is a convex set for any $\gamma \in \Gamma_L$, we may apply Michael's selection theorem (e.g., Aliprantis and Border, 1994, page 489). Hence, there exists a continuous selection $v_L : \Gamma_L \to \mathbb{R}_+^{\mathcal{L}}$ with $v_L(a, \gamma) \in W(a, \gamma)$, $\gamma \in \Gamma_L$. (Composing these selections over $L \subset \mathcal{L}$ we now have established a home production plan $v : \Gamma \to \mathbb{R}_+^{\mathcal{L}}$.) For a vector $x \in \mathbb{R}^{\mathcal{L}}$, let $\widetilde{x}$ denote its restriction to $\mathbb{R}^L$.

For any $\gamma \in \Gamma_L$ we introduce

$$\mathbb{A}_\gamma = \left\{ \hat{f}_v \in \mathbb{R}_+^{\mathcal{L} \times A} \;\middle|\; \begin{array}{c} \sum_{a \in A} [f(a) + r_\gamma(a)] + c(\gamma) \leqq \sum_{a \in A} \widetilde{v}_L(a, \gamma) + y, \\ \text{for some } y \in Y_\gamma \end{array} \right\} \tag{4}$$

as the set of all attainable allocations of carriers in the economy under γ. (Recall the definition of the $\hat{f}_v$ notation on page 9). Note that

$$\overline{Y}_\gamma := \mathbb{R}_+^L \cap \left[Y_\gamma + \left\{ \sum_{a \in A} \widetilde{v}_L(a, \gamma) \right\} \right] \neq \varnothing$$

is a compact and convex set because of the conditions on the production set Y_γ. This implies that $\mathbb{A}_\gamma$ is compact and non-empty.

Since the restricted setup cost function c_L and restricted access cost function $r_L(a, \cdot)$ are continuous on Γ_L it follows from Theorem 3.1.10 in Engelking (1989, page 125) that

$$\left\{ \sum_{a \in A} r_L(a, \gamma) + c(\gamma) \;\middle|\; \gamma \in \Gamma_L \right\} \subset \mathbb{R}_+^L$$

is a compact set. Thus, for every $i \in L$ we may define

$$m_L^i = \min \left\{ x^i \;\middle|\; x = \sum_{a \in A} r_\gamma(a) + c(\gamma), \text{ some } \gamma \in \Gamma_L \right\} \geqq 0. \tag{5}$$

Let $m_L = (m_L^1, \ldots, m_L^L) \in \mathbb{R}_+^L$.

We now show that the correspondence which assigns to each $\gamma \in \Gamma_L$ the set $\overline{Y}_\gamma$ is continuous. The first step is to observe that by the continuity and compact-valuedness of the correspondence W_L it follows that there is a compact set around the origin, C, such that replacing Y_γ by $Y_\gamma \cap C$ and $\mathbb{R}^L_+$ by $\mathbb{R}^L_+ \cap C$ in the definition of $\overline{Y}_\gamma$ does not change the definition. The two correspondences $\gamma \longmapsto \mathbb{R}^L_+ \cap C$ and $\gamma \longmapsto (Y_\gamma \cap C) + \left\{\sum_{a\in A} \widetilde{v}_L(a,\gamma)\right\}$ are compact-valued. As a result, the correspondence $\gamma \longmapsto \overline{Y}_\gamma$ is upper hemi-continuous, by Propositions 11.21(a) and 11.27(a) in Border (1985).

To show the lower hemi-continuity of $\gamma \longmapsto \overline{Y}_\gamma$ we introduce the following lemma, which may be of independent mathematical interest.

Lemma 1 *Let X be a metric space and let $F : X \to\to \mathbb{R}^\ell$ be some lower hemi-continuous correspondence. Then the correspondence $x \longmapsto \overline{F}(x)$, where $\overline{F}(x)$ is the closure $F(x)$ of in $\mathbb{R}^\ell$, is also lower hemi-continuous.*

Proof. Let $x_k \to x \in X$ be some convergent sequence and let $z \in \overline{F}(x)$. We now show that there exist for every $k \in \mathbb{N}$ some $z_k \in \overline{F}(x_k)$ such that $z_k \to z$.

Let $m \in \mathbb{N}$.

Then there exists an open neighborhood $V_m \subset X \times \mathbb{R}^\ell$ of (x,z) such that $\operatorname{diam} V_m < \frac{1}{m}$. Then there exists some $\hat{z}_m \in F(x)$ with $(x,\hat{z}_m) \in V_m$ by definition of $\overline{F}(x)$ as the closure of $F(x)$. Since F is lower hemi-continuous there exists some convergent sequence $z^m_k \in F(x_k)$ such that $z^m_k \to \hat{z}_m$, i.e., there exists $k_m \in \mathbb{N}$ large enough such that $\left(x_{k_m}, z^m_{k_m}\right) \in V_m$.

This defines a subsequence $\left(x_{k_m}, z^m_{k_m}\right)_{m\in\mathbb{N}}$ such that $z^m_{k_m} \in \overline{F}(x_{k_m})$ and $z^m_{k_m} \to z$. It is clear that we can easily extend the subsequence $\left(x_{k_m}, z^m_{k_m}\right)_{m\in\mathbb{N}}$ to a full sequence $(x_k, z_k)_{k\in\mathbb{N}}$ by filling in the blanks with appropriate choices from $\overline{F}(x_k)$ for $k \neq k_m$ for some m.

