Optimal Selling Mechanisms for Multiproduct Monopolists:
Incentive Compatibility in the Presence of Budget Constraints.

by

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Abstract

We demonstrate the existence of an optimal, individually rational, and incentive compatible selling mechanism for a multiproduct monopolist facing a market populated by consumers *with budget constraints*. Our main contribution is to show via examples and our existence result that, in general, when facing consumers with budget constraints the monopolist is able to maximize profits over the set of individually rational and incentive compatible selling mechanisms only if other goods are available and only if the monopolist's goods are nonessential relative to other goods.

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1 Introduction.

In designing an optimal selling mechanism the monopolist naturally seeks to extract as much surplus from each consumer as possible. Under conditions of incomplete information, however, the monopolist's ability to extract surplus is limited by the fact that the monopolist does not know each consumer's type. In particular, the monopolist must design the mechanism so that each consumer is induced to choose the price/quantity pair intended for his type. Thus under incomplete information the monopolist is forced to trade off surplus extraction for incentive compatibility. This self-selection or incentive compatibility problem lies at the heart of the mechanism design problem faced by the incomplete information monopolist. In this paper we prove the existence of an optimal, individually rational, and incentive compatible selling mechanism for a multiproduct monopolist facing a market populated by consumers *with budget constraints*. In the nonlinear pricing literature the question of existence has not been fully addressed and models do not include budget constraints. The standard nonlinear pricing model assumes that consumers' utility functions are increasing in the monopolist's goods and linearly decreasing in prices (or transfer payments) (see for example Mirman and Sibley (1980), Maskin and Riley (1984), Laffont, Maskin, and Rochet(1985), McAfee and McMillan (1988), Page (1992), Spulber (1993), and Armstrong (1996)). Thus, it is implicitly assumed that consumers have infinite wealth - and thus that the monopolist's profit potential is limited only by the disutility associated with higher prices. Here we demonstrate existence within the context of a general model of a multiproduct monopolist. In addition to including budget constraints, in our model consumers' utility functions' are only required to be continuous in goods and measurable in types, while the monopolist profit function is only required to be upper semicontinuous in prices and quantities and measurable in types. Consumer type descriptions are also completely general (for example type descriptions are allowed to be multidimensional - a case recently considered by Armstrong (1996)). In the presence of budget constraints nonexistence is the rule rather than the exception. In order to illustrate this we present several examples. These examples indicate that in the presence of budget constraints the monopolist is often unable to completely exhaust the profit potential available in the market while at the same time maintaining incentive compatibility. Thus the presence of budget constraints introduces serious nonexistence problems. More importantly, these examples together with our main result demonstrate that there is an intimate connection between existence and the availability and desirability of other goods. In particular, it follows from our analysis that the nonexistence problems caused by the presence of budget constraints are eliminated *only if* other goods are available to consumers and *only if the monopolist goods are nonessential relative to other goods in the following sense:*

- each consumer finds that other goods are desirable (i.e., utility increasing) even when consumed without the monopolist goods; and
- each consumer finds that consuming nothing is at least as preferred as consuming only the monopolist's goods.

The main contribution of our paper is to show that, in general, when facing consumers with budget constraints the monopolist is able to maximize profits over the set of individually rational and incentive compatible selling mechanisms only if other goods are available and only if the monopolist's goods are nonessential relative to these goods. Thus, in the presence of budget constraints nonessentiality is required for existence. We also examine the relationship between the monopolist's profit and consumers' private information. In particular, we show that if the monopolist profit function does not depend *directly* upon consumers' private information (i.e., types) then under the optimal selling mechanism the monopolist earns a nonnegative profit from every consumer type. Moreover, we show via an example that if the monopolist profit function *does depend directly* on consumers' private information, then under the optimal selling mechanism the monopolist may earn a negative profit on some consumer types (i.e., there is cross-subsidization between consumer types). The mathematical difficulties encountered in proving the existence of an optimal, individually rational, incentive compatible selling mechanism have much in common with the difficulties encountered in proving existence of an equilibrium in incomplete markets with uncountably many states. These difficulties stem from two sources: (i) the pointwise nature of the incentive compatibility and individual rationality constraints and (ii) the need to equip the set of individually rational and incentive compatible mechanisms (i.e. functions) with a topology with respect to which this set is compact and with respect to which the monopolist's profit function is upper semicontinuous and consumer's utility functions are continuous. In functions spaces there are two natural topologies with respect to which compactness can be defined: the weak topology and the product topology. Given the pointwise nature of the constraints, the weak topology is inadequate since it does not behave well with respect to pointwise limits. The product topology is also problematic since sequential convergence of measurable functions(i.e., mechanisms) with respect to the product topology does not guarantee the preservation of measurability in the limit. In addition, the monopolist's profit function fails to be upper semicontinuous with respect to the product topology. We resolve these inherent technical difficulties by combining the Fatou's lemma approach of Monteiro(1996) and the menu approach to existence introduced in Page(1992).

2 The Framework.

Basic ingredients.

Let X be a compact subset of R^L_+ (the nonnegative orthant of R^L) representing all possible commodity bundles the monopolist can offer consumers. Also let Y be a closed bounded interval of the nonnegative real numbers R_+ representing all possible consumption levels of other goods. We assume that

A-1 $(0,0) \in X \times Y$ and that the cost of other goods is fixed at $q > 0$.

Thus, the cost of consuming y units of other goods is $q \cdot y$. Now let T denote the set of consumer types and equip T with a $\sigma-$ field Σ and a probability measure $\mu(\cdot)$. We shall assume that the probability measure $\mu(\cdot)$ represents the monopolist's beliefs concerning consumer types. In particular, given any $E \in \Sigma$, $\mu(E)$ is the fraction of the total number of consumers the monopolist believes are of type $t \in E$. For each consumer type $t \in T$, let $u(t, \cdot, \cdot) : X \times Y \rightarrow R$ denote the consumer's utility function defined over the set of all possible consumption bundles in $X \times Y$. We assume the following:

A-2 $\begin{cases} \text{(a) for each } t \in T, u(t, \cdot, \cdot) \text{ is continuous on } X \times Y; \\ \text{(b) for each } (x, y) \in X \times Y,\ t \longrightarrow u(t, x, y) \text{ is } \Sigma \text{ measurable;} \end{cases}$

For each consumer type $t \in T$, let $w(t)$ denote his income level. We assume that

A-3 $w(\cdot) : T \longrightarrow [0, \overline{w}]$ for some $\overline{w} > 0$.

Finally, let $D = [0, \overline{d}]$ be a closed bounded interval of R_+ denoting all possible prices, p, the monopolist can charge for commodity bundles in X.

Definition 1 (Nonessentiality) *We say that the monopolist's goods $x \in X$ are nonessential if*

(a) *for each $t \in T, y \longrightarrow u(t, 0, y)$ is strictly increasing in $y \in Y$; and*

(b) *for each $(t, x) \in T \times X, u(t, x, 0) \leq u(t, 0, 0)$.*

Part (a) of the definition of nonessentiality means that consumers find other goods desirable (i.e. utility increasing) even when consumed without the monopolist goods. Part (b) of the definition means that consumers find that consuming nothing is at least as preferred as consuming only the monopolist's goods. Thus by part (b) in order for the monopolist's goods to have utility beyond the reservation level, monopolist's goods must be consumed in combination with positive amounts of other goods. We shall assume that

A-4 the monopolist's goods are nonessential

Induced preferences and nonessentiality.

