Quasiconcavity and the kernel of a separable utility.

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Abstract

In this paper I prove that a quasiconcave separable utility defined on an atomless space is concave.

1 Introduction.

The assumption of a separable utility function is the commonest in economic theory. The discounted utilities so useful in growth theory are separable utilities. Any von Neumann-Morgenstern utility is a separable utility. In general equilibrium with an infinite number of goods, separable utilities also appear, sometimes in a negative fashion: In Bewley(1972) equilibrium existence paper as an example of a Mackey continuous utility which is in general not weakly continuous. Separable utilities satisfying Inada's condition furnishes the simplest examples of preference relations that are not uniformly proper in an essential way. The known proofs of incomplete markets equilibrium existence with a continuum of states asks for separable utilities.

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It is easy to give examples of concave separable utilities functions. It is not so easy to give examples of quasiconcave separable utilities that are not concave. In this paper I show that this is in the nature of things: a quasiconcave separable utility defined on an atomless space (like $L^p[0,1]$) is concave. Moreover its kernel (i.e the integrand) is therefore almost everywhere concave.

2 Basic definitions.

I denote by R the set of real numbers. Let (Ω, A, μ) denote a complete measure space. I denote by $L^p = L^p(\Omega, A, \mu)$, $p \in (0,\infty)$, the space of $p > 0$ integrable, A measurable functions. If $p = \infty$, L^p is the space of essentially bounded measurable functions[1]. The set L^p_+ is the positive cone of L^p. The norm of an element $x \in L^p$ is denoted $|x|_p$.

Definition 1 (Separable utility) *A function $U : L^p_+ \to R$ is a separable utility if there exists $v : \Omega \times R_+ \to R$ such that:*

(a) *v is $A \times B(R_+)$ measurable;*

(b) *for every $x \in L^p_+$ the function $w \to v(w, x(w))$ is A measurable and integrable;*

(c) *For every $x \in L^p_+$, $U(x) = \int_\Omega v(w, x(w))d\mu(w)$.*

The function v is called the kernel of U. Given a kernel v a functional operator is naturally defined. Define $\rho : R_+^{\Omega} \to R^{\Omega}$ by $\rho(x)(w) = v(w, x(w))$ for $w \in \Omega$.

Remark 1 *A well known condition for the measurability needed in (b) is that for almost every $w \in \Omega$, $v(w, \cdot)$ to be continuous in R_+. If this happens I will say that the kernel is almost everywhere continuous.*

Theorem 1 (Continuity of separable utilities.) *Suppose (Ω, A, μ) is a complete measure space and $1 \le p \le \infty$. Suppose also that $U : L^p_+ \to R$ is a separable utility with an almost everywhere continuous kernel. Then U is norm continuous.*

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[1] more precisely equivalence classes of functions.

Proof. Suppose to obtain a contradiction that U is not continuous at $x \in L^p_+$. Then there is a sequence $x^n \in L^p_+$ and an $\varepsilon > 0$ such that $|U(x^n) - U(x)| \geq \varepsilon$ and $|x^n - x|_p < 1/n^2$. Since $\sum_{n=1}^{\infty} |x^n - x|_p < \infty$ the series $\tilde{z} = x + \sum_{n=1}^{\infty} |x^n - x|$ converges in L^p. Define $z = \tilde{z}\chi_{\{w;\tilde{z}(w)<\infty\}}$. The correspondence $S : \Omega \to R_+$, $S(w) = [0, z(w)]$ has an $A \times B(R_+)$ measurable graph since $w \to d(x, S(w)) = \min_{y \in S(w)} |x - y| = (x - z(w))^+$ is measurable for every $x \in R_+$. Define the correspondence $\Gamma : \Omega \to R_+$,

$$\Gamma(w) = \{x \in [0, z(w)]; |v(w, x)| = \max\{|v(w, y)|; y \in [0, z(w)]\}\}.$$

