Pricing Derived Securities Under an Edgeworthian Process

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Abstract

The purpose of this paper is twofold. First, it introduces a new version of the Edgeworth process with trading activities centered around self-interested enterprising arbitragers; and second, it examines how the prices of the derived securities are determined under this process. We show that the proposed process is stable and the resulting equilibria are Pareto optimal. Pareto optimality notwithstanding, this process has the tendency to distribute the welfare gains resulting from the introduction of derived securities in favor of the arbitragers. JEL No.: 022,313,314. Keywords: Edgeworth Process, Derived Security Prices.

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1. Introduction

The publication of the "The Pricing of Options and Corporate Liabilities" by Black and Scholes (1973) thrust the problems of asset pricing into the center stage of financial economics. Yet after two decades of research and a bulging literature, our understanding of this problem is still incomplete. The key unresolved issue is the correct pricing of those derived securities with return distributions not spanned by the existing securities. In order to ascertain the prices of these securities, the prevailing theory must rely on the assumption that the prices of the underlying elementary securities[2] are known. However, no satisfactory explanation is given as to how the elementary security prices are determined. This paper bridges the gap by identifying and describing the market process through which all derived security prices, including those of the elementary securities, are determined.

The market process we envisage centers around arbitragers who create and trade derived securities in order to serve their own self-interests. We believe that the security market is set in motion through the initiatives of these arbitragers and that the market is capable of trading both in and out of equilibrium. Through trial and error, the arbitragers steer the market towards an equilibrium. Whenever trading takes place in disequilibrium situations, the market will continue to adjust. The adjustment follows a modified Edgeworth process. At each stage of the process, trading takes place as long as both the arbitrager and his trading partner become better off. The process terminates whenever mutually beneficial gains are exhausted. At this point, the equilibrium prices of the derived securities, including those of elementary securities, are determined.

In order to set the stage for a formal analysis, this paper begins with a section which describes the composition and the working of the security market. It then presents two models which describe the market adjustment process and shows that this process is stable and the resulting equilibria are Pareto optimal. The structure of the two models are identical, except that they involve different numbers of arbitragers. The first model with only one arbitrager enables us to describe the trading process clearly and to show how the security market functions. The second model with two arbitragers allows us to demonstrate that an increase in the number of arbitragers in the security market does not in any way alter the nature of the market process. Equilibrium properties and the implications of the models are given in the last section.

2. The Security Market

This section describes the composition of the security market and also provides a heuristic description of the market process.

2.1 Characterization of the Security Market

There are two types of securities, the primitive and the derived. Primitive securities are issued by firms which use the proceeds to finance production; they serve as certificates of claim to the firms' assets and profits. Here, the existence of these primitive securities are taken as given and so are their prices. In doing so, we in fact have taken the firms' production policies and their payoff distributions as given. The derived securities, on the other hand, are pure financial obligations; their existence is not justified by production but rather by their ability to shift wealth over time and across the states of nature. These securities help to generate specific income streams desired by the consumers and, therefore, play a role limited only to facilitating exchange, not production. One of the major purposes of this paper is to provide a reason for the existence of these derived securities and to determine their prices.

2.2 The Role of Derived Securities

Let a set of J firms be indexed by $j = 1,...,J$ and let Ω be the finite set of states of nature indexed by ω. Firm j issues a primitive security r_j with payoff pattern $r_j = (r_{\omega j})_{\omega \in \Omega}$, where $r_{\omega j}$ denotes the dividend paid out by firm j in state $\omega \in \Omega$. The payoff pattern for the J securities is denoted by $r = (r_{\omega j})_{\omega \in \Omega, j \in J}$, and the set of securities which yields r is denoted by R.

Given R, the consumer uses the security market to arrange for a portfolio which yields for him the optimal distribution of consumptions across the various states of nature. The ability of any individual to shift wealth from one state to another depends crucially on the number of independent securities available in the security market and the number of the states of nature. Assuming $J < |\Omega|$, then there are not enough securities to implement perfect arbitrage of wealth among all states of nature. Only income lying within the J-dimensional subspace spanned by the primitive securities can be reached. However, since failure to shift wealth outside the J-dimensional subspace is caused by a shortage of independent securities, it is natural to ask why the market cannot simply supply the needed securities.

It is well known that the ability of the market to supply the needed derived securities depends crucially upon whether the J-primitive securities are capable of identifying or resolving more than J states of nature (John 1984). Assume that the set of securities R can identify more states (say m states) than the number of independent securities in R. For simplicity, let

$m = |\Omega|$; then it is possible for the market to augment $(m - J)$ independent derived securities. In doing so, the J primitive securities and the $(m - J)$ independent derived securities together can span an m-dimensional payoff subspace within which wealth can be shifted freely from one state to another.

Thus, if the primitive securities are capable of identifying more than J states, then there exists the potential to use the derived securities to shift wealth to points heretofore unable to be reached. The existence of such possibilities induces the consumers to demand derived securities in order to obtain better allocation for their income and wealth. The existence of such demand also gives rise to profit opportunities for any arbitrager who is able to create such financial instruments.