Given preferences $u(t,\cdot,\cdot)$ over consumption bundles in $X \times Y$, the induced preferences of a type t consumer over $X \times D$ are defined via

$$v(t,x,p) := u(t,x,\frac{w(t)-p}{q}).$$

Note that if the monopolist's goods are nonessential then $v(t,0,p) < v(t,0,0)$ for all $p > 0$. This follows from part (a) of the definition of nonessentiality. Thus under the assumption of nonessentiality, for any consumer type consuming nothing and paying nothing to the monopolist is strictly preferred to paying the monopolist something for nothing. Note also that if the monopolist's goods are nonessential then,

$$v(t,x,w(t)) \leq v(t,0,0) \text{ for all } x \in X. \tag{1}$$

This follows from part (b) of the definition of nonessentiality. Thus under the assumption of nonessentiality, for any consumer type consuming nothing and paying nothing to the monopolist is at least as preferred as paying to the monopolist an amount equal to total income for any consumption bundle x.

3 The monopolist's mechanism design problem.

Direct selling mechanisms.

A direct selling mechanism for the monopolist is a pair of functions

$$x(\cdot) : T \longrightarrow X \text{ and } p(\cdot) : T \longrightarrow D.$$

Given selling mechanism $(x(\cdot),p(\cdot))$, the monopolist intends for a type $t \in T$ consumer to purchase bundle $x(t)$ and pay an amount $p(t)$ for this bundle. Now let $\mathcal{M}(T,X)$ denote the set of all $(\Sigma,\mathcal{B}(X))$ measurable functions $x(\cdot) : T \longrightarrow X$ and $\mathcal{M}(T,D)$ denote the set of all $(\Sigma,\mathcal{B}(D))$ measurable functions $p(\cdot) : T \longrightarrow D$.[1] We shall assume that the feasible set of all selling mechanisms is given by the set of all pairs $(x(\cdot),p(\cdot)) \in \mathcal{M}(T,X) \times \mathcal{M}(T,D)$. Moreover, given any selling mechanism $(x(\cdot),p(\cdot))$, we shall refer to the function $x(\cdot) \in \mathcal{M}(T,X)$ as the *direct* quantity function (since it is defined on types) and the function $p(\cdot) \in \mathcal{M}(T,D)$ as the *direct* pricing function (since it too is defined on types).

The mechanism design problem.

Suppose now that the monopolist's payoff function is given by

$$\pi(\cdot,\cdot,\cdot) : T \times X \times D \longrightarrow R$$

where

[1] A function $x(\cdot) : T \longrightarrow X$ is $(\Sigma,\mathcal{B}(X))$ measurable iff $\{t \in T : x(t) \in E\} \in \Sigma$ for every $E \in \mathcal{B}(X)$. In a similar manner we define $(\Sigma,\mathcal{B}(D))$ measurability.

A-5 $\begin{cases} \text{(a) } \pi(\cdot,\cdot,\cdot) \text{ is } \Sigma \times \mathcal{B}(X) \times \mathcal{B}(D) \text{ measurable}^2; \\ \text{(b) } \pi(t,\cdot,\cdot) \text{is upper semicontinuous on } X \times D \text{ for each } t \in T^3; \\ \text{(c) } \pi(\cdot,\cdot,\cdot) \text{ is integrably bounded on } T \times X \times D.^4 \end{cases}$

The monopolist's mechanism design problem can now be stated as follows:

$$\max_{(x(\cdot),p(\cdot))\in\mathcal{M}(T,X)\times\mathcal{M}(T,D)} \int_T \pi(t,x(t),p(t))d\mu(t),$$

subject to the following constraints:

1. the *incentive compatibility* (IC) constraints: for all t and $t' \in T$

$$v(t,x(t'),p(t')) \leq v(t,x(t),p(t)), \text{provided } p(t') \leq w(t)$$

2. the *individual rationality* (IR) constraints: for all $t \in T$

$$v(t,x(t),p(t)) \geq v(t,0,0).$$

3. the *budget constraints*: for all $t \in T$, $p(t) \leq w(t)$.

Note that it follows from the incentive compatibility constraints that if a type t consumer prefers a commodity bundle $x(t')$ not intended for his type, then it must be true that the commodity bundle is too expensive, i.e. $p(x(t')) > w(t)$. Also note that since $(0,0) \in X \times D$, the monopolist can design the selling mechanism so as to induce certain consumers types to abstain from doing business with the monopolist. In particular, given a mechanism $(x(\cdot),p(\cdot))$ satisfying the incentive compatibility, voluntary participation, and budget constraints, the set $\{t \in T; (x(t),p(t)) = (0,0)\}$ consists of consumers types that will abstain.

4 The existence of an optimal selling mechanism.

The existence result.

Let $\Im \subset \mathcal{M}(T,X)\times\mathcal{M}(T,D)$ denote the subset of direct selling mechanisms satisfying the IC, IR, and budget constraints. Note first that $\Im\neq\emptyset$ since the selling mechanism $(\widehat{x}(\cdot),\widehat{p}(\cdot))$ such that $(\widehat{x}(t),\widehat{p}(t)) = (0,0)$ for all $t \in T$ belongs to $\Im$. The monopolist design problem can be written compactly as

$$\max_{(x(\cdot),p(\cdot))\in\Im} \int_T \pi(t,x(t),p(t))d\mu(t). \tag{2}$$

Theorem 1 (Existence) *Suppose [A-1]-[A-3] and [A-5] hold. If the monopolist's goods are nonessential(i.e. [A-4] holds), then there exists an optimal direct selling mechanism solving the monopolist problem (2).*

Proof. Let $\{(x^n(\cdot),p^n(\cdot))\}_n \subset \Im$ be such that

$$\lim_{n\to\infty}\int_T \pi(t,x^n(t),p^n(t))d\mu(t) = \sup_{(x(\cdot),p(\cdot))\in\Im}\int_T \pi(t,x(t),p(t))d\mu(t).$$

Denote by $S^n = cl\{(x^n(t),p^n(t)); t \in T\}$ the closure of the range of the mechanism $(x^n(\cdot),p^n(\cdot))$. Each S^n belongs to the collection $\mathcal{P}_f(X \times D)$ of all non-empty closed subsets of $X \times D$ (where as before, $X \times D$ is the set of feasible commodity bundle/price pairs).

[2] Here, $\mathcal{B}(X)$ denotes the Borel σ-field in X and $\mathcal{B}(D)$ the Borel σ field in D.

[3] $\pi(t,\cdot,\cdot)$ is upper semicontinous is $(x_n,p_n) \to (x,p)$ implies that $\limsup_n \pi(t,x_n,p_n) \leq \pi(t,x,p)$.

