A Note on Auctions with Endogenous Participation*

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Abstract

In this paper, we study an auction where bidders only know the number of potential applicants. After seeing their values for the object, bidders decide whether or not to enter the auction. Players may not want to enter the auction since they have to pay participation costs.

We characterize the optimal bidding strategies for both first- and second-price sealed-bid auction when participation is endogenous. We show that only bidders with values greater than a certain cut-off point will bid in these auctions. In this context, both auctions generate the same expected revenue.

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We also show that, contrarily to the predictions of the fixed-*n* literature, the seller's expected revenue may decrease when the number of potential participants increases. In addition, we show that it is optimal for the seller to charge an entry fee, which contrasts greatly with results from the existing literature on auctions with entry.

1 Introduction

In the standard Independent Private Values (IPV) auction model it is assumed that bidders submit bids knowing how many opponents they will face. This assumption is clearly not appropriated for sealed-bid auctions, where participants may know the pool of potential applicants, but by definition cannot know how many bidders have submitted bids (unless this information is released by the auctioneer). In oral auctions, even though bidders have a more precise idea about the number of competitors, they only know who has bid before them and not who will bid in the future.

We departure from this hypothesis and examine an auction where bidders only know the number of potential applicants. After seeing their values for the object, bidders decide whether or not to enter the auction. Players may not want to enter the auction since they have to pay participation costs. (In the context of procurement auctions, participation costs can be interpreted as the costs of preparing a bid.)

We characterize the optimal bidding strategies for both first- and second-price sealed-bid auctions when participation is endogenous. We show that only bidders with values greater than a certain cut-off point will bid in these auctions. In this framework, both auctions generate the same expected revenue.

Contrarily to the prediction of the fixed-*n* literature, the seller's expected revenue may decrease when competition increases: when the number of potential participants increases so does the cut-off value. Thus, for some distributions of values, it may be optimal for the seller to engage in policies to reduce the number of potential participants. In addition, we show that it is optimal for the seller to charge an entry fee. This contrasts greatly with results from the existing literature on auctions with entry.

This paper is organized as follows. In the next section we provide a brief survey of auction models which consider entry. In section 3, we specify the model and the notation, derive the optimal bidding strategies and compute the expected revenue for both first- and second-price auctions. In section 4, we compare the expected revenue generated by the two auction formats and provide some new insights on the effects of increased competition and entry fees on expected revenue. In section 5, we examine the case when bidders learn the number of opponents before submitting their bids. Section 6 concludes.

2 Auctions with Entry: A Brief Survey

The question of entry in auctions has been addressed before. For instance, McAfee and McMillan (1987-a) consider a model where participants have to incur a cost before learning their values for the object. These authors use the revelation principle to show that a first-price auction induces the optimal number of bidders to enter. Engelbrecht-Wiggans (1987, 1993) also examines auctions with entry, assuming that potential bidders use pure entry strategies. That is, if only $n < N$ bidders enter the auction, then an asymmetric equilibrium is considered in which $N-n$ bidders stayed out. Note that neither McAfee and McMillan nor Engelbrecht-Wiggans identify the process by which identical bidders are divided into those who participate and those who do not participate in the auction.

In contrast, Levin and Smith (1994) explicitly consider a mechanism by which individuals decide whether or not to participate. Their model is different from ours in at least two dimensions. First, bidders incur a fixed cost of entry before seeing their values for the object. Second, a bidder who enters the auction knows how many opponents she faces. In their model, if there are N potential bidders, but only $n < N$ could enter the auction and make nonnegative profits, then each potential bidder enters with probability q and stays out with probability $1-q$. In equilibrium, n varies stochastically between 0 and N with probabilities exogenously determined by the auction format. For the IPV model they provide a revenue-equivalence theorem. They also show that in the IPV auctions with entry, the seller should not set a reservation prices or charge entry fees, since the seller has no reasons to discourage entry.

As Levin and Smith, we also explicitly consider a mechanism by which individuals decide whether or not to participate. We characterize the optimal bidding strategies when agents learn their values before deciding whether or not to incur a bid preparation cost - a very reasonable assumption in procurement auctions for example. We consider two cases, namely, when bidders know how many opponents they face and when they do not know. As Levin and Smith, we find that, for risk-neutral bidders, the expected revenue does not depend on whether or not the seller reveals the actual number of bidders.

Moreover, since in our model bid preparation costs screen low valuation bidders, entry fees are optimal and increased competition may actually reduce expected revenue. This should be contrasted to the unrestricted-entry result in Levin and Smith. The difference between the two models lies on the fact that, in our model, the participation decision is such that only induces high types to participate.