A method commonly used to create derived securities is the selling of options. In this paper, we shall not examine how options are created, since this procedure is examined in depth by the literature (Ross 1976). Instead, we shall simply point out that through unbundling of primitive securities, derived securities can be created.[3] Unbundling takes place whenever a security is decomposed into multiple parts with the sum of the parts equal to the whole. For example, an asset $r = (2,7,12)$ can be decomposed into $r_1 = (1,6,5)$ and $r_2 = (1,1,7)$ or into $r'_1 = (1,4,7)$, $r'_2 = (0,3,5)$ and $r'_3 = (1,0,0)$. Thus, the unbundling is not unique. However, in general, the unbundled securities can be expressed always as a linear combination of the basic securities[4] $D_1 = (2,0,0)$, $D_2 = (0,7,0)$ and $D_3 = (0,0,12)$. This is true whether r is decomposed into r_1 and r_2 or into r'_1, r'_2, and r'_3.

From this example, it is evident that independent securities are indeed created through unbundling. Moreover, since the primitive security $r = (2, 7, 12)$ identifies 3 states of nature, two independent derived securities, say $D_1 = (2, 0, 0)$ and $D_2 = (0, 7, 0)$, can be created.[5] The three securities r, D_1 and D_2 together span a 3-dimensional return subspace. In general, the creation of the independent derived securities can help the primitive securities to span an augmented return subspace with a dimensionality equal to the number of states identified by the primitive securities. Now, the consumer is able to trade in the security market for a portfolio which enables him to attain the best possible consumption pattern. Yet, in order to identify this consumption pattern, all security prices must be known. Since prices for the basic securities do not exist at the outset, it is not possible to identify either the consumer's optimal portfolio or the market equilibrium. We can only assert that as long as an exchange of derived securities improves the traders' welfare at the prevailing prices, trading of these securities will take place. Because these trades may be suboptimal, after their consummation, other mutually beneficial trading opportunities still exist at different prices and arbitraging profits are still available. Trading activities will continue. Equilibrium is attained only when opportunities for arbitraging are exhausted in the security market.

2.3 The Market Process

In this section, we present a trading process that involves arbitragers who effectively perform the unbundling services, and demonstrate that this trading process does in fact lead to one of the many potential equilibria in the security market and that each of the equilibria is Pareto optimal. The process begins with the emergence of at least one arbitrager. The arbitrager performs his role by purchasing primitive securities, unbundles them into derived securities, and then sells them to the consumer. As we have mentioned above, the consumer is willing to pay a premium for the derived securities because these securities enable him to shift income and wealth across the states of nature heretofore unattainable through the use of primitive securities alone. The implied profit potential spurs the enterprising arbitragers into action.

In this paper, for expositional convenience, we shall assume as an initial condition that the arbitragers have already bought up all the primitive securities. As a consequence, the primitive securities are out of circulation and are used solely to support the creation of derived securities. Only derived securities are traded in the market place. This simplifying assumption enables us to avoid the complexity associated with the simultaneous trading of both primitive and derived securities and, thus, allows us to examine single mindedly how prices of the derived securities are determined. Of course, for greater realism, we must eventually present a model which permits simultaneous trading of both primitive and derived securities. This task is, however, deferred to a later date.

Since the arbitrager must first acquire primitive securities and unbundle them, it is implausible to postulate that the arbitrager's objective is to swap derived securities for primitive securities which he already owns. Instead, it is more reasonable to assume that he intends to sell or swap derived securities for gains. Consequently, the best way to model the arbitrager's activities is by treating him as a retailer and as a broker of derived securities. The crucial issue is: Given that the arbitrager's activities are also motivated by his desire to allocate his own income optimally across the states of nature, in what way are the prices of the derived securities determined.

Our first inclination is to assume that the arbitrager will price the derived securities in order to maximize his own expected utility of consumption. The difficulty associated with this approach is two-fold. First, since the arbitrager does not know his trading partners' preferences, an enormous amount of information is needed before he can make an estimate of the demand for his derived securities and then set the utility maximization prices. This information may be too costly to acquire and hence renders pricing decisions based upon the maximizing principle impractical. Second, even if sufficient (albeit imperfect) information is available, making pricing decisions according to the maximizing principle still may not be desirable. Specifically, optimal prices selected *ex ante* based upon imperfect information may not be optimal *ex post*; some mitigating adjustment must be made *ex post*. The cost involved in these *ex post* adjustments may be too high to make the adherence of the maximizing principle worthwhile. Some may suggest that in order to insure that the selected prices are optimal both *ex ante* and *ex post*, a tâtonnement process must be followed. The rub is that no market institution has emerged to support effectively such a market process.

With these caveats in mind, we opt to adopt a bounded rationality approach and characterize the arbitrager not as a maximizer but a satisficer. We begin by simply assuming that the arbitrager perceives the following situation:

> If the prices of the derived securities were set in such a way as to make the terms of trade too favorable to the arbitrager, customers would trade too few of these securities and thus reduce the arbitrager's utility of consumption. If, on the other hand, the prices of the derived securities were set in such a way as to make the terms of trade unfavorable to the arbitrager, customers would wish to trade too many of these securities and leave few units for the arbitrager. This shortage would hamper the arbitrager's efforts to allocate his own income across the states of nature.

Satisficing under this situation implies that initially the arbitrager will use his own judgement to ascertain the appropriate prices. Transactions will take place at these prices as long as they enhance the utilities of both the buyer and the seller. In the case where the *ex ante* selected prices have proven to be unsatisfactory to the arbitrager *ex post*, as we shall see in the next section, some mitigating adjustment can still be made. The arbitrager can maintain satisfaction by resorting to rationing. After tradings have taken place under rationing, the arbitrager knows that there still exists excess demand (supply) in some of the basic securities and, hence, this round of transactions will not lead to an equilibrium. Profit opportunities remain, and additional transactions will take place at a new set of prices.