[4] $\pi(\cdot,\cdot,\cdot)$ is integrably bounded on $T \times X \times D$ if $|\pi(t,x,p)| \leq \xi(t)$ for all $(t,x,p) \in T \times X \times D$ where $\xi(\cdot)$ is a real valued, μ integrable function defined on T .

Since $X \times D$ is compact, $\mathcal{P}_f(X \times D)$ equipped with the Hausdorff metric is a compact metric space (see Berge(1963)).[5] Without loss of generality, we assume that $\{S^n\}_n$ converges to some $S \in \mathcal{P}_f(X \times D)$ under the Hausdorff metric. Thus, we have

$$\liminf_n S^n = \limsup_n S^n = S. \tag{3}$$

By Fatou's lemma in several dimensions (see Artstein(1979)), there exists $(\overline{x}(\cdot), \overline{p}(\cdot)) \in \mathcal{M}(T, X) \times \mathcal{M}(T, D)$ and a null set N (i.e. $N \in \Sigma, \mu(N) = 0$) such that

$$\begin{aligned} (\overline{x}(t), \overline{p}(t)) &\in \mathrm{Ls}\{(x^n(t), p^n(t))\} \text{ for all } t \in T \backslash N, \text{ and} \\ \int_T \pi(t, \overline{x}(t), \overline{p}(t)) d\mu(t) &= \sup_{(x(\cdot), p(\cdot)) \in \Im} \int_T \pi(t, x(t), p(t)) d\mu(t). \end{aligned}$$

Since the mapping $t \longrightarrow \mathrm{Ls}\{\{(x^n(t), p^n(t))\}$, is measurable with nonempty compact values, it follows from Theorem 5.1(the Kuratowski, Ryll-Nardzewski theorem) in Himmelberg(1975) that there exists a mechanism $(x^*(\cdot), p^*(\cdot)) \in \mathcal{M}(T, X) \times \mathcal{M}(T, D)$ such that $(x^*(t), p^*(t)) = (\overline{x}(t), \overline{p}(t))$ for all $t \in T \backslash N$. Thus

$$\int_T \pi(t, \overline{x}(t), \overline{p}(t)) d\mu(t) = \int_T \pi(t, x^*(t), p^*(t)) d\mu(t).$$

To complete the existence proof, we must show that $(x^*(\cdot), p^*(\cdot)) \in \Im$. To begin, observe that $(x^*(\cdot), p^*(\cdot))$ satisfies the IR and budget constraints for all consumer types. To show that $(x^*(\cdot), p^*(\cdot))$ satisfies the IC constraints let t and t' be in T and $p(t') \leq w(t)$. Consider the following cases:

I. Suppose $p^*(t') = w(t)$. It then follows from nonessentiality (see (1)) that

$$v(t, x^*(t'), p^*(t')) \leq v(t, 0, 0),$$

and by the IR constraints $v(t, 0, 0) \leq v(t, x^*(t), p^*(t))$. Thus for case I,

$$v(t, x^*(t'), p^*(t')) \leq v(t, x^*(t), p^*(t)).$$

II. Suppose $p^*(t') < w(t)$. For some subsequence $\{(x^{n_j}(t), p^{n_j}(t))\}_j$, we have

$$(x^{n_j}(t), p^{n_j}(t)) \longrightarrow (x^*(t), p^*(t)) \text{ and } p^{n_j}(t) \leq w(t) \text{ for all } j.$$

Since $(x^*(t'), p^*(t')) \in S$ and $\liminf_j S^{n_j} = \limsup_j S^{n_j} = S$ there is a sequence $\{(x^{n_j}, p^{n_j})\}_j$ with $\{(x^{n_j}, p^{n_j})\}_j \in S^{n_j}$ for each j, such that $(x^{n_j}, p^{n_j}) \longrightarrow (x^*(t'), p^*(t'))$ and $p^{n_j} < w(t)$ for all j sufficiently large. Now suppose that $v(t, (x^*(t'), p^*(t')) > v(t, x^*(t), p^*(t))$. By the continuity of $v(t, \cdot, \cdot)$ we have for sufficiently large j,

$$v(t, x^{n_j}, p^{n_j}) > v(t, x^{n_j}(t), p^{n_j}(t)).$$

Moreover since $(x^{n_j}, p^{n_j}) \in S^{n_j} = cl\{(x^{n_j}(t), p^{n_j}(t)); t \in T\}$ there exists $t'' \in T$ such that

$$p^{n_j}(t'') < w(t), \text{ and } v(t, x^{n_j}(t''), p^{n_j}(t'')) > v(t, x^{n_j}(t), p^{n_j}(t))$$

contradicting the incentive compatibility of $(x^{n_j}(\cdot), p^{n_j}(\cdot))$. Thus for case II

$$v(t, x^*(t'), p^*(t')) \leq v(t, x^*(t), p^*(t)).$$

Q.E.D.

[5] The Hausdorff metric h is given by $h(A, B) = \max\{\sup_{s \in A} d(s, B), \sup_{s \in B} d(s, A)\}$ where d is the euclidean metric in R^{L+1} , $d(s, U) = \inf_{u \in U} d(s, u)$. If S^n converges in this metric then (3) is true.

The indirect pricing function.

Given nonessentiality (and in particular part (a) of the definition), it follows that for any selling mechanism $(x(\cdot), p(\cdot)) \in \Im, x(t) = x(t')$ implies that $p(t) = p(t')$. Thus it follows that corresponding to the direct selling mechanism $(x(\cdot), p(\cdot)) \in \Im$ there is a function $d(\cdot) : X \to D$, call it the indirect pricing function[6], such that $p(t) = d(x(t))$ for all $t \in T$.

5 Existence and Nonessentiality: Examples.

Example 1 (Existence and nonessentiality) *In this our main example we show that part (b) of the definition of nonessentiality is critical to existence (example 3 below illustrates why part (a) of the definition is required). We summarize this fact in the following theorem:*

Theorem 2 *Suppose [A-1]-[A-3] and [A-5] hold. Suppose also that consumer's utility function satisfy part (a) of the definition of nonessentiality (i.e., for each $t \in T, u(t, 0, \cdot)$ is strictly increasing in y on Y). Then, in general, existence requires that consumer's utility functions satisfy part (b) of the definition of nonessentiality; that is, existence requires that for each $(t, x) \in T \times X,\ u(t, x, 0) \leq u(t, 0, 0)$.*