Our results should be seen as complementary to those of Levin and Smith. While we study the effects of entry when agents learn their values prior to incurring bid preparation costs, they examine the case where agents have to incur the cost prior to learning their values. One can think of examples of auctions where our model is more appropriate and vice-versa.

A different class of model is examined by Harstad et al. (1990), Matthews (1987) and McAfee and McMillan (1987-b) who have made the number of bidders random, with a known distribution. The last two papers adopt the IPV model, and analyze effects of bidder risk aversion across auction institutions. The institutional choice is whether the seller reveals the number of bidders, known by him in advance. Their analysis is indirect in the sense that equilibrium bid functions are not derived. For a first-price auction with bidders having constant absolute risk aversion, McAfee and McMillan show that the expected selling price is higher when bidders do not know how many other bidders there are than when they do know. Thus, the seller should conceal the number of bidders. Matthews has derived a similar conclusion when bidders have decreasing absolute risk aversion. Harstad et al. retain the hypothesis of risk-neutrality and investigate a general independent model which allows for asset value uncertainty. Assuming that the probability of participation is fixed, they show that the following five auctions are revenue-equivalent: FPSB and SPSB auctions, each with the number of bidders known or uncertain, and oral auctions. We replicate their results for the case of endogenous participation.

Finally, our model is very similar to that of Samuelson (1985), who considers an indirect approach for a generic mechanism satisfying some properties and finds that equilibrium entry achieves a welfare optimum but that increased competition may cause the expected revenue to fall. In contrast, we characterize equilibrium bidding strategies, including the cut-off value, for both first-price and second-price sealed-bid auctions when agents do not know the number of opponents they face and when they do. We compute the expected revenue generated by the two auctions and show that they are equal. We obtain a similar result to that of Samuelson with respect to the effects of increased potential competition and provide a new result with respect to entry fees. The advantage of our direct approach of characterizing bidding strategies is to allow empirical tests of auction theory with entry and to provide proper theoretical foundations for laboratory experiments.

3 The Model

We consider the sale of a single indivisible good through a sealed-bid auction. We assume that the reserve price is equal to zero. We denote by I the finite set of potential risk neutral participants, with $\#I = n \in N$, where # denotes the cardinality of the set. Accordingly to the IPV assumption, Bidder $i \in I$ knows her own value (v_i) for the object but only knows the distribution $F(v_j)$, $\forall j \neq i$, of other bidders' values. It is assumed that values are independently drawn from the continuous distribution F with support $[0, \overline{v}]$.

Bidders face participation costs c,which might be interpreted as the costs of preparing a bid. Given their values, each bidder decides whether or not to submit a bid (and pay c) without knowing how many bidders will submit bids. In what follows, we characterize the individual participation decision and derive the optimal bidding strategy for both first- and second-price sealed-bid auctions.

3.1 First-Price Sealed-Bid Auctions

To derive the optimal bidding strategy for Bidder i, let us define first i 's expected profits from participation:

(1) $\pi_i(v_i, b_i, b_{j,\, j \neq i}) =$

$$\sum_{H \subset I \setminus \{i\}} (v_i - b_i) E\left[\chi_{b_i > \max\limits_{j \in H} b_j(v_j)} \; \chi_{v_j > v_{\rho_F},\, \forall j \in H} \chi_{v_j < v_{\rho_F},\, \forall j \in (I \setminus \{i\} - H)} \right] - c$$

Where b_i denotes i's bid, and b_j, Player j's bidding strategy, for $j \neq i$. H denotes the set of participants and v_ρ is such that $\pi_i(v_\rho, b^*) = 0$ in a first-price sealed-bid auction, i.e., the cut-off value when all bidders use the equilibrium strategy b^*.[1] Equation (1) states that Player i's expected profits is equal to the difference between her value and her bid times the probability that she wins with a bid equal to $b_i(.)$. For every possible set of additional participants H, this probability is

[1] In the appendix we show that such strategy indeed exists. Note that Levin and Smith (1994), for example, assume the existence of a unique increasing symmetric Nash equilibrium bidding function.

simply the probability that i's bid is greater than the maximum bid among the $\#H$ other participants. In addition, Player i must take into account the fact that, in equilibrium, other bidders will participate if and only if their value are greater than the cut-off point v_ρ. This cut-off point is such that if a participant has a value v_ρ, then she is indifferent between entering and not entering the auction. Thus, v_ρ solves

(2) $v_\rho F(v_\rho)^{n-1} - c = 0$

where v_ρ denotes her profits conditional on winning and given that her bid is equal to zero and $F(v_\rho)^{n-1}$ the probability that she wins with a zero bid. That is, a participant with value v_ρ wins only if she is the sole participant.