Prices are adjusted on the basis of the excess demands; that is, the price of a security whose excess demand is positive will rise, while the price of a security whose excess demand is negative will fall. At the new set of prices, if the excess demands for the securities are not all zero, then the adjustment process (including rationing) will continue in the above described manner until such time that the excess demand for each and every basic security becomes zero.

When that happens, the consumers' portfolios are all optimal, no arbitraging profit remains and no further trade will take place. The equilibrium prices for the derived securities, including those of the basic securities, emerge at last.

At this point, it is pertinent to raise the question of whether an increase in the number of arbitragers will affect the nature of the market process in the security market. In order to answer this question, we shall divide the presentation of the model in two parts. First, in section 3, we shall present the basic model with a single arbitrager and demonstrate that the proposed process is stable and the resulting equilibria are Pareto optimal. Then, in section 4, we shall present the expanded model with two arbitragers and show that an increase in the number of arbitragers does not destroy either the stability of the process or the Pareto optimality of the equilibria.

3. The Basic Model: The Single Arbitrager Case

There are two dates, today and tomorrow, indexed by 0 and 1. Today is known with certainty and tomorrow is uncertain; this uncertainty is described by a set of finite states of nature Ω. In addition, there is one commodity at each date; and there are n agents indexed by $i=1,...,n$ and J primitive securities, indexed by $j=1,...,J$. As we have mentioned earlier, the jth primitive security is denoted by $r_j = (r_{\omega j})_{\omega \in \Omega}$ where $r_{\omega j}$ represents the dividend paid by security j in state ω, $\omega \in \Omega$. Let $d_\omega = \Sigma_{j=1}^{J} r_{\omega j}$. We write

$D = (d_\omega)_{\omega \in \Omega}$.

The derived security D can be unbundled into basic derived securities D_ω that pays d_ω in state ω and zero in all other states. We assume that only these basic securities and their linear combinations are traded in the security market. Specifically, we postulate that an agent, say agent n, is designated as the arbitrager. The arbitrager buys up all the J primitive securities, combines them into security D, then unbundles D into basic securities D_ω, and sells either D_ω or linear combinations of D_ω, $\omega \in \Omega$, in the security market.

The consumer trades the commodity (designated as the numeraire good) as well as the derived securities in date 0 in order to arrange for his consumption plan $c^i = (c_0^i, c_\Omega^i)$, i = 1,...,n, where $c_\Omega^i = (c_\omega^i)_{\omega\in\Omega}$ and c_ω^i is his consumption of the commodity in date 1 and state ω. We follow the convention of the standard Edgeworth process and assume that date 0 is subdivided into many time periods, tradings take place in each period, and no consumption is allowed while trading is in progress. Consumption can take place only after the security market has reached an equilibrium.

3.1 The Agent's Choice Problem

Let agent i's preferences be represented by a von Neumann-Morgenstern utility function $U^i(c^i)$, where U^i is strictly concave and twice continuously differentiable and $c^i = (c_0^i, c_\Omega^i)$, i=1,...,n. Agent i will trade e_0^i and D_ω, $\omega \in \Omega$, in date 0 in order to obtain a bundle c^i which maximizes his utility; here e_0^i is the quantity of the commodity endowed to agent i in date 0.

The following notations will be used throughout the paper.

(i) Let δ_0 denote the quantity of the commodity consumed in date 0 and let $\delta_\Omega = (\delta_\omega)_{\omega\in\Omega}$ denote a portfolio of basic securities. Denote $\delta = (\delta_0,\delta_\Omega)$.

(ii) Let agent i's endowment of the commodity in date 1 and state ω be denoted by e^i_ω, $\omega \in \Omega$, and his endowment of the commodity as well as a portfolio of basic securities in date 0 be denoted by $\bar{\delta}^i = (\bar{\delta}^i_0, \bar{\delta}^i_\Omega)$.

(iii) Let q_ω be the price of the basic security D_ω and $q_\Omega = (q_\omega)_{\omega\in\Omega}$ be the price vector of the basic securities. The price vector of the commodity as well as the basic securities can be written as $q = (1, q_\Omega)$.

Because $c^i_\Omega = (e^i_\omega + \delta^i_\omega d_\omega)_{\omega\in\Omega}$, it is evident that through trading agent i's choice of δ^i (depending on q and $\bar{\delta}^i$) automatically induces a consumption plan $c^i = (c^i_0, c^i_\Omega)$ with $c^i_0 = \delta^i_0$, provided that trading activities are conducted within the budget constraint
$q\delta^i \leq q\bar{\delta}^i$.

Thus, agent i will choose $\delta^i(q, \bar{\delta}^i)$ in order to

$$\max\ U^i(\delta^i_0, e^i_\Omega + \delta^i_\Omega \times D)$$

$$\text{s.t.}\quad q\delta^i \leq q\bar{\delta}^i,$$

$$\delta^i_0 \geq 0$$

$$0 \leq \delta^i_\omega \leq 1,\ \omega \in \Omega$$

where $\delta^i_\Omega \times D$ denotes the vector $(\delta^i_\omega d_\omega)_{\omega\in\Omega}$. Because $e^i = (e^i_0, e^i_\Omega)$ and D are given and fixed, we may derive a utility function W^i from U^i, where W^i is a function of δ^i only; that is, $W^i(\delta^i) = U^i(\delta^i_0, e^i_\Omega + \delta^i_\Omega D)$. Thus, agent i's maximization problem becomes