Proof. To prove Theorem 2 it suffices to construct an example satisfying [A-1]-[A-3], [A-5], and condition (a) of Definition 1 but *not satisfying* condition (b) of Definition 1 for which no optimal, individually rational and incentive compatible selling mechanisms exists. Consider the following example: Let $X = Y = D = [0, \overline{m}]$ where $\overline{m} > 1$. Suppose that there are two consumers. Consumer 1 is of type $t_1 = 2$ and Consumer 2 if of type $t_2 = 1$. Thus $T = \{t_1, t_2\}$ and each consumer type is equally likely(i.e. $\mu(t_i) = \frac{1}{2}$ for each i). Suppose that consumers have utility functions given by

$$u(t_i, x, y) = t_i\sqrt{x} - \frac{w(t_i)}{q} + y.$$

and incomes $w(t_1) = \frac{1}{2}$ and $w(t_2) = 1$. Finally suppose that the price of other goods is $q = 1$ and that the monopolist profit function is given by

$$\pi(t, x, p) = p - x.$$

Since $q = 1$, consumers have indirect utility functions given by

$$v(t_i, x, p) = t_i\sqrt{x} - p.$$

Note that $v(t_i, 0, 0) = 0$ and that for each i there is an $x \in X = [0, \overline{m}]$ such that $v(t_i, x, w(t_i)) > 0$. Thus, while this example satisfies [A-1]-[A-3],[A-5] and (a) of Definition 1, it does not satisfy (b) of Definition 1. The monopolist's mechanism design problem can be written as:

$$\max p(t_1) + p(t_2) - x(t_1) - x(t_2) \tag{4}$$

subject to the constraints, $(x(t_i), p(t_i)) \in X \times D$ for all i,

$$\text{IR constraints:} \begin{cases} t_1\sqrt{x(t_1)} - p(t_1) \geq 0 \\ t_2\sqrt{x(t_2)} - p(t_2) \geq 0 \end{cases} \tag{5}$$

$$\text{IC constraints:} \begin{cases} t_1\sqrt{x(t_1)} - p(t_1) \geq t_1\sqrt{x(t_2)} - p(t_2) \text{ if } p(t_2) \leq w(t_1) \\ t_2\sqrt{x(t_2)} - p(t_2) \geq t_2\sqrt{x(t_1)} - p(t_1) \end{cases} \tag{6}$$

and

$$p(t_1) \leq w(t_1) \text{ and } p(t_2) \leq w(t_2). \tag{7}$$

Lemma 1 (Nonexistence) *There does not exist a solution to the monopolist's problem (4)-(7).*

[6] We say that $d(\cdot) : X \to D$ is an indirect pricing function because it is defined on bundles X rather than on types T.

Proof. See the appendix. For the monopolist problem (4)-(7), a supremum is achieved via the mechanism

$$x^*(t_1) = (\frac{1}{4})^2, p^*(t_1) = \frac{1}{2}$$
$$x^*(t_2) = \frac{1}{4}, p^*(t_2) = \frac{1}{2}.$$

This mechanism, however, is not incentive compatible. The existence problem can be summarized as follows: Consider the collection of mechanisms

$$x(t_1) = (\frac{1}{4})^2, p(t_1) = \frac{1}{2}$$
$$x(t_2) = p^2(t_2), p(t_2) \in (\frac{1}{2}, 1].$$

These mechanisms are individually rational and incentive compatible. As the monopolist lowers the price $p(t_2)$ for the type t_2 consumer, the supremum is approached (i.e., profits increase) and individual rationality and incentive compatibility are maintained. However, while the supremum is achieved at $p(t_2) = \frac{1}{2}$, incentive compatibility is lost. Thus, no optimal, individually rational and incentive compatible mechanism exists for the monopolist's mechanism design problem (4)-(7).

5.1 Example 2 (Existence and budget constraints)

The example above illustrates the importance of the nonessentiality assumption in problems with budget constraints. In our next example, we remove the budget constraints from the example above and we show that existence is restored(even though nonessentiality is not satisfied). Moreover, we calculate the monopolist's optimal, individually rational and incentive compatible selling mechanism. The monopolist's problem without budget constraints is

$$\max p(t_1) + p(t_2) - x(t_1) - x(t_2)$$

subject to the constraints $(x(t_i), p(t_i)) \in X \times D$ for all i,

$$2\sqrt{x(t_1)} - p(t_1) \geq 0$$
$$\sqrt{x(t_2)} - p(t_2) \geq 0$$
$$2\sqrt{x(t_1)} - p(t_1) \geq 2\sqrt{x(t_2)} - p(t_2)$$
$$\sqrt{x(t_2)} - p(t_2) \geq \sqrt{x(t_1)} - p(t_1).$$

Lemma 2 (Existence without budget constraints.) *The optimal solution to the monopolist problem (4)-(7) without budget constraints is*

$$x^*(t_1) = 1, p^*(t_1) = 2$$
$$x^*(t_2) = 0, p^*(t_2) = 0.$$

The optimal profit is $p^*(t_1) + p^*(t_2) - x^*(t_1) - x^*(t_2) = 1.$

In the table below we compare individually rational and incentive compatible mechanisms and monopoly profits for the example with budget constraints (Example 1) to the optimal, individually rational and incentive compatible mechanism and corresponding monopoly profits for the example without budget constraints (Example 2).

Comparison Table

	with budget constraints	without budget constraints
mechanisms	$x(t_1) = (\frac{1}{4})^2, p(t_1) = \frac{1}{2}$ $x(t_2) = p^2(t_2), p(t_2) \in (\frac{1}{2}, 1]$	$x^*(t_1) = 1, p^*(t_1) = 2$ $x^*(t_2) = 0, p^*(t_2) = 0$
monopoly profits	$\sum_i (p(t_i) - x(t_i)) < \frac{11}{16}$	$\sum_i (p^*(t_i) - x^*(t_i)) = 1$

Observe that in the model with budget constraints the monopolist serves all consumer types. This is not the case in the model without budget constraints. The monopolist serves only consumer type t_1. Note also that the monopolist's profits are strictly higher in the model without budget constraints. This is because in the model without budget constraints the monopolist is not forced to trade off as much profit to maintain incentive compatibility as he is in the model with budget constraints.

Example 3 (Nonsensical solutions and nonessentiality) *Part (a) of the definition of nonessentiality rules out nonsensical solutions to the monopolist's problem as the following simple example shows: Suppose that $X = Y = [0, 2]$. Suppose also that we have only one consumer type with utility function $u(x, y) = \sqrt{xy}$ and income $w = 1$. Finally suppose that the price of other goods is $q = 1$ and that the monopolist profit function is given by $\pi(t, x, p) = p - x$. The indirect utility function is given by*

$$v(t, x, p) = \sqrt{x \cdot (1 - p)}$$

Note that the utility function $u(\cdot, \cdot)$ fails to satisfy part (a) of the definition of nonessentiality (recall that part (a) of Definition 1 requires that, $u(t, 0, \cdot)$ be strictly increasing in $y \in Y$). In this example the monopolist optimal selling mechanism is given by $(x^*(t), p^*(t)) = (0, 1)$. Moreover note that the consumer is indifferent between paying his entire income of 1 to the monopolist in return for nothing and paying nothing in return for nothing. By assuming nonessentiality (and in particular by assuming part (a) of Definition 1), we eliminate such nonsensical solutions.

