Simultaneous Pooled Auctions*

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Abstract

Suppose a seller wants to sell k similar or identical objects and there are $n > k$ potential buyers. Suppose that buyers want only one object. (This is a reasonable assumption in the sale of condominiums or in the sale of government-owned residential units to low-income families). In this case, we suggest the use of a simultaneous auction that would work as follows. Players are asked to submit sealed bids for one object. The individual with the highest bid chooses an object first; the individual with the second highest bid chooses the next object; and this process continues until the individual with the k^{th} highest bid receives the last object. Each individual pays the equivalent to his/her bid.

When objects are identical, we show that the proposed auction generates the same revenue as a first-price sealed-bid sequential auction. When objects are perfectly correlated, there is no known solution for sequential auctions, whereas we can characterize bidding strategies for the proposed pooled auctions. Moreover, the pooled auction is optimal since it satisfies a straightforward generalization of the revelation principle (Myerson, 1981) to k perfectly correlated objects. Thus, if the first-price sequential auction is optimal then it generates the same revenue as the pooled auctions. Otherwise, it generates less revenue. Therefore, the first-price sequential auction generates at most as much revenue than the pooled auction for identical and perfectly correlated objects. In addition, the pooled auction may be easier and cheaper to run, and bidders' strategies are simpler to compute since there are no interdependencies between sales as in the case of sequential auctions, i.e., the strategy space is smaller.

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JEL Classification: D44

1 Introduction

In some areas of the US and in most parts of Australia auctions are a primary method for selling real estate. According to Kravets (1993), the number of properties sold by auction "is growing at a geometric rate." While auctions were previously viewed as a way to dispose of distressed properties, the current view now is that real estate auctions are "an acceptable and profitable way for the real estate person to do business." (Sherman and Bussio (1994).) Although precise numbers are not available, it has been said that real estate sold at auction in 1988 reached more than $2.5 billion. (Chicago Tribune, November 13, 1988.)

This expansion in the use of auctions to sell real estate has been accompanied by a diversification of the auction terms being used. Kravets (1993) distinguishes between pooled unit bidding[1] and sequential bidding[2]. For example, Vanderporten (1992) reports the use of particular pooled auction in the sale of similar condominium units and of like-sized adjoining tracts of land. In this auction, a lot is formed with all items for sale. Oral bids are submitted with the highest bidder winning the right to choose one item from among the objects in the lot. A second auction follows for the right to choose another object from the remaining unclaimed objects. This procedure is repeated until all objects are disposed.

Auctions can also be described by their rules of bidding. Sherman and Bussio (1994), for example, list four types of rules; (i) public or oral auctions - buyers attend the auction site and bid against each other until someone wins (as in the auction reported by Vanderporten). The most popular of such auctions is known as the English auction where the auctioneer requests bids in an increasing order; (ii) sealed bid auctions - bids are mailed in, and each buyer is notified by mail of the highest bid; (iii) spot bid auction - buyers bring sealed bids to the auction site where the auctioneer announces the winning bid; and (iv) negotiated sales - written and telephone offers are taken by the auctioneer before the auction date. The highest offer is accepted on the auction date.

Thus, one can think of various combinations between auction terms and rules of bidding. However, not all combinations are used in practice.[3] In this paper we explore the properties of a simultaneous pooled auction that would work as

[1]Similar properties form a lot and the highest bidder chooses any one property in the pool.
[2]Objects are auctioneed off one at at time in a pre-arrenged order.
[3]For a survey of the theory and practice of real estate auctions, see Quan (1994).

follows. Suppose a seller wants to sell k similar or identical objects and there are $n > k$ potential buyers. Suppose buyers want only one object.[4] Players are asked to submit sealed bids for the right to choose an object. The individual with the highest bid chooses an object first; the individual with the second highest bid chooses the next object; and this process continues until all objects are sold.