If we assume that everyone else except Player i uses the same strategy b, we can rewrite (1) as follows:

(3) $$\pi_i(v_i, b_i, b) = (v_i - b_i)\sum_{k=0}^{n-1}\left[(F(v_\rho))^{n-1-k}\, C^k_{n-1}\left(F(b^{-1}(b_i)) - F(v_\rho)\right)^k\right] - c$$

From the binomial expansion formula we have:

(4) $$\pi_i(v_i, b_i, b) = (v_i - b_i)\left[F(v_\rho) + F(b^{-1}(b_i)) - F(v_\rho)\right]^{n-1} - c$$

Thus, maximizing π_i with respect to b_i yields:

(5) $$b^*(v) = \begin{cases} \dfrac{\int_{v_\rho}^{v}(n-1)xF(x)^{n-2}f(x)dx}{F(v)^{n-1}}, & v \geq v_\rho \\ 0, & v < v_\rho \end{cases}$$

According to our definition of v_ρ, we choose the normalization $b(v_\rho) = 0$. [2]

Next we compute the expected revenue generated by the first-price sealed-bid auction when bidders face participation costs. The expected revenue, R^1, is simply the expected value of the highest bid among those players who decide to participate:

[2] It is straightforward to check that $b(v_\rho) = 0$ is the correct normalization. Consider the expected profits of a player who has a value $v \leq v_\rho$. If this player enters the auction and bids zero, her expected profits are equal to $\pi(v, 0) = vF(v_\rho) - c \leq v_\rho F(v_\rho) - c = 0$, where the non-participation profits are equal to 0. On the other hand, if a player has value $v \geq v_\rho$, then we can prove that her expected profits from participation are greater than zero. In this case,

$$\pi(v, b(v)) = (v - b(v))\,F(v)^{n-1} - c = vF(v)^{n-1} - c - (n-1)\int_{v_\rho}^{v} xF(x)^{n-2}f(x)dx \geq$$
$$vF(v)^{n-1} - c - (n-1)v\int_{v_\rho}^{v} F(x)^{n-2}f(x)dx = vF(v)^{n-1} - c - (n-1)v\left(\frac{F(v)^{n-1} - F(v_\rho)^{n-1}}{n-1}\right)$$
$$= vF(v_\rho)^{n-1} - c = vF(v_\rho)^{n-1} - v_\rho F(v_\rho)^{n-1} \geq 0.$$

(6) $R^1 = \sum_{k=1}^{n} C_n^k \left(F(v_\rho)\right)^{n-k} E\left[b^*(v_1 \vee ... \vee v_k)\chi_{v_j \geq v_\rho,\, 1\leq j\leq k}\right]$

Since this is a symmetric problem, we can rewrite (6) as follows:

(7)$R^1 = \sum_{k=1}^{n} C_n^k \left(F(v_\rho)\right)^{n-k} kE\left[b^*(v_1)(F(v_1) - F(v_\rho))^{k-1}\chi_{v_1 \geq v_\rho}\right]$

Taking the expected value we obtain:

(8) $R^1 = \int_{v_\rho}^{\bar{v}} b^*(x) \sum_{k=1}^{n} C_n^k \left(F(v_\rho)\right)^{n-k} k(F(x) - F(v_\rho))^{k-1} f(x)dx$

Since $\sum_{k=1}^{n} kC_n^k \left(F(v_\rho)\right)^{n-k} \left(F(x) - F(v_\rho)\right)^{k-1} = n(F(v_\rho) + F(x) - F(v_\rho))^{n-1}$, we can write (8) as:

(9) $R^1 = \int_{v_\rho}^{\bar{v}} b^*(x) nF^{n-1}(x) f(x)dx$

From the previous expression, we can conclude that allowing participation to be endogenous reduces the auctioneer's revenue given a fixed population of potential bidders. This is, per se, quite straightforward. The revenue comparison and the other results of the next section are not. We now solve the individual problem and compute the expected revenue when a second-price sealed-bid auction is used.