6 Information and Profits.

In this section we shall assume that

$$\text{A-6} \begin{cases} \text{there exists } (x'', p'') \in X \times D \text{ such that for all } t \in T \\ \pi(t, x'', p'') \geq 0 \\ p'' \leq w(t) \text{ and } v(t, x'', p'') \geq 0. \end{cases}$$

Note that any mechanism $(x''(\cdot), p''(\cdot))$ such that $(x''(t), p''(t)) = (x'', p'')$ for all $t \in T$ belongs to $\Im$. Note also that if $\pi(t, 0, 0) = 0$ for all $t \in T$, then A-6 is automatically satisfied. Our first result show that if A-6 is satisfied and the monopolist's profit function does not depend directly on consumer types (i.e., the monopolist's profit function is independent of consumers' private information) then the optimal selling mechanism generates a nonnegative profit for almost all consumer types. An example, however, demonstrates that if the monopolist's profit function *does depend* on consumers' private information, then under the optimal selling mechanism profit may be negative for some consumer types.

Theorem 3 (Information independence and the nonegativity of profits) *Suppose [A-1]-[A-5] hold. If the monopolist's profit function is independent of consumer types, then the optimal selling mechanism is such that monopolist's profit is nonnegative for almost all consumer types.*

Proof. See appendix. Summarizing the proof, we show that if $(x^*(\cdot), p^*(\cdot)) \in \Im$ is any feasible mechanism with $\pi(x^*(t), p^*(t)) < 0$ for types in some set of positive $\mu-$ measure, then $(x^*(\cdot), p^*(\cdot))$ can be redefined for those types in such a way as to generate a nonnegative profit and at the same time preserve voluntary participation, incentive compatibility, and budget constraints. Thus, if the monopolist profit function is independent of private information, we can always construct from the mechanism $(x^*(\cdot), p^*(\cdot)) \in \Im$ another *feasible* mechanism$(x'(\cdot), p'(\cdot)) \in \Im$ that generates for the monopolist a strictly higher profit. Given assumptions A-1, A-5, we can conclude then that if the monopolist's profit function is independent of private information and if $(x^*(\cdot), p^*(\cdot)) \in \Im$ is an optimal selling mechanism, then $\pi(x^*(t), p^*(t)) \geq 0$ a.e.[μ]. As the following example illustrates, however, this conclusion does not survive if the monopolist profit function is dependent on consumers' private information.

Example 4 (Information dependence and the negativity of profit) *Consider a monopolist who sells a single good to a population of consumers with types given by $T = [0, 1]$. Suppose that for some $\overline{d} > 0, X \times D = [0, \overline{d}] \times [0, \overline{d}]$, and that the monopolist's probability beliefs are given via Lebesgue measure on T. Suppose also that consumer's induced preferences $v(t, \cdot, \cdot)$ are given by $v(t, x, p) = x - h(t)p$, where*

$$h(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq \frac{1}{2} \\ \frac{1}{2} & \text{for } \frac{1}{2} < t \leq 1 \end{cases}$$

Finally, suppose that the monopolist's profit function is given by $\pi(t, x, p) = g(t)p - x$, where

$$g(t) = \begin{cases} a & \text{for } 0 \leq t \leq \frac{1}{2} \\ b & \text{for } \frac{1}{2} < t \leq 1, \end{cases}$$

$a = 2\frac{3}{4}$ and $b = \frac{1}{4}$.

In order to avoid complications with budget constraints, assume that the set of all possible prices the monopolist can charge for bundles in X, given by $D = [0, \overline{d}]$, is such that $\overline{d} < w(t)$ for all $t \in T$. We construct the monopolist's optimal selling mechanism $(x^*(\cdot), p^*(\cdot)) \in \Im$ as follows: First, in order for $(x^*(\cdot), p^*(\cdot))$ to be incentive compatible for types $0 \leq t \leq \frac{1}{2}$, we must have for some constant G

$$x^*(t) - p^*(t) = G \text{ for all types } 0 \leq t \leq \frac{1}{2}.$$

Moreover, G must be nonnegative in order to satisfy the VP constraints. Thus,

$$p^*(t) = x^*(t) - G \text{ for types } 0 \leq t \leq \frac{1}{2}. \tag{8}$$

Similarly, for types $\frac{1}{2} < t \leq 1$ we must have for some nonnegative constant B:

$$x^*(t) - \frac{1}{2}p^*(t) = B.$$

Thus,

$$p^*(t) = 2(x^*(t) - B) \text{ for types } \frac{1}{2} < t \leq 1. \tag{9}$$

Now let $t \leq 1/2$ and $1/2 < t'$. Incentive compatibility requires that

$$x^*(t) - p^*(t) = G \geq x^*(t') - p^*(t')$$

From (9) we have $x^*(t') - p^*(t') = x^*(t') - 2(x^*(t') - B)$. Thus, by incentive compatibility $G \geq x^*(t') - 2(x^*(t') - B)$ for types $\frac{1}{2} < t'$, and thus

$$x^*(t) \geq 2B - G \text{ for types } \frac{1}{2} < t \leq 1.$$

Now suppose $\frac{1}{2} < t$ and $t' \leq \frac{1}{2}$. Incentive compatibility requires that

$$x^*(t) - \frac{1}{2}p^*(t) = B \geq x^*(t') - \frac{1}{2}p^*(t')$$

From (8) we have

$$x^*(t') - \frac{1}{2}p^*(t') = x^*(t') - \frac{1}{2}(x^*(t') - G).$$

Thus by incentive compatibility

$$B \geq x^*(t') - \frac{1}{2}(x^*(t') - G) \text{ for types } t' \leq \frac{1}{2},$$

and thus

$$x^*(t) \leq 2B - G \text{ for types } 0 \leq t \leq \frac{1}{2}. \tag{10}$$

Now consider the monopolist's profit given by

$$\int_0^{\frac{1}{2}} [(x^*(t) - G)a - x^*(t)]dt + \int_{\frac{1}{2}}^1 [(2x^*(t) - 2B)b - x^*(t)]dt$$

$$= \int_0^{\frac{1}{2}} [(a-1) \cdot x^*(t) - aG]dt + \int_{\frac{1}{2}}^1 [(2b-1) \cdot x^*(t) - 2bB]dt.$$

Since $(a - 1) > 0$, the monopolist will set $x^*(t) = 2B - G$ for types $0 \leq t \leq \frac{1}{2}$ (see (10) above), and since $(2b - 1) < 0$, the monopolist will set $x^*(t) = 2B - G$ for types $\frac{1}{2} < t \leq 1$. Thus, the monopolist profit stated in terms of B and G is given by

$$\int_0^{\frac{1}{2}} [(a-1) \cdot x^*(t) - aG]dt + \int_{\frac{1}{2}}^1 [(2b-1) \cdot x^*(t) - 2bB]dt$$

$$= \int_0^{\frac{1}{2}} [(a-1)(2B-G) - aG]dt + \int_{\frac{1}{2}}^1 [(2b-1) \cdot (2B-G) - 2bB]dt \tag{11}$$

$$= (a + b - 2)B - (a + b - 1)G.$$

From (11) it follows that the monopolist will choose B as large as possible and set $G = 0$. Thus, $B = \frac{\overline{d}}{2}$ and the optimal selling mechanism is given by

$$(x^*(t), p^*(t)) = (2B - G, 2(B - G)) = (\overline{d}, \overline{d}) \text{ for all } t \in T.$$

Note that for types $0 \leq t \leq \frac{1}{2}$, $\pi(t, x^*(t), p^*(t)) = a\overline{d} - \overline{d} > 0$, and for types $\frac{1}{2} < t \leq 1$,$\pi(t, x^*(t), p^*(t)) = b\overline{d} - \overline{d} < 0$.