When objects are identical, we show that the proposed auction generates the same revenue as a first-price sealed-bid sequential auction.[5] When objects are perfectly correlated, however, there is no known solution for sequential auctions,[6] whereas we can characterize bidding strategies for the simultaneous pooled auctions. Moreover, the pooled auction is optimal since it satisfies a straightforward generalization of the revelation principle (Myerson, 1981) to k perfectly correlated objects. Thus, if the first-price sequential auction is optimal then it generates the same revenue as the pooled auction. Otherwise, it generates less revenue. Therefore, the first-price sequential auction generates at most as much revenue as the proposed pooled auction for identical and perfectly correlated objects. In addition, the pooled auction may be easier and cheaper to run, and bidders' strategies are simpler to compute since there are no interdependencies between sales as in the case of sequential auctions, i.e., the strategy space is smaller. Ultimately, the choice of the auction mechanism will be determined by the comparison of expected revenue and expected costs. One of the advantages of our direct approach of characterizing bidding strategies is to allow empirical tests of the theory and to provide proper theoretical foundations for laboratory experiments.

Notice that we are not arguing that auctions must be used instead of traditional broker channels.[7]Instead, we are pointing out that whenever a decision has been made to use an auction, the proposed simultaneous pooled auction is an attractive option. We have to add the following caveats. This method is particularly appropriate when potential buyers want to buy only one unit, and are reasonably

[4]For condominium auctions of similar units, for example, this assumption is realistic. (See Vanderporten (1992).) This assumption is also particularly realistic in the sale of government-owned residential units to low-income families.

[5]For an analysis of sequential auctions of identical objects, see Weber (1983).

[6]In the next section we survey the existing papers on sequential pooled auctions. There are no general results for these auctions either. The reasons are analogous to the inexistence of results for sequential auctions, namely, the interdependencies between sales. See the discussion in Menezes and Monteiro (1995).

[7]This issue is addressed, for example, by Mayer (1995) and Vandell and Riddiough (1992). These authors emphasize a trade-off between the speed of the sale and expected revenue. The underlying idea is that auctions are faster than a traditional sale by a broker but generate less revenue because a quick sale implies a poorer "match", on average, between property and buyers.

indifferent between the units they buy, but expect to pay a lower price for a less valuable unit. Moreover, units must be either identical or individuals must agree on how to rank them. For example, in the sale of apartments to first-time home-buyers, our model would be appropriate if buyers agree that, *ceteris paribus*, first-floor apartments should be cheaper than second-floor apartments, and so on. Thus, this method may be a viable alternative to be used by the Resolution Trust Corporation or by the various local housing trust funds in Australia in the disposition of properties with the characteristics above.

This paper is organized as follows. In the next section we review the existing research on (sequential) pooled auctions. Section 3 formalizes our model of simultaneous pooled auctions, while in Section 4 we derive the equilibrium bidding strategies. Section 5 contains the revenue comparison for the case of identical objects and the expected revenue in the case of perfectly correlated objects. We also develop in this section a comparative statics exercise examining what happens to the expected revenue when the degree of correlation between objects changes. Our conclusions are summarized in Section 6.

2 Literature Review

Vanderporten (1992) was, to the best of our knowledge, the first author to specifically analyse pooled or right-to-choose auctions. He examined auctions of the type used to sell condominiums, where oral bids are solicited by the auctioneer with the highest bidder winning the right to choose one condominium among the ones being offered for sale. After the choice is made, a second round follows for the right to choose another condominium from the remaining units. The auction continues until all units are sold.

Vanderporten considers an auction with two homes for sale and two potential bidders. Each bidder wants to purchase only one of the two homes and will drop out of the auction if that home is sold to the other bidder. A simple discrete probability distribution (a binomial distribution) for the bidders's values is considered. For a numerical example, it is shown that the expected revenue to the seller from the pooled auction can never be greater than that from a sequential English auction and may be as much as 14% less. Moreover, the variance of the expected revenue is lower in the sequential pooled auctions than in the sequential English auctions. The author argues that this lower variance reduces the risk to the seller of default by the buyer.

Vanderporten's model provides an insightful approach to sequential pooled auctions. However, his conclusions may not hold under alternative assumption (e.g., when the two objects are stochastically identical, i.e., when valuations for the two objects are drawn from the same distribution; or when there are more bidders than objects).