Finally, by Proposition 11.11(b) in Border (1985) and the above, it follows that $\overline{F}$ is indeed lower-hemi continuous. ■

Next consider the following modification of $\overline{Y}_\gamma$:

$$\overset{\circ}{Y}_\gamma := \mathbb{R}^L_{++} \cap \left[Y_\gamma + \left\{\sum_{a\in A} \widetilde{v}_L(a,\gamma)\right\}\right].$$

By a result of Yannelis and Prabhakar (Border, 1985, Proposition 11.21(c)), the correspondence $\gamma \longmapsto \overset{\circ}{Y}_\gamma$ is lower hemi-continuous. Observing that the pointwise closure of this correspondence is the correspondence $\gamma \longmapsto \overline{Y}_\gamma$, we conclude from the hypothesis that Γ_L is a metric space and Lemma 1 that this last correspondence is lower hemi-continuous. This establishes that $\gamma \longmapsto \overline{Y}_\gamma$ is a continuous correspondence.

The continuity of $\gamma \longmapsto \overline{Y}_\gamma$ implies the existence of a uniform upper bound $B_L \in \mathbb{R}_+^L$ such that $x \leqq B_L$ for every $x \in \cup_{\gamma \in \Gamma_L} \overline{Y}_\gamma$, by using the convexity of $\overline{Y}_\gamma$ for every γ and the maximum theorem.

Next we define the set

$$\mathbb{A}_L = \left\{ \hat{f}_v \in \mathbb{R}_+^{\mathcal{L} \times A} \;\middle|\; \sum_{a \in A} f(a) + m_L \leqq B_L \right\}. \tag{6}$$

From the definitions above, it follows that $\mathbb{A}_L$ is nonempty and compact. Furthermore, by construction $\mathbb{A}_\gamma \subset \mathbb{A}_L$ for any $\gamma \in \Gamma_L$. Finally define $X_L := \cup_{\gamma \in \Gamma_L} \mathbb{A}_\gamma \subset \mathbb{A}_L$.

Lemma 2 *X_L is a closed subset of $\mathbb{A}_L$.*

Proof. Suppose not. Then there is a sequence $\hat{f}_v^n \to \hat{f}_v$ with $\hat{f}_v^n \in \mathbb{A}_{\gamma_n}$ for some $\gamma_n \in \Gamma_L$ and $\hat{f}_v \notin X_L$, i.e., $\hat{f}_v \notin \mathbb{A}_\gamma$ for any $\gamma \in \Gamma_L$.

For every $\gamma \in \Gamma_L$ define

$$z(\gamma) := \min_{i \in L} \min_{x \in \overline{Y}_\gamma} \left[\sum_{a \in A} \left(f^i(a) + r_\gamma^i(a) \right) + c^i(\gamma) - x^i \right]. \tag{7}$$

Then $z(\gamma)$ is well defined since $\overline{Y}_\gamma$ is compact and $f > 0$. Moreover, since $\hat{f}_v \notin \mathbb{A}_\gamma$ it follows that $z(\gamma) > 0$. By the maximum theorem it also follows that z is a continuous function on Γ_L and thus $Z := \min_{\gamma \in \Gamma_L} z(\gamma) > 0$ is well defined. Next define for every $n \in \mathbb{N}$

$$z_n(\gamma) := \min_{i \in L} \min_{x \in \overline{Y}_\gamma} \left[\sum_{a \in A} \left(f^{n,i}(a) + r_\gamma^i(a) \right) + c^i(\gamma) - x^i \right].$$

Again z_n is well defined and continuous on Γ_L. Remark that by definition, $\hat{f}_v^n \in \mathbb{A}_{\gamma_n}$ implies that $z_n(\gamma_n) \leqq 0$.

Since $\hat{f}_v^n \to \hat{f}_v$ it is clear that $z_n(\gamma) \to z(\gamma)$ for every $\gamma \in \Gamma_L$. Thus, for n large enough, it holds that $z_n(\gamma) \geqq \frac{1}{2} Z > 0$ for every $\gamma \in \Gamma_L$. In particular, $z_n(\gamma_n) > 0$. But, as described above, this contradicts that $\hat{f}_v^n \in \mathbb{A}_{\gamma_n}$, proving the claim. ■

Finally define

$$X := \bigcup_{L \subset \mathcal{L}} X_L = \bigcup_{L \subset \mathcal{L}} \bigcup_{\gamma \in \Gamma_L} \mathbb{A}_\gamma \subset \mathbb{R}^{\mathcal{L}}$$

Now the set X is compact since by Lemma 2 for every $L \subset \mathcal{L}$, X_L is a closed subset of the compact set $\mathbb{A}_L$, and $\mathcal{L}$ is a finite collection implying that X is the union of a finite number of compact sets. In particular, this implies that X is a sequentially compact space.

Now we define for $f, g \in X$ that $f \sim g$ if and only if $U(a, f(a)) = U(a, g(a))$ for every $a \in A$. Since preferences are continuous, $\sim$ is a closed equivalence relation on X. Thus, $\mathbb{X} := X/\sim$, being the quotient space with respect to $\sim$, is sequentially compact. (This follows from Alexandroff's Theorem 3.2.11, page 141, and Theorem 3.10.32, page 209, in Engelking, 1989.)

For all equivalence classes $f, g \in \mathbb{X}$ we define $f \succsim g$ if and only if $U(a, f(a)) \geqq U(a, g(a))$ for every $a \in A$. Now $\succsim$ is a partial ordering on $\mathbb{X}$. Indeed, $\succsim$ is reflexive as well as transitive, and by definition of $\mathbb{X}$ as the quotient space with respect to $\sim$ it is also anti-symmetric: if $f \succsim g$ as well as $f \precsim g$ it follows that $U(a, f(a)) = U(a, g(a))$ for every $a \in A$, i.e., $f \sim g$.