3.2 Second-Price Sealed-Bid Auctions

It is not difficult to show that the optimal strategy in a second-price sealed-bid auction with endogenous participation is to bid the true value. (The proof is identical to the case of a fixed number of participants.) In the appendix we show that such a strategy is an equilibrium. Thus, we can write Bidder i 's expected profits in equilibrium as a function of her own value and the value of the other opponents who participate:

(10) $\pi_i(v_i, v_{j,j\neq i}) =$

$$\sum_{H\subset I\setminus\{i\}} E\left[\chi_{v_j > v_{\rho_S},\, \forall j\in H}\chi_{v_j < v_{\rho_S},\, \forall j\in(I\setminus\{i\}-H)}(v_i - \max_{j\in H} v_j)^+\right] - c$$

Where v_{ρ_s} is the cut-off value for the second-price auction. Since this is a symmetric problem, we have:

(11) $\pi_i(v_i, v_{j,j\neq i}) = v_i \left(F(v_{\rho_s})\right)^{n-1} - c +$

$$\sum_{k=1}^{n-1} C_{n-1}^k \left(F(v_{\rho_s})\right)^{n-k-1} kE_{v_i}[(v_i - v_2)\chi_{v_i \geq v_2 \geq v_{\rho_S}} \left(F(x) - F(v_{\rho_s})\right)^{k-1}]$$

Taking the expected value we obtain:

(12) $\pi_i(v_i, v_{j,j\neq i}) = v_i\left(F(v_{\rho_s})\right)^{n-1} - c +$

$$(n-1)\int_{v_{\rho_S}}^{v_i}(v_i - v_2)(F(v_2))^{n-2}f(v_2)dv_2$$

Integrating by parts yields:

(13) $\pi_i(v_i, v_{j,j\neq i}) = v_{\rho_S}\left(F(v_{\rho_s})\right)^{n-1} - c + \int_{v_{\rho_S}}^{v_i}(F(x))^{n-1}dx$

The cut-off point in the case of second-price auctions is obtained by solving the following equation:

(14) $v_{\rho_S}\left(F(v_{\rho_s})\right)^{n-1} = c$

Therefore, $v_{\rho_S} = v_\rho$. We now compute the auctioneer's expected revenue, R^2. It is simply the expected value of the second highest valuation among those who participate ($Y_H^{(2)}$):

(15) $R^2 = E\left[\sum_{H\subset I,\ \#H\geq 2} Y_H^{(2)}\chi_{\substack{v_i\geq v_\rho,\ i\in H \\ v_i<v_\rho,\ i\notin H}}\right]$

By symmetry we can write:

(16) $R^2 = \sum_{k=2}^{n} k(k-1)C_n^k\left(F(v_\rho)\right)^{n-k} E\left[(1-F(v_2))v_2\left(F(v_2)-F(v_\rho)\right)^{k-2}\ \chi_{v_2>v_\rho}\right]$

Taking the expected value, writing the sum inside the brackets and using a binomial expansion yields:

(17) $R^2 = n(n-1)\int_{v_\rho}^{\bar{v}}(1-F(x))x(F(x))^{n-2}f(x)dx$

The expected revenue generated by a second-price auction is trivially smaller than in the case of a fixed number of bidders equal to n. In the next section we compare the expected revenue from the two auctions when participation is endogenous and obtain some insights into the effects of increasing the number of potential participants and charging entry frees.

4 Revenue Comparison, Effects of increased Competition and Entry Fees

We are now in a position to provide a direct proof of the revenue equivalence between the two auction formats.

Proposition 1 *In an IPV model with a fixed number of potential players where participation is endogenous, first-price and second-price sealed-bid auctions generate the same revenue.*

Proof: Define $h(v_\rho) = R_1 - R_2 = \int_{v_\rho}^{\bar{v}} b^*(x)nF^{n-1}(x)f(x)dx - n(n-1)\int_{v_\rho}^{\bar{v}}(1 - F(x))x(F(x))^{n-2}f(x)dx.$

Notice that $h(0) = 0$. Moreover,

$$h'(v_\rho)) = \int_{v_\rho}^{\bar{v}} \frac{(n-1)}{(F(x))^{n-1}} nv_\rho \left(F(v_\rho)\right)^{n-2} f(v_\rho) \left(F(x)\right)^{n-1} f(x)dx -$$
$$nb(v_\rho) \left(F(v_\rho)\right)^{n-2} f(v_\rho) + n(n-1)\left(1 - F(v_\rho)\right) v_\rho \left(F(v_\rho)\right)^{n-2} f(v_\rho) = 0.\square$$

The above result can be explained in the context of the Revelation principle: we know that if the two mechanisms yield the same allocation of the object - in this case the high type wins, and if this type at least equals a reservation type, then the two mechanisms should generate the same revenue. As in the case we examine above the reservation type is the same for both auctions, the revelation principle can be applied to prove revenue equivalence.

The advantage of our direct approach of characterizing bidding strategies is to allow empirical tests of auction theory with entry and to provide proper theoretical foundations for laboratory experiments. Moreover, this direct approach allows us to obtain some new insights on the effects of increased potential competition and entry fees on IPV auctions.