Gale and Hausch (1994) consider the sale of two stochastically identical objects but allow a bidder's valuation to be dependent across objects.[8] They examine two second-price sealed-bid auction formats: standard sequential auctions and sequential pooled auctions. In their model, the two buyers decide whether to enter prior to each auction. For the standard sequential auction, if one of the buyers bids in the first auction and loses, then she wins the second auction and pays zero. If none of the buyers enters the first auction, then they bid their true valuations for the second object. There are three possible equilibrium bids in the first auction, depending on the relationship between the values of the two objects for a particular bidder: (1) she does not enter the first auction; (2) she enters the auction and submits a bid of zero ; or (3) she bids the difference between the first and second object. A bidder may want to bid zero in order to guarantee the first object (her least preferred object) for a price of zero in the case her opponent does not participate in the first auction. Gale and Hausch refer to this property of the equilibrium bid as bottom-fishing.

The analysis of sequential pooled auctions is more straightforward since the loser of the first auction is guaranteed to win the second object for a price equal to zero. Gale and Hausch show that both bidders submit a bid equal to half of the difference between their values for the two objects in the first auction. (The intuition is that a bidder will shade her bid in the first auction by the expected profits from participating in the last auction. In this context, If bidder i loses the first auction, her opponent is equally likely, by assumption, to choose either object. Then Bidder i wins the second object and pays zero. Therefore, the net surplus from losing the first auction is equal to one half of the sum of the value of the two objects and her bid in the first auction is equal to the difference between her value and this net surplus.) As a result, the sequential pooled auctions generate more revenue than the standard sequential auctions whenever the latter exhibits declining expected revenue.

The above result depends crucially on the assumption of two bidders and two objects. For example, if we have three bidders and two objects, the expected rev-

[8]That is, Buyer i has valuations x_i and y_i, which are privately known and drawn independently from a strictly positive density function; x_i and y_i may be correlated, and x_j and y_j may be correlated, $i \neq j$, but all other pairs such as x_i and x_j are independent.

enue from the second pooled auction is different from zero. Moreover, by bidding zero in the first round of the sequential auction, a bidder does not hedge against not receiving her most-preferred object. In this case, bidders may bid more aggressively in the first auction depending on the degree of correlation between the two objects for each bidder. There are no general results for the case of k objects and n bidders ($n > k$) and the difficulties arise from the interdependencies between the results of the auction. (Since first-round bids may provide information regarding bidders' values, bidders' behavior may be very complex). In contrast, these interdependencies between sales are - by definition - inexistent in the simultaneous pooled auction that we examine in the next section.

3 The Model.

Suppose a seller wants to sell k units to n potential buyers ($n > k$). Suppose, further, that each buyer only wants one object. For simplicity, we consider the case where the seller's reserve price is zero. We assume that objects are correlated as follows. If the value of the first object for buyer i is equal to $x \in [0,1]$ [9], then the value of the second object is $V_2(x)$, the third $V_3(x)$,... , the k^{th} $V_k(x)$. This function is assumed to be nonincreasing in the following sense: $x = V_1(x) \geq V_2(x) \geq ... \geq V_k(x)$. The analysis of this paper can be generalized to the case of nondecreasing values and to the case of players with distinct functions V. In our framework, each player knows his own vector of values, but only knows the distribution of his opponents' values. The value x^i is draw from the distribution F with continuous density $f(x) > 0$ and support $[0,1]$.

Each bidder submits only one bid and is allocated one object if the bid is among the k highest bids. The object he receives depends on the ranking of his bid. The bidder with the highest bid receives his most-valued object and so on. Thus, when submitting their bids, players must take into account the possibility that they may receive any of the objects (or none if the bids falls below the k^{th} highest bid). Let's examine the game from the perspective of an arbitrary player, say Player 1. Player 1's expected profit given that his value is equal to x, he submits a bid equal to l, and everyone else submits a bid $b(x_i)$, $\forall i \neq 1$, is equal to:
(1) $\pi_1(x,l) = (V_1(x) - l)\Pr\left(l > (b(x_i))_{\forall i \neq 1}\right) + (V_2(x) - l)\Pr($ largest bid among all $i \neq 1 > l \geq$ second largest bid among all $i \neq 1) + ... + (V_k(x) - l)\Pr(k - 1^{th}$

[9] This restriction is just for normalization purposes. Our results can be generalized for any distribution with bounded support.