Next we claim that each chain in $\mathbb{X}$ has an upper bound. Indeed let $(f_n)_{n \in I}$ be a net in $\mathbb{X}$ such that $f_m \succsim f_n$, $m \geqq n$, $m, n \in I$. Since $\mathbb{X}$ is sequentially compact, (f_n) has a subnet, converging to, say, $f \in \mathbb{X}$. Then it follows that $f \succsim f_n$ for every $n \in I$.

By the Kuratowski-Zorn Lemma (Engelking, 1989, page 8, or Aliprantis and Border, 1994, Theorem 1.4, page 13) there exists a maximal element in $\mathbb{X}$ with respect to $\succsim$, say f^*. Then there is a trade infrastructure $\gamma^* \in \Gamma$ with $f^* \in \mathbb{A}_{\gamma^*}$. It can immediately be concluded that $(f^*, \gamma^*, y^*, v(\cdot, \gamma^*))$ is a Pareto efficient allocation in $\mathbb{E}$ where $y^* \in Y_{\gamma^*}$ is well chosen to make the allocation feasible.

Proof of Theorem 2

Proof of part (a)

Let $(\gamma, f, y, \hat{w}_\gamma)$ be a valuation equilibrium with an equilibrium price system p and valuation function V. We must show that $(\gamma, f, y, \hat{w}_\gamma)$ is Pareto efficient.

Suppose to the contrary that $(\gamma, f, y, \hat{w}_\gamma)$ is not Pareto efficient. Then there exists a feasible allocation $(\delta, g, z, \hat{v}_\delta)$ meeting condition (2) of page 9 with for all a in A

$$U_{\delta,v}(a, g(a)) \geqq U_{\gamma,w}(a, f(a)),$$

and for at least one $b \in A$

$$U_{\delta,v}(b, g(b)) > U_{\gamma,w}(b, f(b)).$$

Since $(\delta, g, z, \hat{v}_\delta)$ is feasible it follows that

$$\sum_{a\in A} g(a) + \sum_{a\in A} r_\delta(a) + c(\delta) = \sum_{a\in A} v_\delta(a) + z. \tag{8}$$

Condition (iii) of the definition of a valuation equilibrium and the monotonicity of the utility functions imply that for all a in A we have that $p(\delta)\cdot g(a) + p(\delta)\cdot r_\delta(a) + V(a,\delta) \geqq p(\delta)\cdot v_\delta(a)$ and for agent b we have $p(\delta)\cdot g(b) + p(\delta)\cdot r_\delta(b) + V(b,\delta) > p(\delta)\cdot v_\delta(b)$. Hence,

$$p(\delta)\cdot \sum_{a'\in A} g(a') + p(\delta)\cdot \sum_{a'\in A} r_\delta(a') + \sum_{a'\in A} V(a',\delta) > p(\delta)\cdot \sum_{a'\in A} v_\delta\left(a'\right). \tag{9}$$

Condition (ii) of the definition of a valuation equilibrium now implies that

$$\sum_{a'\in A} V(a',\gamma) + p(\gamma)\cdot [y - c(\gamma)] \geqq \sum_{a'\in A} V(a',\delta) + p(\delta)\cdot [z - c(\delta)] \tag{10}$$

Since equation (8) can be written as

$$\sum_{a'\in A} [v_\delta(a') - r_\delta(a') - g(a')] - c(\delta) + z = 0, \tag{11}$$

we conclude that

$$\begin{aligned} 0 &= \textstyle\sum_{a'\in A} V(a',\gamma) + p(\gamma)\cdot [y - c(\gamma)] \geqq && \text{by (10)} \\ &\geqq \textstyle\sum_{a'\in A} V(a',\delta) + p(\delta)\cdot [z - c(\delta)] > && \text{by (9)} \\ &> p(\delta)\cdot \textstyle\sum_{a'\in A} (v_\delta(a') - r_\delta(a') - g(a')) + && \\ &\qquad\qquad + p(\delta)\cdot [z - c(\delta)] = && \text{by (11)} \\ &= p(\delta)\cdot 0 = 0. && \end{aligned}$$

This is a contradiction, proving part (a) of the assertion.

Proof of part (b)

Let $(\gamma, f, y, \hat{w}_\gamma)$ be a Pareto efficient allocation in $\mathbb{E}$, and let $a \in A$ and $\delta \in \Gamma$ be arbitrary. We define for every $v \in W(a,\delta)$

$$\begin{aligned} F(a,\delta,v) &= \left\{ g \in \mathbb{R}^{L(\delta)}_{++} \,\middle|\, U_{\delta,v}(a,g) > U_{\gamma,w}(a, f(a)) \right\}, \quad \text{and} \\ \overline{F}(a,\delta,v) &= \left\{ g \in \mathbb{R}^{L(\delta)}_{+} \,\middle|\, U_{\delta,v}(a,g) \geqq U_{\gamma,w}(a, f(a)) \right\}. \end{aligned}$$

Next we let

$$\begin{aligned} F(a,\delta) &= \bigcup_{v\in W(a,\delta)} \left[F\left(a,\delta,v\right) - \{\tilde{v}_\delta\}\right], \quad \text{and} \\ \overline{F}(a,\delta) &= \bigcup_{v\in W(a,\delta)} \left[\overline{F}\left(a,\delta,v\right) - \{\tilde{v}_\delta\}\right]. \end{aligned}$$

Note that from Example 1 it is evident that it is not necessarily true that $F(a,\delta)\neq\emptyset$. This justifies the introduction of the following:

$$\begin{aligned} A^L(\delta) &= \{a\in A \mid U_{\delta,v}(a,0) > U_{\gamma,w}(a,f(a)) \text{ for some } v\in W(a,\delta)\} \\ A^H(\delta) &= \left\{a\in A \,\middle|\, \overline{F}(a,\delta,v)=\emptyset \text{ for all } v\in W(a,\delta)\right\} \end{aligned}$$

Agents in $A^L(\delta)$ can be interpreted as those agents in A that like the trade infrastructure δ so much that any bundle obtained in δ is superior to consuming $f(a)$ in γ. On the other hand, agents in $A^H(\delta)$ dislike the trade infrastructure δ so much that any bundle obtained in δ is strictly inferior to consuming $f(a)$ in γ.