The effect of increased competition in a standard fixed-n IPV auction is straightforward: expected revenue increases as n increases. Thus, the seller has the incentive to use policies - e.g., advertisement, to boost the number of bidders attending the auction. When bidders face bid preparation costs, however, there is a potential trade-off between the number of potential participants and expected revenue since the cut-off value increases as n increases. The intuition for the possible existence of a trade-off is as follows. Consider the effect of raising the number of potential players from n to $n+1$; we are comparing the second-order statistics of $n+1$ draws from a fixed distribution truncated at some point $\bar{x}$ with the second-order statistics of n draws from the same distribution truncated at some point $\bar{y} < \bar{x}$. The next examples will clarify this point.

Example 1 *Let's assume there are n potential players with values uniformly distributed on the interval [0,1]. In this case,* $v_\rho = c^{\frac{1}{n}}$ *and the expected revenue given*

by equation (17) can be rewritten as:
$\Psi(n) = \frac{n-1}{n+1}\left(1-(n+1)c+nc^{\frac{n+1}{n}}\right)$
Thus,
$\Psi'(n) = \frac{2}{(n+1)^2}\left(1-(n+1)c+nc^{\frac{n+1}{n}}\right)+\frac{n-1}{n+1}\left(-c+c^{\frac{n+1}{n}}-\frac{1}{n}c^{\frac{n+1}{n}}\log c\right)$
Let $g(c) = \Psi'(n)$. Note that $g(1) = 0$. The first and second derivative of g are given by:
$g'(c) = -1+c^{\frac{1}{n}}\left(1-\frac{n-1}{n^2}\log c\right)$
$g''(c) = c^{\frac{1-n}{n}}(\frac{1}{n^2}-\frac{n-1}{n^3}\log c) \geq 0$
We can then conclude that $g'(c) \leq g'(1) = 0$ and $g(c) \geq g(1) = 0$. That is, we have shown that, when values are uniformly distributed on [0,1], the expected revenue increases in n for any c between 0 and 1.

Example 2 *Suppose that the n players are represented by random draws from the distribution $F(x) = x^a, a > 0, x \in [0,1]$. The cutoff v_ρ for this distribution when there are n potential participants is $v_\rho = c^{\frac{1}{1+a(n-1)}}$. The expected revenue is (after simplification):*

$$R_n = n(n-1)\int_{c^{\frac{1}{1+a(n-1)}}}^{1} a(1-x^a)x^{a(n-1)}dx$$

Letting $y = x^a$ yields:

$$R_n = \int_{c^{\frac{a}{1+a(n-1)}}}^{1} n(n-1)y^{\frac{1}{a}+n-2}(1-y)dy.$$

The limit of R_n as $a \to \infty$ is:

$$r_n = \int_{c^{\frac{1}{n-1}}}^{1} n(n-1)y^{n-2}(1-y)dy = 1-nc+(n-1)c^{\frac{n}{n-1}}.$$

Thus

$$r_n - r_{n+1} = c+(n-1)c^{\frac{n}{n-1}}-nc^{\frac{n+1}{n}} = c\left(1+(n-1)c^{\frac{1}{n}}-nc^{\frac{1}{n}}\right).$$

Therefore $r_n > r_{n+1}$ if and only if $g(c) = \left(1+(n-1)c^{\frac{1}{n}}-nc^{\frac{1}{n}}\right) > 0$. Since $g\prime(c) = c^{\frac{1}{n}-1}-c^{\frac{1}{n}-1} = c^{\frac{1}{n}-1}(c^{\frac{1}{n-1}-\frac{1}{n}}-1) < 0$ we have that $g(c) > g(1) = 0$ if $c < 1$.

Hence we conclude that for sufficiently large a and cost $c < 1$ *the expected revenue decreases when the number of potential participants increases. For example, if* $a = 7$ *and* $c = 0.1$, r_n *is maximized when* $n = 4$.

The first example conforms with the fixed-*n* literature, where increased competition increases revenue. (e.g., Holt,1979.) In the second example, however, increased potential competition decreases expected revenue because its effect on raising the cut-off value dominates the effect of having a larger pool of participants. In this case, it may be optimal for the seller to engage in policies that limit the number of potential participants. This is summarized by the following proposition.