The correspondence Γ has from Lemma III.39 (and its application), page 86 of Castaing and Valadier(1977) book, a measurable selection $f : \Omega \to R_+$. Since $0 \leq x^n(w) \leq z(w)$ for almost every $w \in \Omega$, $|\rho(x^n)| \leq |\rho(f)|$. Since $f \in L^p$ we have by hypothesis that $\rho(f) \in L^1$. We can therefore apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{n\to\infty} U(x^n) = \lim_{n\to\infty} \int v(w, x^n(w))d\mu = \int v(w, x(w))d\mu = U(x),$$

a contradiction. Therefore U is continuous. QED

The following theorem proved by Krasnoselskii(1964) for the interval $[0, 1]$ can be proved with the technique above.

Theorem 2 (Krasnoselskii) *Suppose the kernel v is almost everywhere continuous and $r \geq 1$. Then if $\rho(L^p_+) \subset L^r$ the operator $\rho|_{L^p_+} : L^p_+ \to L^r$ is continuous in the norm topology.*

Proof. For the original proof I refer the reader to Krasnoselskii(1964) book, theorem 2.1, pages 22-25. A proof along the lines of theorem's 1 proof can easily be done and will be omitted.

3 Quasiconcavity and separable utilities.

In this section I prove the main result of this paper.

Theorem 3 *Suppose (Ω, A, μ) is an atomless complete measure space and $1 \leq p \leq \infty$. Suppose $U : L^p_+ \to R$ is a quasiconcave separable utility with an almost everywhere continuous kernel. Then U is concave.*

Proof. Suppose $U : L^p_+ \to R$ is a quasiconcave separable utility function with an almost everywhere continuous kernel v. Suppose $x, y \in L^p_+$, $\lambda \in (0,1)$ and $\theta = 1-\lambda$. I will prove that $U(\lambda x+\theta y) - \lambda U(x) - \theta U(y) \geq 0$. First take a sequence of A measurable simple functions $x^N := \sum_{n=1}^{m_N} x_{nN}\chi_{E^N_n} \geq 0, y^N := \sum_{n=1}^{m_N} y_{nN}\chi_{E^N_n} \geq 0$ such that $|x^N - x|_p \to 0$ and $|y^N - y|_p \to 0$. Since μ is atomless, using Lyapunov's theorem we can suppose without loss of generality that $\lim_{N\to\infty} \max_{n\leq m_N} \mu(E^N_n) = 0$. Consider for each $N \geq 1$ the function

$$\phi : R^{m_N}_+ \to R\ , \phi(x) = \sum_{n=1}^{m_N} \phi_n(x_n),\ \text{where } \phi_n(x_n) = \textstyle\sum_{n=1}^{m_N} \int_{E^N_n} v(w, x_n)d\mu.$$

Since $\phi(x) = U(\sum_{n=1}^{m_N} x_n\chi_{E^N_n})$ the quasiconcavity of ϕ follows from the quasiconcavity of U. A quasiconcave function which is a sum of m_N functions of independent variables is by a theorem of Debreu-Koopmans(1982) a sum of at least $m_N - 1$ concave summands. More precisely at most one of the summands ϕ_n is not a concave function. Define for a given N, $E^N_0 = \emptyset$ and define $k_N = 0$ if all summands are concave. If not all summands are concave define $k_N \in \{1,\ldots,m_N\}$ as the index of the unique non-concave summand. Then we have the following: $U(\lambda x^N + \theta y^N) - \lambda U(x^N) - \theta U(y^N) =$

$$\begin{aligned}
&\sum_{n\neq k_N} (\phi_n(\lambda x_{nN} + \theta y_{nN}) - \lambda\phi_n(x_{nN}) - \theta\phi_n(y_{nN})) + \\
&\phi_{k_N}(\lambda x_{k_N N} + \theta y_{k_N N}) - \lambda\phi_{k_N}(x_{k_N N}) - \theta\phi_{k_N}(y_{k_N N}) \geq \\
&\phi_{k_N}(\lambda x_{k_N N} + \theta y_{k_N N}) - \lambda\phi_{k_N}(x_{k_N N}) - \theta\phi_{k_N}(y_{k_N N}) = \\
\int_{E^N_{k_N}} &\left[v(w, \lambda x_{k_N N} + \theta y_{k_N N}) - \lambda v(w, x_{k_N N}) - \theta v(w, y_{k_N N})\right] d\mu = \qquad (1) \\
&U((\lambda x^N + \theta y^N)\chi_{E^N_{kN}}) - \lambda U(x^N\chi_{E^N_{kN}}) - \theta U(y^N\chi_{E^N_{kN}}).
\end{aligned}$$