For all agents $a\in A\setminus A^H(\delta)$ by compactness and convexity of $W(a,\delta)$ and strict monotonicity of $U_{\delta,v}$ on $\mathbb{R}_+^{L(\delta)}$ for every selection $v\in W(a,\delta)$, it follows that $F(a,\delta)$ is nonempty and open as well as comprehensive in the sense that $\mathbb{R}_{++}^{L(\delta)}+F(a,\delta)\subset F(a,\delta)$. $F(a,\delta)$ is also bounded from below due to compactness of $W(a,\delta)$. Also, for every agent $a\in A$, $\overline{F}(a,\delta)$ is closed and bounded from below. Finally, from the quasi-concavity of U it follows that for *every* agent $a\in A$, $F(a,\delta)$ as well as $\overline{F}(a,\delta)$ are convex; we show this next.

We will show that $\overline{F}(a,\delta)$ is convex; a similar proof applies to $F(a,\delta)$. Let $f,g\in\overline{F}(a,\delta)$ and consider $h=\lambda f+(1-\lambda)g$, where $\lambda\in(0,1)$. Then there are $v_1,v_2\in W(a,\delta)$ with $f+\tilde{v}_1\in\overline{F}(a,\delta,v_1)$ and $g+\tilde{v}_2\in\overline{F}(a,\delta,v_2)$. In other words, $U_{\delta,v_1}(a,f+\tilde{v}_1)\geqq U_{\gamma,w}(a,f(a))$ and $U_{\delta,v_2}(a,g+\tilde{v}_2)\geqq U_{\gamma,w}(a,f(a))$. Note that $\lambda\widehat{f}_{v_1}+(1-\lambda)\widehat{g}_{v_2}=\widehat{h}_v$ where $v=\lambda v_1+(1-\lambda)v_2$, because δ is fixed throughout (so $L(\delta)$ is fixed, as well as $\mathcal{L}\backslash L(\delta)$). Now, by the quasi-concavity of U, we have:

$$\begin{aligned} U_{\delta,v}(a,h+\tilde{v}_\delta) &= U\left(a,\widehat{h}_0+v\right) = U\left(a,\lambda\left(\widehat{f}_0+v_1\right)+(1-\lambda)\left(\widehat{g}_0+v_2\right)\right) \geqq \\ &\geqq \min\left\{U\left(a,\widehat{f}_0+v_1\right),U\left(a,\widehat{g}_0+v_2\right)\right\} = \\ &= \min\left\{U_{\delta,v_1}(a,f+\tilde{v}_1),U_{\delta,v_2}(a,g+\tilde{v}_2)\right\} \geqq U_{\gamma,w}(a,f(a)). \end{aligned}$$

Hence, since $v=\lambda v_1+(1-\lambda)v_2\in W(a,\delta)$ by the convexity of $W(a,\delta)$, we have that $h+\tilde{v}_\delta\in\overline{F}(a,\delta,v)$ and, thus, $h\in\overline{F}(a,\delta)$.

Finally, we introduce auxiliary notation: $A^*(\delta) := A \setminus \left[A^L(\delta) \cup A^H(\delta)\right]$.

The proof of part (b) of the assertion now proceeds with showing two lemmata.

Lemma 3 *For every $\delta \in \Gamma$ there exist a price $p(\delta) \in \Delta$, a production plan $y(\delta) \in Y_\delta$, a home production plan $\{\hat{w}_\delta(a) \in W(a,\delta) \mid a \in A\}$, and consumption bundles $\left\{x(a,\delta) \in \mathbb{R}_+^{L(\delta)} \;\middle|\; a \in A\right\}$ such that:*

(a) $p(\delta) \cdot x(a,\delta) = \inf p(\delta) \cdot F(a,\delta,\hat{w}_\delta(a))$, *$a \in A \setminus A^H(\delta)$.*

(b) $p(\delta) \cdot y(\delta) = \sup p(\delta) \cdot Y_\delta$.

(c) *For every $a \in A \setminus A^L(\delta)$, $p(\delta) \cdot \tilde{w}_\delta(a) = \max p(\delta) \cdot W_\delta(a)$, i.e., $p(\delta) \cdot \tilde{w}_\delta(a) \geqq p(\delta) \cdot v$ for any $v \in W_\delta(a)$.*

(d) *If $A^H(\delta) = \emptyset$ and $A \setminus A^L(\delta) \neq \emptyset$, then $\sum_{a\in A} x(a,\delta) + c(\delta) \geqq \overline{w}_\delta + y(\delta)$, where $\overline{w}_\delta = \sum_{a\in A}[w_\delta(a) - r_\delta(a)]$.*

(e) *If $A^H(\delta) \neq \emptyset$ or $A \setminus A^L(\delta) = \emptyset$, then $p(\delta) \gg 0$.*

(f) *$x(a,\gamma) = f(a)$, $a \in A$, $y(\gamma) = y$, and $\hat{w}_\gamma$ is chosen equal to the home production plan in the allocation under consideration.*

(g) *For every $a \in A^L(\delta)$: $x(a,\delta) = 0$ and $\hat{w}_\delta \in W(a,\delta)$ is such that $U_{\delta,\hat{w}_\delta}(a,0) > U_{\gamma,w}(a,f(a))$.*

Proof of Lemma 3.