$$\max \; W^i(\delta^i)$$

$$\text{s.t. } q\delta^i \le q\bar{\delta}^i,$$

$$\delta^i_0 \ge 0$$

$$0 \le \delta^i_\omega \le 1, \; \omega \in \Omega.$$

The above maximization problem yields for agent i a vector of excess demand functions denoted by $Z^i(q,\bar{\delta}^i) = (Z^i_0(q,\bar{\delta}^i), Z^i_\Omega(q,\bar{\delta}^i))$, where $Z^i_0(q,\bar{\delta}^i) = \delta_0(q,\bar{\delta}^i) - \bar{\delta}^i_0$ and $Z^i_\Omega(q,\bar{\delta}^i) = (Z^i_\omega(q,\bar{\delta}^i))_{\omega\in\Omega}$ with $Z^i_\omega(q,\bar{\delta}^i) = \delta^i_\omega(q,\bar{\delta}^i) - \bar{\delta}^i_\omega$ for all $\omega \in \Omega$.

3.2 Trading Process

Let $\delta^{-n} = (\delta^i)^{n-1}_{i=1}$, $\delta = (\delta)^n_{i=1}$, $Z^{-n}(q,\delta) = \Sigma^{n-1}_{i=1} Z^i(q,\delta^i)$, and $Z(q,\delta) = \Sigma^n_{i=1} Z^i(q,\delta^i)$. Denote the price vector and agent i's portfolio at time t (in date 0) by $q(t)$ and $\delta^i(t)$, respectively, for $i=1,2,...,n$. For notational convenience, let

$Z_\omega(t) = Z_\omega(q(t),\delta(t))$ and $Z^i(t) = Z^i(q(t),\delta(t))$. The dynamic system for the trading process becomes:

$$(*) \begin{cases} \dot{q}_\omega(t) = Z_\omega(t), \quad \omega \in \Omega \\ \dot{q}_0(t) = 0 \\ \dot{\delta}^i(t) = \lambda_i(t) Z^i(t), \; i=1,2,...,n-1 \\ \dot{\delta}^n(t) = -\sum_{i=1}^{n-1} \lambda_i(t) Z^i(t) \end{cases}$$

where $\lambda^{-n}(t) = (\lambda_1(t),...,\lambda_{n-1}(t))$ is an element in the set $\Lambda(t)$ such that there does not exist any $\lambda^{-n} \in \Lambda(t)$ with $\lambda^{-n} \leq \lambda^{-n}(t)$ and $\lambda^{-n} \neq \lambda^{-n}(t)$. Here, $\Lambda(t)$ is the set of nonnegative vectors $\lambda^{-n} = (\lambda_1,...,\lambda_{n-1}) \in [0,1]^{n-1}$ such that

$$\delta^n - \sum_{i=1}^{n-1} \lambda_i Z^i(t) \geq 0$$

and

$$W^n(\delta^n - \sum_{i=1}^{n-1} \lambda_i Z^i(t)) \geq W^n(\delta^n - \sum_{i=1}^{n-1} \lambda_i' Z^i(t))$$

for all $\lambda'^{-n} = (\lambda_1',...,\lambda_{n-1}') \in [0,1]^{n-1}$ satisfying the above inequalities. Elements in $\Lambda(t)$ will be called *rationing vectors.*

Note that the above trading process satisfies the following feasibility condition:

(†) No production is involved; that is, at any t

$$\Sigma_{i=1}^{n} \dot{\delta}^i(t) = \Sigma_{i=1}^{n-1}\lambda_i(t)Z^i(t) - \Sigma_{i=1}^{n-1}\lambda_i(t)Z^i(t) = 0.$$

3.3. Interpretation of the Adjustment Rules

The above adjustment rules reflect the arbitrager's bounded rationality approach to price and quantity adjustments. As we have mentioned in the introduction, under the condition of uncertainty, the arbitrager will use his judgement of the market situation to set and adjust prices. Since exchange takes place only between the arbitrager and the consumers, judgement here means that the arbitrager must be certain that the prices selected by him will afford him the opportunity to derive from his trading activities a satisfactory level of utility. Bounded rationality also implies that once the prices are set and the excess demands are revealed, whenever $Z^n(t) \neq -\sum_{i=1}^{n-1} Z^i(t)$ the arbitrager still has an opportunity to protect his lot by rationing the quantity exchanged to the various consumers. He does so by choosing the rationing coefficients $\lambda^{-n} \in \Lambda$ so that $W^n(\delta^n - \sum_{i=1}^{n-1} \lambda_i Z^i(t))$ is made as large as possible.

To see this, we first observe that the prices selected by the arbitrager induce for him a *trading set*

$$T(t) = \left\{ -\sum_{i=1}^{n-1} \lambda_i Z^i(t) \,|\, (\lambda_1,\ldots,\lambda_{n-1}) \in [0,1]^{n-1} \right\}$$

which embodies all of his potential net trades. There are two possible ways to carry out the *ex post* allocative adjustments. First, when $\alpha Z^n(t) \in T(t)$ for some $\alpha \in (0,1]$, then $\alpha Z^n(t) = -\sum_{i=1}^{n-1} \lambda_i Z^i(t)$ for some rationing vector $\lambda^{-n} \in [0,1]^{n-1}$; the arbitrager will ration the consumers according to the rationing coefficients $(\lambda_1,\ldots,\lambda_{n-1})$ and will self-ration by allowing himself to receive only α fraction of his excess demand $Z^n(t)$. Only when $Z^n(t) \in T(t)$ will the arbitrager exempt himself from rationing. Second, when $\alpha Z^n(t) \notin T(t)$ for any $\alpha \in (0,1]$, the arbitrager cannot obtain any excess demand vector proportional to $Z^n(t)$. However, there may still exist an $Z \in T(t)$ such that $W^n(\delta^n(t) + Z(t)) > W^n(\delta^n(t))$. In this case, the arbitrager will choose a utility maximizing Z in $T(t)$ to trade. In any event, once the rationing coefficients are selected, the actual portfolio adjustment for the agents are determined; namely, $\dot{\delta}^i(t) = \lambda_i(t) Z^i(t)$, i=1,...,n-1, and $\dot{\delta}^n(t) = -\sum_{i=1}^{n-1} \lambda_i(t) Z^i(t)$ as shown in (*).