largest bid among all $i \neq 1 > l \geq k^{th}$ largest bid among all $i \neq 1$).
Let's set $g = b^{-1}$ and assume that b is increasing and differentiable (we will show that in the next section). Thus, we can rewrite (1) as follows:
(2) $\pi_1(x,l) = (V_1(x) - l)\Pr\left(g(l) > (x_i)_{\forall i \neq 1}\right) + (V_2(x) - l)\Pr($ largest value among all $i \neq 1 > g(l) >$ second largest value among all $i \neq 1) + ... + (V_k(x) - l)\Pr(k-1^{th}$ largest value among all $i \neq 1 > g(l) > k^{th}$ largest value among all $i \neq 1)$.
Using symmetry the above expression can be written as:
(3) $\pi_1(x,l) = (V_1(x) - l)(F(g(l)))^{n-1} + (V_2(x) - l)(n-1)(1 - F(g(l))(F(g(l))^{n-2} + ... + (V_k(x) - l)C_{n-1}^{k-1}(1 - F(g(l))^{k-1}(F(g(l))^{n-k}$
Expression (3) can be written more economically as:
(4) $\pi_1(x,l) = \sum_{t=1}^{k}(V_t(x) - l)C_{n-1}^{t-1}(1 - F(g(l))^{t-1}(F(g(l))^{n-t}$
The problem that our arbitrary player faces is to choose l (given x) to maximize the above expression. This problem will be dealt with in the next section.

4 Equilibrium Bidding Strategies.

In the next proposition we characterize the equilibrium bidding strategy in a symmetric equilibrium. First, however, we need the following lemma. Define $\Psi_t(x) = (1 - F(x))^{t-1}(F(x))^{n-t}$.

Lemma 1 *For any $l, 1 \leq l \leq n$, and every $a \in R^{l+1}, a_{l+1} = 0$:*

$$\sum_{t=1}^{l} a_t C_{n-1}^{t-1}\Psi_t'(x) = f(x)\sum_{t=1}^{l}(a_t - a_{t+1})(n-t)C_{n-1}^{t-1}(1 - F(x))^{t-1}(F(x))^{n-t-1}.$$

Proof: The proof is easy and will be omitted.
Proposition 1: In a symmetric equilibrium, bidding strategies are given by**:**

$$(5)\ b(x) = \frac{\int_0^x \sum_{t=1}^{k} V_t(y)C_{n-1}^{t-1}\left((1 - F(y))^{t-1}(F(y))^{n-t}\right)' dy}{\sum_{s=1}^{k} C_{n-1}^{s-1}(1 - F(x))^{s-1}F(x)^{n-s}}$$

Proof: In the symmetric equilibrium, it must be the case that for any x, l is such that maximizes profits. Thus, we must have $\frac{\partial \pi_1}{\partial l} = 0$, i.e.(recall Ψ_t definition and

(4)):

(6) $\sum_{t=1}^{k}(V_t(x)-l)C_{n-1}^{t-1}\Psi_t'(g(l))g'(l)-\sum_{t=1}^{k}C_{n-1}^{t-1}\Psi_t(g(l))=0.$

We can now use the fact that in a symmetric equilibrium l must be set equal to b, and we can replace $g'(l)$ in expression (6) by $\frac{1}{b'(x)}$. Moreover, since by definition $g(l)=b^{-1}(l)$, we can replace $g(l)$ by x. Condition (6) then becomes:

(7) $\sum_{t=1}^{k}(V_t(x)-b(x))C_{n-1}^{t-1}\Psi_t'(x)\frac{1}{b'(x)}=\sum_{t=1}^{k}C_{n-1}^{t-1}\Psi_t(x).$

Notice that we can write this expression as a first-order differential equation as follows:

(8) $b'(x)\ \sum_{t=1}^{k}C_{n-1}^{t-1}\Psi_t(x)+b(x)\sum_{t=1}^{k}C_{n-1}^{t-1}\Psi_t'(x)=\sum_{t=1}^{k}V_t(x)C_{n-1}^{t-1}\Psi_t'(x).$

We have:

$$\left(b(x)\ \sum_{t=1}^{k}C_{n-1}^{t-1}\Psi_t(x)\right)'=b'(x)\sum_{t=1}^{k}C_{n-1}^{t-1}\Psi_t(x)+b(x)\ \sum_{t=1}^{k}C_{n-1}^{t-1}\Psi_t'(x)=\sum_{t=1}^{k}V_t(x)C_{n-1}^{t-1}\Psi_t'(x)$$

Integrating and choosing $b(0)=0$ we obtain $b(x)$.