Now suppose $p < \infty$. Then from $|x^N\chi_{E^N_{kN}}|_p \leq |x^N - x|_p + |x\chi_{E^N_{k_N}}|_p \to 0$ and by the same reasons, $|y^N\chi_{E^N_{kN}}|_p \to 0$, it follows from the continuity of U that

$$U((\lambda x^N + \theta y^N)\chi_{E^N_{kN}}) - \lambda U(x^N\chi_{E^N_{kN}}) - \theta U(y^N\chi_{E^N_{kN}}) \to 0.$$

Therefore $U(\lambda x+\theta y) - \lambda U(x) - \theta U(y) = \lim_{N\to\infty} U(\lambda x^N + \theta y^N) - \lambda U(x^N) - \theta U(y^N) \geq 0$. Suppose now that $p = \infty$. Take $M \geq |x^N|_\infty$ for all N. Then, for example,

$$|\int_{E^N_{k_N}} v(w, x^N_{k_N})d\mu| \leq \int_{E^N_{k_N}} |v(w, x^N)|d\mu \leq \int_{E^N_{k_N}} |v(w, f(w))|d\mu \to 0.$$

Here f is such that $|v(w, f(w))| = \max\{|v(w,y))|; 0 \le y \le M\}$. Analogously we consider the other summands in (1). Therefore $U(\lambda x+\theta y) - \lambda U(x) - \theta U(y) \ge 0$ as well. QED

Remark 2 *The discussion of convexity index in Debreu-Koopmans has been generalized and simplified by Crouzeix-Lindberg(1986) to which I refer the reader.*

A quasiconcave utility may not be concave if the space has atoms. The example below is valid in any measure space which has at least two finite measure atoms.

Example 1 (Quasiconcave but not concave separable utility) *Let us consider* $U : R_+^2 \to R$ *given by* $U(x,y) = -e^{-x} + (y+1)^2$. *This is a strictly monotonic utility. It is not concave since* $U_{yy} = 2 > 0$. *To check that it is quasiconcave consider for a given* $C > 0$ *the indifference curve*

$$U^{-1}(C) = \{(x, y(x)) : -e^{-x} + (y(x)+1)^2 = C\}.$$

I will show that $y(x)$ *is convex. Note that* $y(x) = \sqrt{C+e^{-x}} - 1$. *The first two derivatives of* y *are:*

$$y'(x) = \frac{-e^{-x}}{2}(C+e^{-x})^{-\frac{1}{2}} \text{ and } y''(x) = \frac{e^{-x}(C+e^{-x})^{-\frac{1}{2}}}{2}[1 - \frac{1}{2}\frac{e^{-x}}{(C+e^{-x})}] > 0.$$

The concavity of U imply as a corollary that the kernel must be for almost every w concave.

Theorem 4 *In an* σ*-finite measure space every concave separable utility has an almost everywhere concave kernel.*

Proof. First it shall be clear that $v(w,\cdot)$ is concave for almost every $w \in A$ if A is an atom of $\mathcal{A}$. So we can suppose without loss of generality μ to be atomless. Since μ is σ-finite there is a sequence $O_i \in \mathcal{A}$ such that $\cup_{i=1}^{\infty} O_i = \Omega$ and $\mu(O_i) < \infty$. If $v(w,\cdot)$ is concave for almost every $w \in O_i$ then the proof will be done. We may therefore without loss of generality suppose that μ is an atomless finite measure space. Take $D \subset R_+$ a countable dense set. And fix $x, y \in D$, $\lambda \in D \cap (0,1)$ and $\theta = 1-\lambda$. Since $\rho(x)$ and $\rho(y)$ are measurable and the space is atomless there is a sequence of partitions of Ω, $\{E_{nN}, 1 \le n \le N\}$ such that