Proposition 2 *In a IPV auction model with endogenous participation, the expected revenue may decrease when the number of potential participants increases.*

Next, we consider a fixed number of potential participants and ask whether is optimal to charge an entry fee or to establish a participation subsidy. Formally, the problem facing the seller is to choose an entry fee/subsidy, δ, so as to maximize total revenue:

$$(18)\ \varphi(\delta) = \underset{-c<\delta<1-c}{Max} \left(n(n-1) \int_{v_\rho(\delta)}^{\overline{v}} (1 - F(x))x(F(x))^{n-2} f(x)dx + n\delta\,(1 - F(v_\rho(\delta))) \right)$$

Where $v_\rho(\delta)$ is defined so that $v_\rho(\delta)\,(F(v_\rho(\delta)))^{n-1} = c + \delta$ and the last term in the brackets denotes the expected revenue from the entry fees, i.e., the entry fee times the expected number of actual bidders.

As one may recall from section 2, when entry is modelled as in Levin and Smith (1994), the seller should never charge entry fees or use a reservation price since unrestricted entry is optimal (Proposition 6 in Levin and Smith) . In our model, however, since the bid preparation costs screen low valuation bidders, it may be optimal to set either an entry fee or a subsidy. We show next that charging an entry fee maximizes expected revenue.

Theorem. *In a IPV auction model with endogenous participation it is optimal to charge an entry fee.*

Proof: It suffices to show that δ^* that maximizes the problem defined in (18) is positive. Taking the derivative of $\varphi(\delta)$ with respect to δ yields:

$$\varphi'(\delta) = \frac{n\left(1 - F(v_\rho(\delta))\right) f(v_\rho(\delta)) F(v_\rho(\delta))^{n-1} - \delta f(v_\rho(\delta))}{F(v_\rho(\delta))^{n-1} + (n-1) f(v_\rho(\delta)) F(v_\rho(\delta))^{n-2}}$$

This expression is positive whenever $\delta \leq 0$. Thus, we conclude that $\delta^* > 0$.
□

The intuition again is in the nature of the trade-off between the effects of an entry fee on raising the cut-off value and expected revenue as determined by the underlying distribution. The next two examples help to illustrate this point.

Example 3 *As in example 1, let's assume there are n potential players with values uniformly distributed on the interval [0,1]. Expected revenue is given by:*

$$\frac{n-1}{n+1}\left(1 - (n+1)(c+\delta) + n(c+\delta)^{\frac{n+1}{n}}\right) + n\delta\left(1 - (c+\delta)^{\frac{1}{n}}\right)$$

When $n = 10$ *and* $c = .2$*, the optimal entry fee is* $\delta^* = .0367$.

Example 4 *Suppose that the n players are represented by random draws from a distribution* $F(x) = x^4, 0 < x < 1$*.expected revenue given by*

$$4n(n-1)\left(\frac{1-c-\delta}{4n-3} - \frac{1-(c+\delta)^{\frac{4n-1}{4n+3}}}{4n+1}\right) + n\delta\left(1-(c+\delta)^{\frac{4n-1}{4n+3}}\right)$$

If $n = 10$ *and* $c = .4$*, the optimal entry fee is* $\delta^* = .2271$.

5 Is it profitable for the seller to reveal the number of actual bidders?

We now modify the benchmark model by examining individual behavior when bidders learn the number of opponents before submitting their bids. The timing now is as follows. Individuals learn their values for the object; based on this information and on how other players behave in equilibrium they decide whether or not to enter the auction and pay the bid preparation cost; bidders observe the number of opponents and then submit a bid. As in Levin and Smith (1994), we find first- and second-price auctions to be revenue-equivalent. Our proof, however, is direct in that we derive equilibrium bidding strategies for both auctions. Moreover, as in Harstad, Kagel and Levin (1990), who consider an auction where the number of bidders is a random variable with a known distribution, we prove that the following five auctions are revenue-equivalent: first-price and second-price auctions, each with the number of bidders known or unknown, and English auctions.

Note that the individual behavior in a second-price auction does not change when bidders learn how many opponents there are. Once they have paid their participation costs, bidding the true values is still a dominant strategy. Thus, the cut-off point is the same as in the previous section as it is the expected revenue. On the other hand, individual behavior in a first-price auction may be quite different, since the amount they shade depends on the number of participants. Next, we derive the optimal strategy and the expected revenue for FPSB auctions under this new informational assumption.