Appendix

Proof of the Lemma 1. Recall that the monopolist's design problem is given by

$$\max p(t_1) + p(t_2) - x(t_1) - x(t_2)$$

subject to the constraints $(x(t_i), p(t_i)) \in X \times D$ for all i,

$$\text{IR constraints:} \begin{cases} t_1\sqrt{x(t_1)} - p(t_1) \geq 0 \\ t_2\sqrt{x(t_2)} - p(t_2) \geq 0 \end{cases}$$

$$\text{IC constraints:} \begin{cases} t_1\sqrt{x(t_1)} - p(t_1) \geq t_1\sqrt{x(t_2)} - p(t_2) \text{ if } p(t_2) \leq w(t_1) \\ t_2\sqrt{x(t_2)} - p(t_2) \geq t_2\sqrt{x(t_1)} - p(t_1) \end{cases}$$

and

$$p(t_1) \leq w(t_1) \text{ and } p(t_2) \leq w(t_2).$$

We will show that the supremum is achieved at a mechanism that is not incentive compatible. To accomplish this we divide the monopolist problem in two parts. In one part we restrict $p(t_2)$ to be in the interval $[0, w(t_1)] = [0, \frac{1}{2}]$. In the other part we restrict $p(t_2)$ to be in the interval $(w(t_1), w(t_2)] = (\frac{1}{2}, 1]$. <u>Part 1</u> $p(t_2) \in (w(t_1), w(t_2)] = (\frac{1}{2}, 1]$. Recalling that $t_1 = 2$ and $t_2 = 1$ and that $w(t_1) = 1/2$ and $w(t_2) = 1$, the monopolist problem is given by

$$\max p(t_1) + p(t_2) - x(t_1) - x(t_2)$$

subject to the constraints $(x(t_i), p(t_i)) \in X \times D$ for all i,

$$\text{IR constraints:} \begin{cases} 2\sqrt{x(t_1)} - p(t_1) \geq 0 \\ \sqrt{x(t_2)} - p(t_2) \geq 0 \end{cases}$$

$$\text{IC constraints} \sqrt{x(t_2)} - p(t_2) \geq \sqrt{x(t_1)} - p(t_1)$$

and

$$\frac{1}{2} < p(t_2) \leq 1 \text{ and } 0 \leq p(t_1) \leq \frac{1}{2}.$$

Our strategy is the following: We prove that the supremum for the above design problem is achieved with $p(t_2) = \frac{1}{2}$, that this supremum is greater than the maximum achievable whenever $p(t_2) \in [0, w(t_1)] = [0, \frac{1}{2}]$, and that incentive compatibility is not satisfied (i.e., that $v(t_1, x(t_2), p(t_2)) > v(t_1, x(t_1), p(t_1))$. To begin, notice that in order to satisfy the constraints we must have $p(t_1) = \min\{\frac{1}{2}, 2\sqrt{x(t_1)}\}$. The monopolist's problem then simplifies to

$$\max\{\min\{\frac{1}{2}, 2\sqrt{x(t_1)}\} + p(t_2) - x(t_1) - x(t_2)\}$$

subject to the constraints$(x(t_i), p(t_i)) \in X \times D$ for all i,

$$\begin{aligned} \sqrt{x(t_2)} - p(t_2) &\geq 0, \\ \sqrt{x(t_2)} - p(t_2) &\geq \sqrt{x(t_1)} - \min\{\frac{1}{2}, 2\sqrt{x(t_1)}\}, \\ 1 &\geq p(t_2) \geq \frac{1}{2}. \end{aligned}$$

If $2\sqrt{x(t_1)} > \frac{1}{2}$ it is profitable and possible to reduce $x(t_1)$. Therefore the maximization problem simplifies to

$$\max 2\sqrt{x(t_1)} + p(t_2) - x(t_1) - x(t_2)$$

subject to the constraints, $(x(t_i), p(t_i)) \in X \times D$ for all i,

$$\sqrt{x(t_2)} - p(t_2) \geq 0, \tag{12}$$

$$\sqrt{x(t_2)} - p(t_2) \geq -\sqrt{x(t_1)} \tag{13}$$

$$1 \geq p(t_2) \geq \frac{1}{2}.$$

$$1 \geq 4\sqrt{x(t_1)}.$$

Since (12) implies that (13) is satisfied we have

$$\max 2\sqrt{x(t_1)} + p(t_2) - x(t_1) - x(t_2)$$

subject to the constraints,$(x(t_i), p(t_i)) \in X \times D$ for all i,

$$\begin{array}{c} \sqrt{x(t_2)} - p(t_2) \geq 0 \\ 1 \geq 4\sqrt{x(t_1)}, \\ 1 \geq p(t_2) \geq \frac{1}{2} \end{array}$$

Now observe that it is possible and profitable to reduce $x(t_2)$ up to $\sqrt{x(t_2)} = p(t_2)$. Hence the monopolist problem reduces to

$$\max 2\sqrt{x(t_1)} + \sqrt{x(t_2)} - x(t_1) - x(t_2)$$

subject to the constraints,

$$1 \quad \geq 4\sqrt{x(t_1)} \tag{14}$$

$$1 \quad \geq \sqrt{x(t_2)} \quad \geq \frac{1}{2}. \tag{15}$$

Now observe that

$$\frac{\partial}{\partial x(t_1)}\left(2\sqrt{x(t_1)} + \sqrt{x(t_2)} - x(t_1) - x(t_2)\right) = \frac{1}{\sqrt{x(t_1)}} - 1.$$

By (14), $\frac{1}{\sqrt{x(t_1)}} - 1 \geq 3 > 0$. Thus, at the optimum $\frac{1}{2} = 2\sqrt{x(t_1)}$. Also observe that

$$\frac{\partial}{\partial x(t_2)}(2\sqrt{x(t_1)} + \sqrt{x(t_2)} - x(t_1) - x(t_2)) = \frac{1}{2\sqrt{x(t_2)}} - 1.$$

By (15),$\frac{1}{2\sqrt{x(t_2)}} - 1 \leq 0$. Thus at the optimum $\sqrt{x(t_2)} = \frac{1}{2}$ and $p(t_2) = \frac{1}{2}$. The monopolist's profit is

$$\sum_i (p(t_i) - x(t_i)) = w(t_1) + w(t_1) - (\frac{w(t_1)}{2})^2 - (w(t_1))^2 = \frac{1}{2} + \frac{1}{2} - (\frac{1}{4})^2 - (\frac{1}{2})^2 = \frac{11}{16}.$$

To check that incentive compatibility fails at the optimum note that

$$v(t_1, x(t_2), p(t_2)) - v(t_1, x(t_1), p(t_1)) = 2\sqrt{x(t_2)} - w(t_1) = \frac{1}{2}.$$