From inspection of (5), equilibrium bids are an average over the set of all possible values so that bidders behave as if they were bidding for an “average ” object. Of course, this auction is very different from a standard single-object independent private value auction, since a bidder’s value for this average object depends on how other bidders behave. However, this analogy will be useful later when we compare the expected revenue resulting from this auction with the expected revenue from a standard sequential auction when objects are identical.

Contrarily to standard auction theory models, the equilibrium bidding function given by (5) may not be increasing in x. For example, when objects are not identical, a player faces a potential trade-off: increasing her bid increases the chances of receiving her most-valued object but it may decrease her profits if she wins. The existence of the trade-off depends on the difference between the values of the objects. [10] Thus, we need an additional assumption to guarantee that any convex combination of the values for the objects is increasing in x. In particular, we assume that $\frac{\partial V_t(x)}{\partial x} \geq 0$. This condition simply says that the t^{th} highest value increases as x increases. For example, in the case of a constant degree of correlation (e.g., t^{th} object is worth λ times the value of the $t-1^{th}$ object, $0 < \lambda < 1$, for any $1 \leq t < k$) this assumption is trivially satisfied. We then have the following proposition.

Proposition 2: If $\frac{\partial V_t(x)}{\partial x} \geq 0, t \geq 1$ then $b'(x) > 0\ \forall x, x \in (0,1)$.

Proof. We can write $b(x)$ as:

$$(9)\ b(x) = \left(\sum_{t=1}^{k} C_{n-1}^{t-1}\Psi_t(x)\right)^{-1} \left(\sum_{s=1}^{k} V_s C_{n-1}^{s-1}\Psi_s(x) - \int_0^x \sum_{s=1}^{k} \frac{\partial V_s(y)}{\partial y} C_{n-1}^{s-1}\Psi_s(y)dy\right).$$

Taking the derivative yields:

$$b'(x) = \left(\sum_{s=1}^{k} V_s C_{n-1}^{s-1}\Psi_s(x) - \int_0^x \sum_{s=1}^{k} \frac{\partial V_s(y)}{\partial y} C_{n-1}^{s-1}\Psi_s(y)dy\right) \left(-\frac{\sum_{t=1}^{k} C_{n-1}^{t-1}\Psi_t'(x)}{\left(\sum_{t=1}^{k} C_{n-1}^{t-1}\Psi_t(x)\right)^2}\right) +$$

$$\left(\sum_{t=1}^{k} C_{n-1}^{t-1}\Psi_t(x)\right)^{-1} \left(\sum_{s=1}^{k} C_{n-1}^{s-1} V_s(x)\Psi_s'(x)\right)$$

By the lemma, $\sum_{t=1}^{k} C_{n-1}^{t-1}\Psi_t'(x) = f(x)C_{n-1}^{k-1}(n-k)(1-F(x))^{k-1}F(x)^{n-k-1} > 0$ if

[10] For example, consider the following example. Let $k = 2$ and $n = 3$. Let the function $V(x)$ be given by:

$$V_1(x) = x$$
$$V_2(x) = \begin{cases} x & 0 \leq x \leq a \\ (2a-x)^+ & a \leq x \end{cases}$$

If we set $a = 1/10$ and use our equilibrium bidding strategy we obtain:

$$b(x) = \begin{cases} \frac{x(3-2x)}{3(2-x)} & 0 \leq x \leq a \\ \frac{-7/325+2x/5-7x^2/5+2x^3}{2x-x^2} & a \leq x \leq 2a \\ \frac{\frac{2}{125}+2x^3/3}{2x-x^2} & 2a \leq x \end{cases}$$

Note that $b'(2a) = -0.04$. Therefore, (5) cannot be an equilibrium bidding strategy for this example.