Let $\delta \in \Gamma$. Select for every agent $a \in A^L(\delta)$ the consumption bundle $x(a,\delta) = 0$ and the home production plan $\hat{w}_\delta(a) \in W(a,\delta)$ such that condition (g) in the assertion is satisfied. Recall that we denote for $a \in A^L(\delta)$ by $\widetilde{w}_\delta(a)$ the restriction of $\hat{w}_\delta(a)$ to the commodity space $\mathbb{R}_+^{L(\delta)}$.

Next, let

$$\begin{aligned}
F(\delta) &= \sum_{a'\in A^*(\delta)} F(a',\delta) + (-Y_\delta) + \left\{c(\delta) + \sum_{a\in A} r_\delta(a) + \sum_{a\in A^L(\delta)} \widetilde{w}_\delta(a)\right\} \subset \mathbb{R}^{L(\delta)}, \\
\overline{F}(\delta) &= \sum_{a'\in A^*(\delta)} \overline{F}(a',\delta) + (-Y_\delta) + \left\{c(\delta) + \sum_{a\in A} r_\delta(a) + \sum_{a\in A^L(\delta)} \widetilde{w}_\delta(a)\right\} \subset \mathbb{R}^{L(\delta)}.
\end{aligned}$$

First, note that the mirrored production set $(-Y_\delta)$ is nonempty, closed, convex, and bounded from below. Moreover, $(-Y_\delta)$ comprehensive in the sense that $\mathbb{R}_+^{L(\delta)} +$

$(-Y_\delta) \subset (-Y_\delta)$. This implies that, if $A^*(\delta) \neq \emptyset$, $F(\delta)$ is also nonempty, open, convex, bounded from below, and comprehensive in the sense that $\mathbb{R}_+^{L(\delta)} + F(\delta) \subset F(\delta)$. An element in $F(\delta)$ represents the excess demand corresponding to a strictly Pareto superior allocation under trade infrastructure $\delta \in \Gamma$. Second, if $A^*(\delta) \neq \emptyset$, $\overline{F}(\delta)$ has the same properties except that it is closed. Since monotonicity implies that the recession cones (Rockafellar 1970, page 61) of the $F(a, \delta, v)$ sets are all contained in $\mathbb{R}_+^{L(\delta)}$, the home production possibilities set $W(a, \delta)$ is compact, and the set $(-Y_\delta)$ is contained in some half-space separating it from $\mathbb{R}_-^{L(\delta)}$ (using Minkowski's separation theorem), Corollary 9.1.1 of Rockafellar (1970, page 74) applies, and hence $\overline{F}(\delta)$ is the closure of $F(\delta)$.

We distinguish two cases:

First, suppose $A^H(\delta) = \emptyset$ and $A \setminus A^L(\delta) \neq \emptyset$. Because $(\gamma, f, y, \hat{w}_\gamma)$ is efficient, we have $0 \notin F(\delta)$. By strict monotonicity of the preferences and $A^H(\delta) = \emptyset$, there exists $\varkappa > 0$ such that $\varkappa e \in F(\delta)$, where $e = (1, \ldots, 1) \in \mathbb{R}^{L(\delta)}$. Hence, there exists $\lambda(\delta) \geqq 0$ with $\lambda(\delta)\, e \in \partial F(\delta) \equiv \overline{F}(\delta) \setminus F(\delta)$.

Consider the case that $\delta \neq \gamma$. For this case we can choose $\{\hat{w}_\delta(a) \mid a \in A \setminus A^L(\delta)\}$, $\{x(a, \delta) \in \overline{F}(a, \delta, \hat{w}_\delta(a)) \mid a \in A \setminus A^L(\delta)\}$, and $y(\delta) \in Y_\delta$ such that

$$\sum_{a' \in A^*(\delta)} x(a', \delta) + c(\delta) - y(\delta) - \overline{w}_\delta = \lambda(\delta)\, e \geqq 0,$$

where $\overline{w}_\delta = \sum_{a \in A} (\widetilde{w}_\delta(a) - r_\delta(a))$ as given in the assertion. (That these choices can be made is immediate from the definition of $\overline{F}(\delta)$.) The definition above implies that

$$\sum_{a \in A} x(a, \delta) = \sum_{a \in A \setminus A^L(\delta)} x(a, \delta) = \sum_{a \in A^*(\delta)} x(a, \delta)$$

and thus condition (d) of Lemma 3 is satisfied for $\delta \neq \gamma$.

For $\delta = \gamma$ it is evident that $A^L(\gamma) = A^H(\gamma) = \emptyset$. Choose $\{x(a, \gamma) \in \overline{F}(a, \gamma) \mid a \in A\}$ and $y(\gamma) \in Y_\gamma$ satisfying condition (f) of Lemma 3. Also let the home production plan be selected equal to w_γ and, thus, $\overline{w}_\gamma = \sum_{a \in A} (w_\gamma(a) - r_\gamma(a))$. By feasibility and efficiency we then have that

$$\sum_{a' \in A} f(a') + c(\gamma) - y - \overline{w}_\gamma = 0 \equiv \lambda(\gamma)\, e$$

implying condition (d) of Lemma 3 for the case $\delta = \gamma$.