Part 2 $p(t_2) \in [0, w(t_1)] = [0, \frac{1}{2}]$: For Part 2 the monopolist design problem is given by

$$\max p(t_1) + p(t_2) - x(t_1) - x(t_2)$$

subject to the constraints,$(x(t_i), p(t_i)) \in X \times D$ for all i,

$$\text{IR constraints:} \begin{cases} 2\sqrt{x(t_1)} - p(t_1) \geq 0 \\ \sqrt{x(t_2)} - p(t_2) \geq 0 \end{cases} \tag{16}$$

$$\text{IC constraints} \begin{cases} 2\sqrt{x(t_1)} - p(t_1) \geq 2\sqrt{x(t_2)} - p(t_2) \\ \sqrt{x(t_2)} - p(t_2) \geq \sqrt{x(t_1)} - p(t_1) \end{cases} \tag{17}$$

and

$$p(t_1) \leq w(t_1) \text{ and } p(t_2) \leq w(t_1). \tag{18}$$

Since $2\sqrt{x(t_1)} - p(t_1) \geq 2\sqrt{x(t_2)} - p(t_2) \geq \sqrt{x(t_2)} - p(t_2) \geq 0$, the first inequality in (16) will be satisfied if the second inequality in (16) is satisfied and if the first inequality in (17) is satisfied. The above problem is therefore equivalent to:

$$\max p(t_1) + p(t_2) - x(t_1) - x(t_2)$$

subject to the constraints, $(x(t_i), p(t_i)) \in X \times D$ for all i,

$$\begin{aligned} &\sqrt{x(t_2)} - p(t_2) \geq 0 \\ &\sqrt{x(t_2)} - \sqrt{x(t_1)} + p(t_1) \geq p(t_2), \\ &2(\sqrt{x(t_1)} - \sqrt{x(t_2)}) + p(t_2) \geq p(t_1), \\ &p(t_1) \leq w(t_1) \text{ and } p(t_2) \leq w(t_1). \end{aligned}$$

Observe now that if $2(\sqrt{x(t_1)} - \sqrt{x(t_2)}) + p(t_2) > w(t_1)$ then it is possible and profitable to reduce $x(t_1)$. Therefore $2\left(\sqrt{x(t_1)} - \sqrt{x(t_2)}\right) + p(t_2) \leq w(t_1)$ and $p(t_1) = 2(\sqrt{x(t_1)} - \sqrt{x(t_2)}) + p(t_2)$. So we can rewrite the monopolist's problem as

$$\max 2(\sqrt{x(t_1)} - \sqrt{x(t_2)}) + 2p(t_2) - x(t_1) - x(t_2)$$

subject to the constraints,

$$\begin{aligned} &\sqrt{x(t_2)} - p(t_2) \geq 0, \\ &\sqrt{x(t_2)} - p(t_2) \geq \sqrt{x(t_1)} - 2(\sqrt{x(t_1)} - \sqrt{x(t_2)}) - p(t_2) \\ &\text{or equivalently } x(t_1) \geq x(t_2), \\ \\ &2(\sqrt{x(t_1)} - \sqrt{x(t_2)}) + p(t_2) \leq w(t_1). \end{aligned}$$

So we have $p(t_2) = \min\{\sqrt{x(t_2)}; w(t_1) - 2(\sqrt{x(t_1)} - \sqrt{x(t_2)})\}$ and $w(t_1) - 2(\sqrt{x(t_1)} - \sqrt{x(t_2)}) \geq 0$. The monopolist problem is then:

$$\max 2(\sqrt{x(t_1)} - \sqrt{x(t_2)}) + 2\min\{\sqrt{x(t_2)}; w(t_1) - 2(\sqrt{x(t_1)} - \sqrt{x(t_2)})\} - x(t_1) - x(t_2)$$

subject to the constraints $x(t_1) \geq x(t_2)$ and

$$w(t_1) + 2\sqrt{x(t_2)} \geq 2\sqrt{x(t_1)} + \min\{\sqrt{x(t_2)}; w(t_1) + 2(\sqrt{x(t_2)} - \sqrt{x(t_1)})\}.$$

We can divide the last problem in two parts. If $\sqrt{x(t_2)} \geq w(t_1) + 2(\sqrt{x(t_2)} - \sqrt{x(t_1)})$ then the maximand in the last problem becomes

$$2w(t_1) + 2(\sqrt{x(t_2)} - \sqrt{x(t_1)}) - x(t_1) - x(t_2).$$

Therefore the maximum

$$\max 2w(t_1) + 2(\sqrt{x(t_2)} - \sqrt{x(t_1)}) - x(t_1) - x(t_2)$$

subject to the constraints

$$\begin{gathered} x(t_1) \geq x(t_2) \geq 0, \\ w(t_1) + 2\left(\sqrt{x(t_2)} - \sqrt{x(t_1)}\right) \geq 0, \\ \sqrt{x(t_2)} \geq w(t_1) + 2\left(\sqrt{x(t_2)} - \sqrt{x(t_1)}\right) \end{gathered}$$

is not greater than

$$\max_{x(t_1)\geq x(t_2)\geq 0}\{w(t_1)+\sqrt{x(t_2)}-x(t_1)-x(t_2)\} \quad \leq \quad w(t_1)+\max_{x(t_2)\geq 0}\{\sqrt{x(t_2)}-2x(t_2)\}$$
$$= w(t_1)+\frac{2}{16} \quad = \quad \frac{10}{16}.$$

Recall that $w(t_1)=\frac{1}{2}$. If $\sqrt{x(t_2)} < w(t_1)+2(\sqrt{x(t_2)}-\sqrt{x(t_1)})$, then we have:

$$\max 2(\sqrt{x(t_1)}-\sqrt{x(t_2)})+2\sqrt{x(t_2)}-x(t_1)-x(t_2)$$

subject to the constraints

$$x(t_1)\geq x(t_2)\geq 0,$$
$$w(t_1)+2\left(\sqrt{x(t_2)}-\sqrt{x(t_1)}\right)\geq 0,$$
$$w(t_1)\geq 2\sqrt{x(t_1)}-\sqrt{x(t_2)}.$$

which is less or equal to

$$\max_{x(t_1)\geq x(t_2)\geq 0}\{w(t_1)+\sqrt{x(t_2)}-x(t_1)-x(t_2)\}\leq \frac{10}{16}.$$

. Q.E.D.