$x \in (0,1)$. By our assumption,
$\int_0^x \sum_{s=1}^{k} \frac{\partial V_s(y)}{\partial y} C_{n-1}^{s-1} \Psi_s(y) dy \geq 0.$
Thus, we can write

(10) $b'(x) \left(\sum_{t=1}^{k} C_{n-1}^{t-1} \Psi_t(x) \right)^2 >$
$- \sum_{t=1}^{k} \sum_{s=1}^{k} C_{n-1}^{s-1} C_{n-1}^{t-1} V_s(x) \Psi_t'(x) \Psi_s(x) + \sum_{t=1}^{k} \sum_{s=1}^{k} C_{n-1}^{s-1} C_{n-1}^{t-1} V_s(x) \Psi_s'(x) \Psi_t(x)$

If we define $w_k(t) = \sum_{t=1}^{k} \sum_{s=1}^{k} C_{n-1}^{s-1} C_{n-1}^{t-1} V_s(x) \left(\Psi_t \Psi_s' - \Psi_s \Psi_t' \right)$, then it suffices to show that $w_k(t) \geq 0$. We will use induction to prove that $w_k(t) \geq 0$. For $k = 1$ we have: $w_1(x) = V_1(x) \left(\Psi_1 \Psi_1' - \Psi_1 \Psi_1' \right) = 0$.
Before we proceed with the induction, we need to compute $(\Psi_t \Psi_s' - \Psi_s \Psi_t')$ for arbitrary t and s: From

(11) $\Psi_s' = (1-F)^{s-2} F^{n-s-1} f \left((n-s)(1-F) - (s-1)F \right)$

we obtain that:

(12) $\Psi_t \Psi_s' = (1-F)^{t+s-3} F^{2n-s-t-1} f \left((n-s)(1-F) - (s-1)F \right)$

(13) $\Psi_s \Psi_t' = (1-F)^{t+s-3} F^{2n-s-t-1} f \left((n-t)(1-F) - (t-1)F \right)$

Using (11), (12) and (13) we obtain:

(14) $(\Psi_t \Psi_s' - \Psi_s \Psi_t') = f(x) \, (t-s) \, (1-F)^{t+s-3} F^{2n-s-t-1}$

Thus, we can rewrite $w_k(t)$:

(15) $w_k(t) = \sum_{t=1}^{k} \sum_{s=1}^{k} C_{n-1}^{s-1} C_{n-1}^{t-1} V_s (1-F)^{t+s-3} F^{2n-s-t-1} f \cdot (t-s).$

The induction step is to assume that $w_k(t)$ as given in (15) is greater or equal to zero for $k < n$ and check whether $w_{k+1}(t)$ is also greater or equal to zero, where $k+1 \leq n$. Using (15) we can write:

(16) $w_{k+1}(t) = \sum_{t=1}^{k+1} \sum_{s=1}^{k+1} C_{n-1}^{s-1} C_{n-1}^{t-1} V_s (1-F)^{t+s-3} F^{2n-s-t-1} f \cdot (t-s)$

But this is equivalent to write:

(17) $w_{k+1}(t) = \sum_{t=1}^{k} \sum_{s=1}^{k} C_{n-1}^{s-1} C_{n-1}^{t-1} V_s (1-F)^{t+s-3} F^{2n-s-t-1} f \cdot (t-s) +$
$\sum_{t=1}^{k} C_{n-1}^{k} C_{n-1}^{t-1} V_{k+1} (1-F)^{t+k-2} F^{2n-k-t-2} f \cdot (t-k-1) +$

$$\sum_{s=1}^{k} C_{n-1}^{k} C_{n-1}^{s-1} V_s (1-F)^{k+s-2} F^{2n-k-t-2} f \cdot (k+1-s)$$

That is,

$$w_{k+1}(t) = w_k(t) + f(x) \sum_{t=1}^{k} C_{n-1}^{k} C_{n-1}^{t-1} (1-F)^{t+k-2} F^{2n-t-k-2} \left((t-(k+1))V_{k+1} + (k+1-t)V_t\right)$$

The above expression can be reduced further to:

$$w_{k+1}(t) = w_k(t) + f \sum_{t=1}^{k} C_{n-1}^{k} C_{n-1}^{t-1} (1-F)^{t+k-2} F^{2n-t-k-2} (k+1-t)(V_t - V_{k+1})$$

The induction is completed since $(V_t - V_{k+1})$ is greater than zero by assumption and $w_k(t)$ is greater than zero by the induction step.