Whenever trading takes place with *ex post* adjustments, unsatisfied demands remain. Further trading at some different prices will benefit both the trading partners. The arbitrager will select a new price for D_ω, $\omega \in \Omega$, according to the rule $\dot{q}_\omega(t) = Z_\omega(t)$. Tradings again will take place at these new prices. The market continues to function in the manner described by the adjustment process (*) until no further gain is possible.

EXAMPLE 1: Consider a security market in which there are two agents, 1 and 2, and there are two states, ω_1 and ω_2. Let Agent 1's and 2's future income be given respectively by $e^1_\Omega = (2,5)$ and $e^2_\Omega = (10,5)$, and the two basic securities be

$D_{\omega_1} = (5,0)$ and $D_{\omega_2} = (0,10)$. Both agents' utility functions over consumption plans take logarithmic forms:

$$U^i(c^i) = \ell n\ c^i_0 + \pi^i_{\omega_1}\ \ell n\ c^i_{\omega_1} + \pi^i_{\omega_2}\ \ell n\ c^i_{\omega_2},\ i=1,2$$

where $\left(\pi^1_{\omega_1}, \pi^1_{\omega_2}\right) = \left(\frac{2}{5}, \frac{3}{5}\right)$ and $\left(\pi^2_{\omega_1}, \pi^2_{\omega_2}\right) = \left(\frac{3}{5}, \frac{2}{5}\right)$ are, respectively, the subjective probability beliefs over future states for agents 1 and 2. Suppose at the beginning of time t, $\delta^1 = \left(10, \frac{1}{5}, \frac{1}{5}\right)$, $\delta^2 = \left(5, \frac{4}{5}, \frac{4}{5}\right)$, and $q_\Omega = (2,4)$. Taking the prices as given, the optimal choices for agent i, i=1,2, are found by the following first order conditions:

$$\delta_0^{i*} = \frac{q\delta^i}{2} + \frac{q_{\omega_1} e_{\omega_1}^i}{2d_{\omega_1}} + \frac{q_{\omega_2} e_{\omega_2}^i}{2d_{\omega_2}}$$

$$\delta_{\omega_1}^{i*} = \frac{\pi_{\omega_1}^i q\delta^i}{2q_{\omega_1}} + \frac{(1 + \pi_{\omega_2}^i) e_{\omega_1}^i}{2d_{\omega_1}} + \frac{\pi_{\omega_1}^i e_{\omega_2}^i q_{\omega_2}}{2d_{\omega_2} q_{\omega_1}}$$

$$\delta_{\omega_2}^{i*} = \frac{\pi_{\omega_2}^i q\delta^i}{2q_{\omega_2}} + \frac{(1 + \pi_{\omega_1}^i) e_{\omega_2}^i}{2d_{\omega_2}} + \frac{\pi_{\omega_2}^i e_{\omega_1}^i q_{\omega_1}}{2d_{\omega_1} q_{\omega_2}}$$

Substituting values of the variables into the above equations, we obtain

$\delta^{1*} = \left(7, 1, \frac{11}{20}\right)$ and $\delta^{2*} = \left(7.9, \frac{37}{100}, \frac{29}{100}\right)$. Thus, $Z^1 = \left(-3, \frac{80}{100}, \frac{35}{100}\right)$ and $Z^2 = \left(2.9, -\frac{43}{100}, -\frac{51}{100}\right)$. Since $-Z^1(t) \neq Z^2(t)$, rationing will take place. Simple calculation would show that $Z^2(t)$ is not proportional to $-Z^1(t)$, and hence $\alpha Z^2(t) \neq -Z^1$ for all $\alpha \in (0,1]$. However, agent 2 (the arbitrager) nonetheless can arrange for himself a consumption plan $c^{2'}$ = (7.15, 6.8, 10.325) by offering a net trade of 0.96 $Z^1(t)$ to agent 1. Otherwise, agent 2′s consumption plan becomes $c^2 = (5,14,13)$ which is based solely on his endowments.

Since $U^2(c^{2'}) = 4.3615$ and is greater than $U^2(c^2) = 4.2188$, the arbitrager's expected utility is increased by trading with agent 1. At the same time, because agent 1's utility function is concave, trading any fraction of $Z^1(t)$ will increase the expected utility of agent 1. Hence, the consumer will accept the arbitrager's trading proposal of 0.96 $Z^1(t)$ and obtain $U^1(c^{1'}) = 4.1346$ which is greater than $U^1(c^1) = 3.9095$. Consequently, trading will take place between the two agents.