Bidder i 's expected profits from participation is given by:

(19) $\pi_i(v_i, b_i, b_{j,\, j\neq i}) = v_i F(\hat{v})^{n-1} + \sum_{\emptyset \neq H \subset I\setminus\{i\}} (v_i - b_i^H(v_i)) \left(F(v_i) - F(\hat{v})\right)^{\#H} F(\hat{v})^{\#(I\setminus\{\{i\}\cup H\})} - c$

Where $\hat{v}$ denotes the value that sets expected profits from participation equal to zero when the equilibrium number of bidders enters the auction and H denotes the set of participants excluding player i. Note that the cardinality of H is at least equal to one. When Player i faces no competition he bids zero. For a given set H, Player i chooses b_i^H in order to maximize (19) yielding

(20) $b_i^H = v_i - \dfrac{\int_{\hat{v}}^{v_i} \left(F(y) - F(\hat{v})\right)^{\#H} dy}{\left(F(v_i) - F(\hat{v})\right)^{\#H}}$

When players bid knowing the set of participants, the amount of shading changes accordingly. If we replace (20) into the individual decision (19) and set it equal to zero to find $\hat{v}$, we obtain:

(21) $\hat{v}F(\hat{v})^{n-1} = c.$

That is, the cut-off point now is identical to the cut-off point for the second-price auction. Next we compute the expected revenue, which can be written as:

(22) $\overline{R}^1 = \sum_{k=2}^{n} C_n^k F(\hat{v})^{n-k} k E\left[b_i^H \left(F(y) - F(\hat{v})\right)^{k-1}\right]$

Replacing (22) into (20) yields:

(23) $\overline{R}^1 = E\left[v_i \sum_{k=2}^{n} C_n^k F(\hat{v})^{n-k} k \left(F(v_i) - F(\hat{v})\right)^{k-1} - \int_{\overline{v}}^{v_i} \sum_{k=2}^{n} C_n^k F(\overline{v})^{n-k} k \left(F(v_i) - F(\overline{v})\right)^{k-1} dy\right]$

After writing explicitly the expected value, integrating by parts and simplifying, we obtain:

(24) $\overline{R}^1 = n(n-1) \int_{\hat{v}}^{\overline{v}} (1 - F(x)) x (F(x))^{n-2} f(x) dx$

This expression is identical to the expression obtained for the second-price sealed-bid auction. As a corollary of proposition 1, this expression is also identical to the expected revenue generated by a first-price sealed-bid auction where bidders do not know how many opponents they actually face. This can be summarized in the following proposition:

Proposition 3. *When participation is endogenous, the four auctions are revenue-equivalent: first- and second-price sealed-bid auctions, each with the number of bidders known and unknown.*

5.1 Oral Auctions

In oral auctions players typically observe the number of opponents. In our formulation, if we allow bidders to submit the number of bids they want and observe the number of opponents at each time (natural assumptions in the case of oral auctions), then we claim that the oral auction will generate the same expected revenue as the first-price sealed-bid auction.[3] We will not present a formal proof, but the argument is quite straightforward. A bidder in a oral auction submits, for every subset H, a bid that is at most equal to the last bid plus a small epsilon. (The epsilon is of course determined by the auctioneer - who calls the bids in discrete intervals.) But this is how individuals bid in first-price auctions. As a consequence, the Revenue Equivalence theorem holds when agents are allowed to observe the number of participants before submitting their bids and the last proposition can be extended to include oral auctions.

6 Conclusion

We examined individual behavior and the expected revenue in first and second-price sealed-bid auctions when participation is endogenous. We found that in a IPV model, when agents decide whether or not to enter the auction after seeing their values and without knowing the exact number of opponents they will face - a natural assumption in procurement auctions, the Revenue Equivalence Theorem holds.

[3] Note that this characterization of oral auctions is different from the one introduced by Milgrom and Weber (1982), who consider a button auction: the last player to remove her finger from the button wins. Once a bidder removes her finger from the button, she cannot go back to the auction.

This result can be explained in the context of the Revelation principle: We know that if the two mechanisms yield the same allocation of the object - in this case the high type wins, and if this type at least equals a reservation type, then the two mechanisms should generate the same revenue. The revelation principle can be applied to prove revenue equivalence in the case we examine since the reservation type is the same for both auctions. The advantage of our direct approach of characterizing bidding strategies is to allow empirical tests of auction theory with entry and to provide proper theoretical foundations for laboratory experiments. Moreover, this direct approach allows us to obtain some new insights on the effects of potential competition and entry fees on IPV auctions.

Analogously to Samuelson (1985), and contrarily to the fixed-n literature, we found that the expected revenue may or may not increase with increased potential competition. The nature of the possible trade-off between expected revenue and the number of potential participants can be understood in the following way. When examining the effects of increasing the number of potential bidders from n to $n+1$, we are comparing the second-order statistics of n independent draws from a fixed distribution truncated at a point $\bar{y}$ with the second-order statistics of $n + 1$ independent draws from the same distribution truncated at a point $\bar{x} > \bar{y}$. Thus, the existence of the trade-off depends on the distribution of bidders' values.