5 The Revenue Comparison

In this section we first show that when objects are identical, the proposed auction generates the same revenue as a first-price sealed-bid sequential auction where k objects are sold one at a time. As in our model, bidders only want one object. Thus, in each round, the remaining bidders (i.e., those who have not received an object) submit sealed bids. The highest bidder is awarded an object and pays his bid. Each remaining bidder knows the sale price. We use Weber's (1983) Theorem 2 that states that the expected revenue in this sequential auction is given by $k \cdot E[x_{(k+1)}]$, i.e., k times the $(k+1)^{th}$ order statistics. The expected revenue from a simultaneous pooled auction when objects are identical is simply the expected value of the sum of the k highest bids (setting $V_t(x) = x, t = 1, \ldots, n,$ in (5)).

Let's denote by $\mathbf{R}_S$ the expected revenue in the sequential auction and by $\mathbf{R}_P$ the expected revenue in the simultaneous pooled auction. We then have the following proposition:

Proposition 3: If the objects are identical, then $R_S = R_P$.

Proof: The density of the l^{th} greatest value is

$$f^l(x) = \sum_{u=0}^{l-1} C_n^u [(1-F(x))^u (F(x))^{n-u}]', 1 \leq l \leq n.$$

Therefore

$$R_S = k \int x \sum_{u=0}^{k} C_n^u [(1-F(x))^u (F(x))^{n-u}]' dx \tag{1}$$

and

$$R_P = \int b(x) \sum_{l=1}^{k} \sum_{u=0}^{l-1} C_n^u [(1-F(x))^u (F(x))^{n-u}]' dx \tag{2}$$

Let us first consider (18). By the lemma with $n+1$ in place of n and $l = k+1$ we have

$$\sum_{u=0}^{k} C_n^u [(1-F(x))^u (F(x))^{n-u}]' = \sum_{u=0}^{k+1} C_n^{u-1} [(1-F(x))^{u-1} (F(x))^{n-u+1}]' = f(x) C_n^k (n-k)(1-F(x))^k F(x)^{n-k-1} \tag{3}$$

Substituting this expression in (1) we obtain

$$R_S = k \int x f(x) C_n^k (n-k)(1-F(x))^k F(x)^{n-k-1} dx$$

Let us consider now the integrand of $\mathbf{R}_P$. From (3) with $l-1$ in the place of k,

$$\sum_{u=0}^{l-1} C_n^u [(1-F(x))^u (F(x))^{n-u}]' = f(x) C_n^{l-1} (n-l+1)(1-F(x))^{l-1} F(x)^{n-l} = n f(x) C_{n-1}^{l-1} (1-F(x))^{l-1} F(x)^{n-l}.$$

Therefore substituting $b(x)$ in (2) we have that

$$R_P = n \int f(x) \left(\int_0^x y \sum_{t=1}^{k} C_{n-1}^{t-1} \Psi_t'(y) dy \right) dx = n \int y(1-F(y)) \sum_{t=1}^{k} C_{n-1}^{t-1} \Psi_t'(y) dy = n \int y f(y) C_{n-1}^{k-1} (n-k)(1-F(y))^k F(y)^{n-k-1} dy.$$

Since $kC_n^k (n-k) = nC_{n-1}^{k-1}(n-k)$ we finish the proof.

The above result seems less surprising if we use the interpretation that bidders behave in a simultaneous pooled auction as if they were bidding for an average object. When objects are identical, the average is equal to the value of the object so that effectively bidders bid as to beat the individual with the marginal valuation. But this is how individuals bid at the last round of the sequential auction which determines the expected price at all rounds.

The above result can also be explained in the context of the revelation principle. For the case of one object, Myerson (1981) shows, as a corollary to the revelation principle, that an auction is optimal if it satisfies two properties, namely, (1) the object always goes to the bidder with the highest value, and (2) bidders with the lowest possible valuation should expect zero profits. A generalization of Myerson's result for the case of k objects (identical or perfectly correlated) would require for an auction to be optimal, in addition to (2), that the highest-valued object should go to the bidder with the highest value, the second-highest-valued object should go the bidder with the second highest value, and so on; when the k objects are identical, the bidders with the k^{th} highest values should each receive an object.