Even though exchange had taken place, because $-Z^1(t) \neq Z^2(t)$, the security market is still not in equilibrium. Trading opportunities remain at different prices. The arbitrager will adjust the security prices according to the rule:

$$\dot{q}_{\omega_1} = Z_{\omega_1} = \left(\frac{80}{100} - \frac{43}{100}\right) = 0.37$$

$$\dot{q}_{\omega_2} = Z_{\omega_2} = \left(\frac{35}{100} - \frac{51}{100}\right) = -0.16$$

The revised prices become $q_{\omega_1} = (1 + 0.37) \times 2 = 3.74$ and $q_{\omega 2} = (1 - 0.16) \times 4 = 3.36$. The agents will trade from their current portfolios $\delta^1 = \left(7.12,\ \frac{96.8}{100},\ \frac{53.6}{100}\right)$ and $\delta^2 = \left(7.88,\ \frac{3.2}{100},\ \frac{46.4}{100}\right)$ at these new prices.[6] The market process continues until all gains are exhausted.

3.4 Stability of the Trading Process

The trading process (*) yields a set of simultaneous differential equations. Given the initial conditions, a solution of these equations is path dependent on time. Since the path taken is not unique, there exists a multiplicity of solutions. In this section, we establish the **proposition** that (1) the trading process is stable in the sense that regardless which time path the adjustment has taken, there always exists an equilibrium allocation supported by a set of market clearing prices; (2) regardless which equilibrium has been obtained, the equilibrium is Pareto optimal.

Thus said, an equilibrium is nothing but a limit point of the trading process. Formally,

D1. Let $(q(0), \delta(0))$ be any initial state and $(q(t), \delta(t))$ be the solution to the trading process. A pair (q^*, δ^*) is an equilibrium of the process if there is a subsequence $(q(t_\lambda)),\delta(t_\lambda)$ such that $(q^*, \delta^*) = \lim_{\lambda \to \infty} (q(t_\lambda), \delta(t_\lambda))$.

Based upon this notion of equlibrium, the above stated proposition can be established in two steps:

First: We prove that under a given initial state, if there does not exist a mutually beneficial trade at any time, then the initial state must be Pareto optimal.

Second: We prove that if any of the equilibria of the trading process is taken as the initial state, then this equilibrium satisfies the above-mentioned property and, hence, is Pareto optimal.

However, before presenting the theorems, we need to make the assumption that the arbitrager prefers any portfolio with positive component of all basic securities to those with zero component of at least one basic security. Formally,

A1. $W^n(\delta^n) < W^n(\delta'^n)$ for any $\delta^n \geq 0$ and $\delta'^n \geq 0$ such that δ^n has at least one zero component while all the components of δ'^n are positive.

THEOREM 1 *Let $(q(0),\delta(0))$ be a given initial state and let $(q(t),\delta(t))$ be the solution to the trading process (*) with the rationing coefficient vectors $(\lambda^{-n}(t))$. If $\lambda^{-n}(t) = 0$ for all t and $\delta^n(0) >> 0$, then $\delta(0)$ is Pareto optimal.*

Proof. Let $V(t) = \sum_{\omega\in\Omega}\left[\frac{W^n_\omega(\delta^n(0))}{W^n_o(\delta^n(0))} - q_\omega(t)\right]^2$, where observe

$W^n_\omega(\delta^n(0)) = \partial W^n(\delta^n(0))/\partial\delta^n_\omega$ for $\omega \in \Omega$ and $W^n_0($ $^n(0))$ $^n($ $^n(0))$ $_0^n$. Firs

that V(t) = 0 only if q(t) is an equilibrium price vector; otherwise, V(t) > 0. Thus, for any $t > 0$, if $q(t)$ is not an equilibrium price vector, then $Z^i(t) \neq 0$ for some trader i. Next, using a tactic δ = ∂W δ /∂δ t, we similar to the one employed by Arrow and Hahn (1971, pp. 331-332), we show that V is monotonically decreasing.

Suppose first that $i \neq n$. Since $\delta^n(0) >> 0$, $\delta^n(0) - \theta Z^i(t) \geq 0$ for a small $\theta > 0$. Because $\lambda^{-n}(t) = 0$ and $Z^i(t) \neq 0$, it must be true that $W^n(\delta^n(0) - \theta Z^i(t)) - W^n(\delta^n(0)) < 0$. For small $\theta > 0$, this difference in utility can be approximated by

$-\theta W_0^n(\delta^n(0))Z_0^i(t) - \theta\, \Sigma_{\omega\in\Omega} W_\omega^n(\delta^n(0))Z_\omega^i(t)$. Thus,

$$W_0^n(\delta^n(0))Z_0^i(t) + \sum_{\omega\in\Omega} W_\omega^n(\delta^n(0))Z_\omega^i(t) > 0. \tag{1}$$

(Note that for those i whose $Z^i(t) = 0$, (1) becomes an equality.) By summing (1) over $i = 1,2,...,n\text{-}1$, we obtain

$$W_0^n(\delta^n(0))Z_0^{-n}(t) + \sum_{\omega\in\Omega} W_\omega^n(\delta^n(0))Z_\omega^{-n}(\tag{2}$$

Now suppose $i=n$. Then, $W^n(\delta^n(0) + \theta Z^n(t)) - W^n(\delta^n(0)) > 0$. As in the above case, for a small proportion $\theta > 0$ of $Z^n(t)$, we may approximate the corresponding difference in utility by the linear terms and obtain

$$W_0^{\,n}(\delta^n(0))Z_0^{\,n}(t) \; + \; \sum_{\omega\in\Omega} \; W_\omega^{\,n}(\delta^n(0))Z_\omega^{\,n}(t) \; > \; 0. \tag{3}$$