$$f_x^N(w) := \sum_{n=1}^{N} \left(\frac{1}{\mu(E_{nN})}\int_{E_{nN}} v(w,x)d\mu\right)\chi_{E_{nN}}(w) \to v(w, x(w))$$

for almost every w. Analogously we define f_y and $f_{\lambda x+\theta y}$ for y and $\lambda x+\theta y$ respectively. From the concavity of U we have $U(\lambda x\chi_{E_{nN}}+\theta y\chi_{E_{nN}}) \geq \lambda U(x\chi_{E_{nN}})+\theta U(y\chi_{E_{nN}})$. Therefore if $w \in E_{nN}$ the following inequality is true:

$$\frac{1}{\mu(E_{nN})}\int_{E_{nN}} v(w,\lambda x+\theta y)d\mu(w) \geq \frac{1}{\mu(E_{nN})}\int_{E_{nN}}(\lambda v(w,x)+\theta v(w,y))d\mu(w).$$

So $f^N_{\lambda x+\theta y}(w) \geq \lambda f^N_x(w)+\theta f^N_y(w)$ for almost every $w \in \Omega$. Hence passing to the limit $N \to \infty$ we have $v(w,\lambda x+\theta y) \geq \lambda v(w,x)+\theta v(w,y)$ for almost every w. Now since D is countable we obtain as well this inequality for every $x,y \in D$ and $\lambda \in D\cap(0,1)$. From the continuity of $v(w,\cdot)$ we finish the proof: $v(w,\lambda x+\theta y) \geq \lambda v(w,x)+\theta v(w,y)$ for almost every w and every $x,y \in R_+, \lambda \in (0,1)$ and $\theta = 1-\lambda$.
QED

In the same vein we prove that if the separable utility is monotonic then the kernel is non-decreasing for almost every w.

Theorem 5 *If the separable utility is monotonic then for almost every $w \in \Omega$ the kernel is non-decreasing. Moreover if U is strictly monotonic the kernel is almost everywhere increasing.*

4 How to make examples of separable utilities.

If there is an $\alpha \in L^1$ and a $\beta > 0$ such that $|v(w,x)| \leq \alpha(w)+\beta x^p$ for almost every $w \in \Omega$ we have that $\rho(L^p_+) \subset L^1$ and therefore that the separable utility with kernel v is well defined. This condition, in atomless spaces, has been proved by Krasnoselskii (theorem 2.3 page 27 of his book), to be also necessary for U to be well defined. However the following condition is probably easier to apply for concave kernels. I denote by $v^+(w,y)$ the right-handed derivative of $v(w,\cdot)$ at $y \in R_+$. And $v^+(y) = (v^+(w,y(w))_{w\in\Omega}$ if $y \in L^p_+$. As usual q is the conjugate of p (i.e. $1/p+1/q = 1$).

Theorem 6 *Suppose the kernel is positive and almost everywhere concave. If there exists an $y \in L^p_+$ such that $\rho(y) \in L^1$ and $v^+(y) \in L^q$ then $\rho(L^p_+) \subset L^1$.*

Proof. Take $x \in L^p_+$. For almost every w, $0 \leq v(w,x(w)) \leq v(w,y(w))+v^+(y)(w)(x(w)-y(w))$. Since $v^+(y) \in L^q$, $v^+(y)(x-y) \in L^1$. Hence $\rho(x) \in L^1$.
QED

Remark 3 *The condition in the theorem above has a nice interpretation if* $y \in L^p_{++}$. *The requirement* $v^+(y) \in L^q$ *is then the requirement that the utility* U *to have a continuous price supporting the endowment* y *(were the utility defined). Then we conclude that the utility is really defined on* L^p_+ *and has a support at* y.

5 References

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