It is easy to verify that when the objects are identical, both the pooled and the sequential auctions are optimal and, therefore, should generate the same revenue. When the objects are perfectly correlated it is still the case that the pooled auction is optimal; we can guarantee that the bidder with the t^{th} highest value receives the t^{th} highest-valued object since strategies are increasing functions. For the case of sequential auctions of perfectly correlated objects, however, there is no known solution. If strategies in each auction are increasing in values, then the revelation principle will guarantee its optimality and the revenue it generates will be equal to the revenue generated by the pooled auction. Otherwise, the sequential auction will generate less revenue than the pooled auction.

Therefore, a sequential auction will generate at most as much revenue as the proposed pooled auction for identical or perfectly correlated objects. In addition, we argue that the total social cost of running pooled auctions is smaller than the total social cost of running sequential auctions. From the auctioneer's perspective, running a simultaneous auction is cheaper since he only collects bids once; whereas the auctioneer has to collect bids k times in a sequential auction. From the bidders' point of view, pooled auctions are faster and they have to spend less time at the auction site. Moreover, bidding strategies are easier to compute for pooled auctions than for sequential auctions when there are independencies between bidding behavior in the various rounds. That is, the strategy space for the pooled auction is smaller than the strategy space for the sequential auction. Nevertheless, further evidence by means of either empirical tests or laboratory experiments is needed and one of the advantages of our direct approach is exactly to allow for that.

Finally, we can compute what happens to the expected revenue when the degree of correlation changes. Consider the sale of two objects and when the value of the first object for Player i is denoted by x_i and the value of the second object is denoted by λx_i, with $x_i \geq \lambda x_i, \forall i, i = 1, \dots, n$. The expected revenue can be written as:

$$\mathbf{R}_P = n(n-1)\int_0^1 f(x)[\int_0^x f(y)y(F(y))^{n-2}\,dy+$$

$$\lambda \int_0^x \left((n-2)\,(F(y))^{n-3} - (n-1)\,(F(y))^{n-2} \right) dy]dx$$

Thus, we can compute the derivative of $\mathbf{R}_P$ with respect to λ as follows:

$$\frac{\partial \mathbf{R}_P}{\partial \lambda} = n(n-1) \int_0^1 f(x) \left(\int_0^x \left((n-2)\,(F(y))^{n-3} - (n-1)\,(F(y))^{n-2} \right) dy \right) dx$$

This can be simplified to:

$$(21)\ \frac{\partial \mathbf{R}_P}{\partial \lambda} = n(n-1) \int_0^1 (1-F(x)) \left((n-2)\,(F(x))^{n-3} - (n-1)\,(F(x))^{n-2} \right) dx$$

From inspection of (21), we can conclude that $\frac{\partial \mathbf{R}_P}{\partial \lambda} > 0$. For any particular distribution F, it is possible to compute the precise value of the derivative. For the case of the uniform distribution, for example, $\frac{\partial \mathbf{R}_P}{\partial \lambda} = 1$, *.i.e.*, if the degree of correlation doubles, the expected revenue also doubles.

6 Conclusion

In this paper we examine a simultaneous pooled auction. When objects are identical, we show that the proposed auction generates the same revenue as a first-price sealed-bid sequential auction. When objects are perfectly correlated, we are able to characterize bidding strategies for the simultaneous pooled auction - which is optimal, whereas there is no known solution for sequential auctions. These strategies are not very different from the case of identical objects. Moreover, we can determine precisely the expected revenue and how it changes as the degree of correlation between objects changes.

Simultaneous pooled auctions may be easier and cheaper to run than sequential auctions since players submit only one bid. Bidders' strategies are simpler to compute given that there are no interdependencies between sales as in the case of sequential auctions. Moreover, our direct approach of characterizing bidding strategies allows empirical tests and provides proper theoretical foundations for laboratory experiments who could offer further evidence of the appropriateness of polled auctions. Our results do provide, however, a *prima facie* case for the use of simultaneous pooled auctions to sell government-owned residential units to low-income home buyers.

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