Finally, sum (2) and (3) we have

$$W_0^{\,n}(\delta^n(0))Z_0(t) \; + \; \sum_{\omega\in\Omega} \; W_\omega^{\,n}(\delta^n(0))Z_\omega(t) \; > \; 0. \tag{4}$$

From the budget constraints, we know that $q(t)Z(t) = 0$; thus, $Z_0(t) = -\Sigma_{\omega\in\Omega}q_\omega(t)Z_\omega(t)$. By substituting $Z_0(t)$ into (4) and observing that $W_0^n(\delta^n(0)) > 0$, (4) becomes

$$\sum_{\omega\in\Omega} \left[\frac{W_\omega^{\,n}(\delta^n(0))}{W_0^{\,n}(\delta^n(0))} \; - \; q_\omega(t)\right]Z_\omega(t) \; > \; 0. \tag{5}$$

Finally, observe that the time derivative of V(t) is

$$\dot{V}(t) \; = \; -2 \sum_{\omega\in\Omega} \left[\frac{W_\omega^{\,n}(\delta^n(0))}{W_0^{\,n}(\delta^n(0))} \; - \; q_\omega(t)\right]\dot{q}_\omega(t).$$

The dynamic system (*) and (5) together imply

$$\dot{V}(t) \; = \; -2 \sum_{\omega\in\Omega} \left[\frac{W^{\,n}(\delta^n(0))}{W_0^{\,n}(\delta^n(0))} \; - \; q_\omega(t)\right]Z_\omega(t) \; < \; 0. \tag{6}$$

In turn, (6) implies that V decreases monotonically in t along the solution path as long as q(t) is not an equilibrium price vector. Since V is monotonically decreasing and is bounded from below by 0, $\lim_{t\to\propto} V(t) = \lim_{t\to\propto} \dot{V}(t) = 0$. However, $\dot{V}(t) = 0$ if and only if $Z(t) = 0$ as shown in (6), thus, we may conclude that $(q(t))$ converges and the limit q^* satisfies $Z^i(q^*,\delta^i(0)) = 0$ for all i. Because no exchange is possible at q*, the fact that $Z^i(q^*,\delta^i(0)) = 0$ for all i also implies that $\delta(0)$ is Pareto optimal. Q.E.D.

THEOREM 2 *Let $(q(0),\delta(0))$ be any given initial state such that $\delta^n(0) >> 0$. Then, the equilibria of the trading process (*) are Pareto optimal.*

Proof. Let $(q(t),\delta(t))$ be the solution of the system. Since $\dot{W}^i(\delta^i(t)) \geq 0$, W^i converges as $t \to \infty$ for all i. Consider the function $W(t) = \Sigma_{i=1}^{n} W^i(\delta^i(t))$. Clearly, W is monotonically increasing and hence converges as $t \to \infty$. Let W^* and W^{i*}, respectively, be the limit of W and W^i for i=1,...,n. Then, $W^* = \Sigma_{i=1}^{n} W^{i*}$. For any limit point (q^*,δ^*) of $(q(t),\delta(t))$, it must be true that $W^i(\delta^{i*}) = W^{i*}$ and, by assumption A1,
$\delta^{n*} >> 0$. Thus, if $(q^*(t),\delta^*(t))$ is the solution of the system initiated at (q^*,δ^*) with rationing vectors $(\lambda^{-n*}(t))$, then $\delta^{i*}(t) = \delta^{i*}$ and $\lambda^{-n*}(t) = 0$ for all $t > 0$. By Theorem 1, δ^* is Pareto optimal. Q.E.D.

4. The Expanded Model: The Two Arbitragers Case[7]

Having presented the basic model, we are now in position to examine the impact of an increase in the number of arbitragers on the nature of equilibrium in the security market. For simplicity, let us assume that there are now two arbitragers n and n + 1, each of whom possesses 50% of the primitive securities[8] and hence 50% of the bundled security D. Each arbitrager performs the unbundling feat and competes with the other to satisfy the consumers' portfolio needs. The basic structure of the model remains intact except that the consumer's maximization problem becomes slightly more complex; namely, each consumer now has the option to arrange for his portfolio from two sources. Let $q^h(t)$ be the price vectors of the basic securities quoted by arbitrager h, h = n, n+1, at time t. Assume that the two arbitragers set their pricing policies independently, then it is possible that at any t $q_\omega^n(t) \neq q_\omega^{n+1}(t)$ for some $\omega \in \Omega$. Let

$$\Omega_1(t) = \{\omega \in \Omega : q^n(t) = q^{n+1}(t)\}$$

$$\Omega_2(t) = \{\omega \in \Omega : q^n(t) > q^{n+1}(t)\}$$

$$\Omega_3(t) = \{\omega \in \Omega : q^n(t) < q^{n+1}(t)\}$$

The following assumptions are in order.

A2. Each agent is permitted to buy and sell securities only once in each time t; no short sale is permitted.

A3. The consumer will buy (sell) D_ω, $\omega \in \Omega$, from (to) the arbitrager who quotes the lower (higher) price. In case the arbitragers quote the same price for some D_ω, $\omega \in \Omega$, we assume that the consumer will purchase (sell) one half of the amount from (to) each arbitrager.