In contrast to existing entry models, we show that charging an entry fee is optimal from the seller's perspective. When entry decisions occur prior to bidders knowing their values, as in Levin and Smith, entry should not be discouraged in a IPV model, since the private gains from further entry corresponds exactly to that of the seller. In our model, since low valuations bidders are screened, an entry fee increases total revenue.

We also prove that revealing the number of bidders does not raise the seller's revenue. If participation is endogenous but agents know the number of opponents when submitting their bids, then the expected revenue generated by both sealed-bid auctions are equivalent. More specifically, the following auctions are revenue-equivalent: first- and second-price sealed-bid auctions, each with the number of bidders known and unknown, and oral auctions.

Finally, the analysis for the case of correlated values remained to be examined; it is of interest to determine whether the ranking of auction formats by the amount of revenue they generate, as it was obtained by Milgrom and Weber (1982), holds when participation is endogenous.

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Appendix 1

(1) **Proof of Existence of Optimal Bidding Strategy for First-Price Auctions**

We will show that if players participate whenever their values are greater than a certain cut-off point $\tilde{v}$, then there is a unique optimal bidding strategy defined by:

$$b(v) = \begin{cases} \dfrac{\int_{\tilde{v}}^{v} (n-1)xF(x)^{n-1}dx}{F(v)^{n-1}}, v \geq \tilde{v} \\ 0, v < \tilde{v} \end{cases} .$$

Let b^* be a bid, set $v^i = v$ and define $I^i = I \backslash \{i\}$. Agent i's profits from participation are given by:

$$\pi_i = (v - b^*) \sum_{H \subset I^i} \Pr\left(v_j \geq \tilde{v},\ \forall j \in H; v_j < \tilde{v},\ \forall j \in I^i \backslash H; b^* > \max_{j \in H} b(v_j)\right) - c$$

The above expression can be reduced even further given the independence assumption and the individual problem can be written as:

$$\max_{b^*} (v - b^*) \left(F(b^{-1}(b^*))\right)^{n-1} - c$$

Since $g(z) = (v - z)(F(z))^{n-1}$ is such that $g(0) = -c$, $g(\tilde{v}) \leq -c$, there is an interior maximum. The first order condition is given by:

$$-\left(F(b^{-1}(b^*))\right)^{n-1} + (x - b^*)(n-1)\left(F(b^{-1}(b^*))\right)^{n-2} F'(b^{-1}(b^*))(b^{-1})'(b^*) = 0$$

Let $b(y) = b^*$, then the first order condition can be rewritten as:

$$-(F(y))^{n-1} + (v - b(y))(n-1)\frac{(F(y))^{n-2} F'(y)}{b'(y)} = 0$$

Replacing the value of b in the above expression and rearranging we obtain that $y = v$ is indeed the maximum. □

(2) **Proof of Existence of Optimal Bidding Strategy for Second-Price Auctions**

We will show that there exists a cut-off value $\bar{x} \in [0, \bar{v}]$ such that, for all $i \in I$, the following strategy is a Nash equilibrium: "agent i participates and bids his/her true value in the auction if and only if $v_i \geq \bar{v}$." We assume that $F(v)$ is continuous and strictly increasing. To simplify the notation, define $I^i = I \backslash \{i\}$ as in the previous proof. Let's define the variable $p \in [0, 1]$. For a given realization $\varpi \in \Omega$ of the state of nature, Player i's expected profits are given by:

$$\pi_i = E\left[\sum_{H \subset I^i} \left(v_i(\varpi) - \max_{j \in H} v_j(\varpi)\right)^+ \prod_{j \in H} \chi_{F(v_j(\varpi)) \geq p} \prod_{j \in I^i \backslash H} \chi_{F(v_j(\varpi)) < p}\right] - c$$

Thus, for $v_i(\varpi) = v$, define

$$\varphi_i(v, p) = E\left[\sum_{H \subset I^i} \left(v - \max_{j \in H} v_j(\varpi)\right)^+ \prod_{j \in H} \chi_{F(v_j(\varpi)) \geq p} \prod_{j \in I^i \backslash H} \chi_{F(v_j(\varpi)) < p}\right] - c$$

Then, we define the function $g : [0, \bar{v}] \to [0, 1]$ as follows:

$$g(p) = (\Pr(\varphi_i(v, p) \leq 0))_{i \in I}$$

Since g is continuous and increasing, the Intermediate Value Theorem guarantees that there exists $\bar{p}$ where $g(\bar{p}) = \bar{p}$ such that $\varphi_i(v, p) = 0$. But $\bar{p} = \Pr(\varphi_i(v, p) \leq 0) = \Pr(v \leq \bar{v}) = F(\bar{v})$.□