For any $\delta \in \Gamma$, by definition $\lambda(\delta)\, e \in \partial F(\delta)$, and thus we may assume without loss of generality that $x(a,\delta) - w_\delta\,(a) \in \partial F\,(a,\delta)$ and $x(a,\delta) \in \partial F\,(a,\delta,\hat{w}_\delta\,(a))$, $a \in A$, and that $y(\delta) \in \partial Y_\delta$. Now let $p(\delta) \in \Delta$ be the normal vector of the supporting hyperplane of $F(\delta)$ in $\lambda(\delta)\, e$, i.e., $\inf\, p(\delta) \cdot F(\delta) \geqq p(\delta) \cdot \lambda(\delta)\, e = \lambda(\delta)$. (The existence of $p(\delta)$ follows from application of a standard supporting hyperplane argument as given by, e.g., Theorem 11.6, Rockafellar (1970, page 100) as well as the comprehensiveness of $F(\delta)$.) Since $p(\delta)$ defines a supporting hyperplane of $F(\delta)$ at $\lambda(\delta)\, e$, it follows immediately that conditions (a), (b) and (c) of Lemma 3 are satisfied.

Second, consider the case that $A^H(\delta) \neq \emptyset$ or $A \setminus A^L(\delta) = \emptyset$. Now choose $p(\delta) \gg 0$ arbitrarily in Δ such that $\arg\max\, p(\delta) \cdot Y_\delta \neq \emptyset$ as well as $\arg\max p\,(\delta) \cdot W_\delta\,(a) \neq \varnothing$, for $a \in A \setminus A^L(\delta)$. (Such a price $p(\delta)$ exists since $W_\delta\,(a)$ is compact and convex, and by assumption $Y_\delta \cap \mathbb{R}_+^{L(\delta)} = \{0\}$ and Y_δ is closed and convex.) This implies that by definition condition (e) of Lemma 3 is satisfied.

We conclude by constructing an appropriate tuple as required in the assertion. First, choose a selection $\hat{w}_\delta$ in $W\,(\cdot,\delta)$ such that

$$p(\delta) \cdot \tilde{w}_\delta\,(a) = \max\, p(\delta) \cdot W_\delta\,(a)\,, \quad a \in A \setminus A^L\,(\delta)\,,$$

and for $a \in A^L\,(\delta)$ as indicated above. It is clear by convexity and compactness of $W_\delta\,(a)$ that such a choice is admitted. This choice also implies condition (c) of Lemma 3.

Finally, we are able to choose consumption bundles $x\,(a,\delta) = 0$ for $a \in A^H(\delta)$ complemented with $\left\{ x(a,\delta) \in \overline{F}\,(a,\delta,\hat{w}_\delta\,(a)) \,\middle|\, a \in A \setminus A^H(\delta) \right\}$ such that

$$p(\delta) \cdot x(a,\delta) = \inf\, p(\delta) \cdot F(a,\delta), \tag{12}$$

as well as a production plan $y(\delta) \in Y_\delta$ such that

$$p(\delta) \cdot y(\delta) = \max\, p(\delta) \cdot Y_\delta. \tag{13}$$

Obviously, from $p(\delta) \gg 0$ and the properties of $F(a,\delta)$, $a \in A$, it immediately follows that $x(a,\delta) \in \overline{F}(a,\delta)$ can be chosen as required. Furthermore, by definition of $p(\delta)$ it follows that such a production plan $y(\delta)$ exists. This implies that conditions (a) and (b) as asserted in Lemma 3 are satisfied.

THIS COMPLETES THE PROOF OF LEMMA 3. ■

Lemma 4 *Let $\delta \in \Gamma$ and let $p(\delta)$, $y(\delta)$, $\hat{w}_\delta$, and $x(\cdot, \delta)$ be given as in Lemma 3. Furthermore, let*

$$\varphi(\delta) = p(\delta) \cdot \left[\sum_{a \in A} x(a, \delta) + c(\delta) - \overline{w}_\delta - y(\delta) \right].$$

(a) *If $A^H(\delta) = \emptyset$ and $A \setminus A^L(\delta) \neq \emptyset$, then $p(\delta) \gg 0$.*

(b) *If $\varphi(\delta) \leqq 0$ and $A^L(\delta) \neq \emptyset$, then $A^H(\delta) \neq \emptyset$.*

Proof of Lemma 4.
First, we show assertion (a). By Lemma 3(d) and $p(\delta) > 0$ it follows that $\varphi(\delta) \geqq 0$. Since $y(\delta) \in \partial Y_\delta$, condition (1) implies that $\overline{w}_\delta \gg c(\delta)$. Thus, we conclude that

$$p(\delta) \cdot \overline{w}_\delta + p(\delta) \cdot y(\delta) > p(\delta) \cdot c(\delta)$$

implying with $\varphi(\delta) \geqq 0$ that

$$p(\delta) \cdot \sum_{a \in A^*(\delta)} x(a, \delta) > 0.$$

since $x(a, \delta) = 0$ for $a \in A^L(\delta) \cup A^H(\delta)$. Hence, there exists $b \in A^*(\delta)$ with $p(\delta) \cdot x(b, \delta) > 0$.