Proof of Lemma 2. Problem (4)-(7) simplifies to

$$\max p(t_1)+p(t_2)-x(t_1)-x(t_2)$$

subject to the constraints,$(x(t_i),p(t_i))\in X\times D$ for all i,

$$\sqrt{x(t_2)}-p(t_2)\geq 0$$

$$2\sqrt{x(t_1)}-p(t_1)\geq 2\sqrt{x(t_2)}-p(t_2),$$
$$\sqrt{x(t_2)}-p(t_2)\geq \sqrt{x(t_1)}-p(t_1) \tag{19}$$

Given the constraints, the monopolist can increase $p(t_1)$ up to

$$p(t_1)=2(\sqrt{x(t_1)}-\sqrt{x(t_2)})+p(t_2).$$

Substituting this into the monopolist's problem above and noting that (19) is now equivalent to $x(t_1)\geq x(t_2)$, we obtain

$$\max 2(\sqrt{x(t_1)}-\sqrt{x(t_2)})+2p(t_2)-x(t_1)-x(t_2)$$

subject to the constraints

$$\sqrt{x(t_2)}-p(t_2)\geq 0$$
$$x(t_1)\geq x(t_2)\geq 0$$

It is clear that at the optimum $p(t_2)=\sqrt{x(t_2)}$. Finally, to maximize

$$2\sqrt{x(t_1)}-x(t_1)-x(t_2)$$

subject to $x(t_1)\geq x(t_2)\geq 0$ we make $x(t_2)=0$ and $x(t_1)=1$. Therefore the optimal solution to the monopolist design problem without budget constraints is

$$x^*(t_1) \quad = \quad 1, p^*(t_1)=2$$
$$x^*(t_2) \quad = \quad 0, p^*(t_2)=0$$

and the optimal profit is $p^*(t_1) + p^*(t_2) - x^*(t_1) - x^*(t_2) = 1$.

Q.E.D.

Proof of Theorem 3:
Let $(x^*(\cdot), p^*(\cdot)) \in \Im$ be an optimal selling mechanism and suppose that $\pi(x^*(t), p^*(t)) < 0$ for consumer types $t \in E$ with $\mu(E) > 0$. Note that the monopolist's profit function does not depend directly on consumer types. Define the sets

$$S^* = cl\{(x^*(t), p^*(t)); t \in T\} \text{ and } S' = \{(x, p) \in S^*; \pi(x, p) \geq 0\} \cup \{(0, 0)\}.$$

The set S^* is the closure of the range of the mechanism $(x(\cdot), p(\cdot)) \in \Im$. By A-6 and the optimality of $(x^*(\cdot), p^*(\cdot))$, $\{(x, p) \in S^*; \pi(x, p) \geq 0\} \neq \emptyset$. Moreover, by the upper semicontinuity of the profit function $\pi(\cdot, \cdot)$ on $X \times D$, $\{(x, p) \in S^*; \pi(x, p) \geq 0\}$ is closed. By theorem 2 is Schal(1974) there exists functions, $(x'(\cdot), p'(\cdot)) \in \mathcal{M}(T, X) \times \mathcal{M}(T, D)$, such that

$$v(t, x'(t), p'(t)) = \max\{v(t, x, p); (x, p) \in S' \text{ and } p \leq w(t)\}.$$

Now define a new mechanism $(\overline{x}(\cdot), \overline{p}(\cdot))$ as follows:

$$(\bar{x}(t), \bar{p}(t)) = \begin{cases} (x^*(t), p^*(t)) & \text{if} \quad \pi(x^*(t), p^*(t)) \geq 0, \\ (x'(t), p'(t)) & \quad\quad \text{otherwise.} \end{cases}$$

It is easy to see that $(\bar{x}(\cdot), \bar{p}(\cdot))$ satisfies the IR and budget constraints. To see that $(\bar{x}(\cdot), \bar{p}(\cdot))$ satisfies the IC constraints let t and t' be any two consumer types in T and consider the following cases.

Case 1. Suppose $\pi(x^*(t), p^*(t)) \geq 0$ and $\pi(x^*(t), p^*(t)) \geq 0$.We have then

$$v(t, \overline{x}(t), \overline{p}(t)) = v(t, x^*(t), p^*(t)), \text{and } v(t, \overline{x}(t'), \overline{p}(t')) = v(t, x^*(t'), p^*(t)).$$

Thus, we must show that $v(t, x^*(t), p^*(t)) \geq v(t, x^*(t'), p^*(t'))$ provided $p^*(t') \leq w(t)$. But this follows from the incentive compatibility of $(x^*(\cdot), p^*(\cdot))$.

Case 2. Suppose $\pi(x^*(t), p^*(t)) \geq 0$ and $\pi(x^*(t), p^*(t)) < 0$. Then we have $v(t, \overline{x}(t), \overline{p}(t)) = v(t, x^*(t), p^*(t))$ and $v(t, \overline{x}(t'), \overline{p}(t')) = v(t, x'(t'), p'(t'))$. Thus we must show that $v(t, x^*(t), p^*(t)) \geq v(t, x'(t'), p'(t'))$ provided $p'(t') \leq w(t)$. But since $S' \subset S^* \cup \{(0, 0)\}$ and $v(t, x^*(t), p^*(t)) \geq v(t, x, p)$ for all $(x, p) \in S^* \cup \{(0, 0)\}$ such that $p \leq w(t)$,this follows from the definition of $(x'(\cdot), p'(\cdot))$.

Case 3. Suppose $\pi(x^*(t), p^*(t)) < 0$ and $\pi(x^*(t'), p^*(t')) \geq 0$. Then we have

$$v(t, \overline{x}(t), \overline{p}(t)) = v(t, x'(t), p'(t)), \text{and } v(t, \overline{x}(t'), \overline{p}(t')) = v(t, x^*(t'), p^*(t)).$$

Thus, we must show that $v(t, x'(t), p'(t)) \geq v(t, x^*(t'), p^*(t'))$ provided $p^*(t') \leq w(t)$. But this follows from the definition of $(x'(\cdot), p'(\cdot))$ and the fact that $(x^*(t'), p^*(t')) \in S'$.

Case 4. Suppose $(\pi(x^*(t), p^*(t)) < 0$ and $\pi(x^*(t'), p^*(t')) < 0$.Then we have $v(t, \overline{x}(t), \overline{p}(t)) = v(t, x'(t), p'(t))$, $v(\overline{x}(t'), \overline{p}(t')) = v(t, x'(t'), p'(t'))$. Thus we must show that $\text{v}(t, x'(t), p'(t)) \geq v(t, x'(t'), p'(t'))$ provided $p'(t') \leq w(t)$. But this from the definition of $(x'(\cdot), p'(\cdot))$. Comparing the monopolist's profit under the mechanisms $(x'(\cdot), p'(\cdot))$ and $(x^*(\cdot), p^*(\cdot))$ we have that

$$\int_T \pi(x'(t), p'(t)) d\mu(t) =$$

$$\int_{\{t; \pi(x^*(t), p^*(t)) \geq 0\}} \pi(x^*(t), p^*(t)) d\mu(t) + \int_{\{t; \pi(x^*(t), p^*(t)) < 0\}} \pi(x'(t), p'(t)) d\mu(t) >$$

$$\int_{\{t; \pi(x^*(t), p^*(t)) \geq 0\}} \pi(x^*(t), p^*(t)) d\mu(t) + \int_{\{t; \pi(x^*(t), p^*(t)) < 0\}} \pi(x^*(t), p^*(t)) d\mu(t) =$$

$\int_T \pi(x^*(t), p^*(t)) d\mu(t)$ contradicting the optimality of $(x^*(\cdot), p^*(\cdot))$. Q.E.D.

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