Now suppose by contradiction that $p_i(\delta) = 0$ for some $i \in L(\delta)$. Now let $g := x(b, \delta) + e^i$, where e^i is the i-th unit vector in $\mathbb{R}^{L(\delta)}$. Then by strict monotonicity of b's preferences $U_{\delta, \hat{w}_\delta}(b, g) > U_{\delta, \hat{w}_\delta}(b, x(b, \delta)) = U_{\gamma, w}(b, f(b))$ and

$$p(\delta) \cdot g = p(\delta) \cdot x(b, \delta) > 0. \tag{14}$$

The continuity of $U_{\delta, \hat{w}_\delta}(b, \cdot)$, $w_\delta(b) \gg r_\delta(b)$, and (14) now imply that there exists $h \in \mathbb{R}_+^{L(\delta)}$ with $U_{\delta, \hat{w}_\delta}(b, h) > U_{\gamma, w}(b, f(b))$ and

$$0 < p(\delta) \cdot h < p(\delta) \cdot g = p(\delta) \cdot x(b, \delta). \tag{15}$$

Hence, $h \in F(b, \delta, \hat{w}_\delta(b))$ and $p(\delta) \cdot h < p(\delta) \cdot x(b, \delta)$, which is a contradiction to Lemma 3(a) regarding $x(b, \delta)$. Thus we conclude that indeed $p(\delta) \gg 0$.

Next we show assertion (b) of the lemma. Suppose to the contrary that $A^H(\delta) = \emptyset$. Then by assertion (a) as shown above and Lemma 3(e) it holds that $p(\delta) \gg 0$. Furthermore, by Lemma 3(d) $\varphi(\delta) \geqq 0$ and, thus, $\varphi(\delta) = 0$. This implies that

$$\sum_{a \in A} x(a, \delta) + c(\delta) = \overline{w}_\delta + y(\delta),$$

i.e., $(\delta, x(\cdot, \delta), y(\delta), \hat{w}_\delta)$ is feasible. For all $a \in A$ we have that $U_{\delta,\hat{w}_\delta}(a, x(a,\delta)) \geqq U_{\gamma,w}(a, f(a))$. For $a \in A^L(\delta)$ by definition

$$U_{\delta,\hat{w}_\delta}(a, x(a,\delta)) = U_{\delta,\hat{w}_\delta}(a, 0) > U_{\gamma,w}(a, f(a)).$$

But then $(\delta, x(\cdot,\delta), y(\delta), \hat{w}_\delta)$ is a Pareto improvement of $(\gamma, f, y, \hat{w}_\gamma)$, which contradicts the latter's Pareto optimality. Thus, $A^H(\delta) \neq \varnothing$.
THIS COMPLETES THE PROOF OF LEMMA 4. ■

Lemma 3 introduces a conjectural price system $p : \Gamma \to \Delta$. In order to define an appropriate valuation system $V : A \times \Gamma \to \mathbb{R}$ we need some preliminaries. For every $\delta \in \Gamma$ with $A^L(\delta) \neq \emptyset$ we define $d(\delta) > 0$ as follows:

- If $\varphi(\delta) > 0$, let $d(\delta) > 0$ be such that
$$d(\delta) < \frac{\varphi(\delta)}{\# A^L(\delta)}. \tag{16}$$
- If $\varphi(\delta) \leqq 0$, using Lemma 4, we have $A^H(\delta) \neq \varnothing$ and we can choose $d(\delta) > 0$ such that
$$d(\delta) < \frac{1}{\# A^L(\delta)} \sum_{a \in A^H(\delta)} p(\delta) \cdot [\widetilde{w}_\delta(a) - r_\delta(a)]. \tag{17}$$
This bound is positive, since $p(\delta) \gg 0$ — as shown in Lemma 4(a) — and $w_\delta(a) \gg r_\delta(a)$ for all $a \in A$.

Now we are in the position to define a valuation system $V(\cdot, \delta)$ by

$$V(a,\delta) = \begin{cases} p(\delta) \cdot [\widetilde{w}_\delta(a) - r_\delta(a) - \ x(a,\delta)] & a \in A^*(\delta) \\ p(\delta) \cdot [\widetilde{w}_\delta(a) - r_\delta(a) - \ x(a,\delta)] + d(\delta) & a \in A^L(\delta) \\ \min\left\{0, \frac{\varphi(\delta)}{\# A^H(\delta)}\right\} & a \in A^H(\delta) \end{cases} \tag{18}$$

Next we show that (p, V) satisfies the conditions as required in Definition 4.

Condition (i).
By definition $A^L(\gamma) = A^H(\gamma) = \emptyset$. Hence, for $a \in A$

$$V(a,\gamma) = p(\gamma) \cdot [\widetilde{w}_\gamma(a) - r_\gamma(a) - \ f(a)] \tag{19}$$

together with feasibility (equation (2)) of $(\gamma, f, y, \hat{w}_\gamma)$ implying that

$$\begin{aligned} \sum_{a \in A} V(a,\gamma) &= p(\gamma) \cdot \sum_{a \in A} [\widetilde{w}_\gamma(a) - r_\gamma(a) - \ f(a)] = \\ &= p(\gamma) \cdot [c(\gamma) - y]. \end{aligned}$$

Condition (ii).

Let $\delta \neq \gamma$. For $z \in Y_\delta$ it holds that

$$\begin{aligned}
\sum_{a\in A} V(a,\delta) \quad + \quad & p(\delta)\cdot[z-c(\delta)] = \\
= \quad & p(\delta)\cdot \sum_{a\in A\setminus A^H(\delta)} [\widetilde{w}_\delta(a) - r_\delta(a) - \, x(a,\delta)] + \\
& +\#\, A^L(\delta)\, d(\delta) + \min\,\{0,\varphi(\delta)\} + p(\delta)\cdot z - p(\delta)\cdot c(\delta) \\
= \quad & p(\delta)\cdot \left[\overline{w}_\delta - c(\delta) + y(\delta) - \sum_{a\in A^*(\delta)} x(a,\delta)\right] + \\
& -p(\delta)\cdot \sum_{a\in A^L(\delta)} x(a,\delta) + \\
& -p(\delta)\cdot \sum_{a\in A^H(\delta)} [\widetilde{w}_\delta(a) - r_\delta(a)] + \\
& +p(\delta)\cdot[z-y(\delta)] + \#\, A^L(\delta)\, d(\delta) + \min\,\{0,\varphi(\delta)\} \\
\leqq \quad & -\varphi(\delta) - p(\delta)\cdot \sum_{a\in A^H(\delta)} [\widetilde{w}_\delta(a) - r_\delta(a)] + \\
& +\#\, A^L(\delta)\, d(\delta) + \min\,\{0,\varphi(\delta)\}\,,
\end{aligned}$$

since $p(\delta)\cdot y(\delta) \geqq p(\delta)\cdot z$ by definition of $y(\delta)$. Next we consider two cases:

- $\varphi(\delta) > 0$: In this case

  $$\begin{aligned}
  \sum_{a\in A} V(a,\delta) + p(\delta)\cdot[z-c(\delta)] &\leqq -\varphi(\delta) - p(\delta)\cdot \sum_{a\in A^H(\delta)} [\widetilde{w}_\delta(a) - r_\delta(a)] \\
  &\qquad +\#\, A^L(\delta)\, d(\delta) \\
  &< -p(\delta)\cdot \sum_{a\in A^H(\delta)} [\widetilde{w}_\delta(a) - r_\delta(a)] \\
  &< 0,
  \end{aligned}$$

  since $\#\, A^L(\delta)\, d(\delta) < \varphi(\delta)$, $w_\delta(a) \gg r_\delta(a)$ for every $a \in A$ and $p\,(\delta) \gg 0$.

- $\varphi(\delta) \leqq 0$: In this case

  $$\begin{aligned}
  \sum_{a\in A} V(a,\delta) + p(\delta)\cdot[z-c(\delta)] &\leqq -p(\delta)\cdot \sum_{a\in A^H(\delta)} [\widetilde{w}_\delta(a) - r_\delta(a)] + \\
  &\qquad +\#\, A^L(\delta)\, d(\delta) \\
  &< 0
  \end{aligned}$$

Condition (iii).

We first recall that $p(\delta) \gg 0$ for all $\delta \in \Gamma$.[10] Now we verify condition (iii) of the definition of a valuation equilibrium. Let $a \in A$. First, from the definition of $V(a,\gamma)$ it directly follows that

$$p(\gamma) \cdot f(a) + p(\gamma) \cdot r_\gamma(a) + V(a,\gamma) = p(\gamma) \cdot \widetilde{w}_\gamma(a). \tag{20}$$

This shows that $\left(\widehat{f}_w(a), \hat{w}_\gamma(a)\right) \in B_\gamma(a)$.

To show that $\left(\widehat{f}_w(a), \hat{w}_\gamma(a)\right)$ is a maximal element in $\cup_{\delta\in\Gamma} B_\delta(a)$ let $\delta \in \Gamma$. We distinguish three cases:

1. If $a \in A^H(\delta)$, then there is no $g \in \mathbb{R}_+^{L(\delta)}$ and $v \in W(a,\delta)$ such that $U_{\delta,v}(a,g) > U(a, \hat{f}_w(a))$. This trivially implies that $\left(\widehat{f}_w(a), \hat{w}_\gamma(a)\right)$ is U-superior to anything in $B_\delta(a)$.

2. If $a \in A^L(\delta)$, then since $x(a,\delta) = 0$ we have that $V(a,\delta) > p(\delta) \cdot [\widetilde{w}_\delta(a) - r_\delta(a)]$ and, thus, by Lemma 3(c): $V(a,\delta) > p(\delta) \cdot [\widetilde{v}_\delta - r_\delta(a)]$ for any $v \in W(a,\delta)$. Hence, $B_\delta(a) = \emptyset$, again implying that $\left(\widehat{f}_w(a), \hat{w}_\gamma(a)\right)$ is U-superior to anything in $B_\delta(a)$.

3. If $a \in A^*(\delta)$, then, since $p(\delta) \gg 0$, for any $g \in \mathbb{R}_+^{L(\delta)}$ and any $v \in W(a,\delta)$, $U_{\delta,v}(a,g) > U_{\gamma,w}(a, f(a))$ implies by definition that $p(\delta) \cdot g > p(\delta) \cdot x(a,\delta)$ and, so, by Lemma 3(c),

$$\begin{aligned} p(\delta) \cdot g + p(\delta) \cdot r_\delta(a) + V(a,\delta) &= \\ p(\delta) \cdot g + p(\delta) \cdot w_\delta(a) - p(\delta) \cdot x(a,\delta) &> p(\delta) \cdot \widetilde{w}_\delta(a) \geqq p(\delta) \cdot \tilde{v}_\delta. \end{aligned}$$

   Thus, $(\widehat{g}_v, v) \notin B_\delta(a)$.

The three cases discussed above can be combined to conclude that $\left(\widehat{f}_w(a), \hat{w}_\gamma(a)\right)$ is a maximal element in $\cup_{\delta\in\Gamma} B_\delta(a)$.

This completes the proof of Theorem 2.

[10] Indeed, for $\delta \in \Gamma$ with $A^H(\delta) \neq \emptyset$ we have $p(\delta) \gg 0$ by definition. Similarly for $A \setminus A^L(\delta) = \emptyset$. Finally for the case that $A^H(\delta) = \emptyset$ and $A \setminus A^L(\delta) \neq \emptyset$, it was shown in Lemma 4(a) that $p(\delta) \gg 0$.