Based upon these assumptions, the choice problem for consumer i, i=1,...,n+1, becomes

$$\max \quad W^i(\delta^i)$$

$$\text{s.t.} \quad \delta_0^i + q_{\Omega_1}\delta_{\Omega_1}^i + q_{\Omega_2}^{n+1}\delta_{\Omega_2}^i + q_{\Omega_3}^{n}\delta_{\Omega_3}^i$$

$$\leq \bar{\delta}_0^i + q_{\Omega_1}\bar{\delta}_{\Omega_1}^i + q_{\Omega_2}^{n}\bar{\delta}_{\Omega_2}^i + q_{\Omega_3}^{n+1}\bar{\delta}_{\Omega_3}^i$$

$$\delta_0^i \geq 0$$

$$0 \leq \delta_\omega^i(t) \leq 1 \qquad \text{for all } \omega \in \Omega.$$

Note that consumer i's budget constraint in terms of the excess demand becomes

$$q^n Z^{i,n} + q^{n+1} Z^{i,n+1} \leq 0$$

where

$$Z_0^{i,n} = Z_0^{i,n+1} = \frac{1}{2}\left(\delta_0^i - \bar{\delta}_0^i\right)$$

$$Z_{\Omega_1}^{i,n} = Z_{\Omega_1}^{i,n+1} = \frac{1}{2}\left(\delta_{\Omega_1}^i - \bar{\delta}_{\Omega_1}^i\right)$$

$$Z_{\Omega_2}^{i,n} = -\bar{\delta}_{\Omega_2}^i \quad \text{and} \quad Z_{\Omega_2}^{i,n+1} = \delta_{\Omega_2}^i(t)$$

$$Z_{\Omega_3}^{i,n} = \delta_{\Omega_3}^i \quad \text{and} \quad Z_{\Omega_3}^{i,n+1} = -\bar{\delta}_{\Omega_3}^i$$

The above maximization problem yields for consumer i during each period of time two vectors of excess demand functions $Z^{i,h}(t) = Z^{i,h}(q^{n+1}(t), q^n(t), \delta^i(t))$ where $i=1,\ldots,n+1$ and $h=n, n+1$.

4.1 Trading Process

Let $Z^{(h)}(t) = \sum_{i=1}^{n-1} Z^{i,h}(t)$ and $Z^{(-h)}(t) = Z^{(h)}(t) - Z^{h,h}(t)$ for h=n, n+1. Given $Z^{(-h)}(t)$, let the set of rationing vectors $\Lambda^h(t)$ for arbitrager h, h=n, n+1, be defined similarly as $\Lambda(t)$. Denote the price vector of arbitrager h by $q^h(t)$. The dynamic system for the trading process becomes:

$$
(**)\begin{cases}
\dot{q}_\omega^h(t) = Z_\omega^{(h)}(t),\ \omega\in\Omega \ \textit{and}\ h = n, n+1 \\
\dot{q}_0(t) = 0 \\
\dot{\delta}^i(t) = \lambda_i^n(t)Z^{i,n}(t) + \lambda_i^{n+1}(t)Z^{i,n+1}(t),\ i = 1,2,\ldots,n-1 \\
\dot{\delta}^h(t) = -\sum_{i\neq h} \lambda_i^h(t)Z^{i,h}(t) + \lambda_h^{h'}(t)Z^{h,h'}(t), h\neq h' \ \textit{and}\ h,h' = n,n+1.
\end{cases}
$$

The above adjustment process can be similarly interpreted as in the single arbitrager case and it satisfies the feasibility condition:

(††) No production is involved; that is,

$$
\sum_{i=1}^{n+1} \dot{\delta}(t) = \sum_{i\neq n} \lambda_i^n(t)Z^{i,n}(t) + \sum_{i\neq n+1} \lambda_i^{n+1}(t)Z^{i,n+1}(t)
$$

$$
- \sum_{i\neq n} \lambda_i^n(t)Z^{i,n}(t) - \sum_{i\neq n+1} \lambda_i^{n+1}(t)Z^{i,n+1}(t) = 0
$$

4.2 Stability of the Trading Process

The stability of the trading process is established by the following two theorems.

THEOREM 3 *Let $(q^{n+1}(0),q^n(0),\delta(0))$ be a given initial state and let $(q^{n+1}(t),q^n(t),\delta(t))$ be the solution of the trading process (**) with the rationing vectors $(\lambda^{-h}(t))$, $h = n,n+1$. If $\lambda^{-h}(t) = 0$ and $\delta^h(0) >> 0$ for all h and all t, $\delta(0)$ is Pareto optimal.*

Proof. Let

$$V(t) = \sum_{h=n}^{n+1} \sum_{\omega\in\Omega} \left[\frac{W_\omega^h(\delta^h(0))}{W_0^h(\delta^h(0))} - q_\omega^h(t) \right]^2 ,$$

where $W_\omega^h(\delta^h(0)) = \partial W^h(\delta^h(0))/\partial\delta_\omega^h$ for $\omega\in\Omega$, $W_0^h(\delta^h(0)) = \partial W^h(\delta^h(0))/\partial\delta_0^h$ and $h=n, n+1$. Again, $V(t) = 0$ only if $q(t) = q^n(t) = q^{n+1}(t)$ is an equilibrium price vector; otherwise, $V(t) > 0$. Next, we show that V is monotonically decreasing.

Consider the following two cases:

Case 1: $q^n(t) \neq q^{n+1}(t)$.

(i) Suppose $Z^{i,h}(t) \neq 0$ for $i\neq h$. Since $\delta^h(0) >> 0$, $\delta^h(0) - \theta Z^{i,h}(t) >> 0$ for small $\theta > 0$. Because $\lambda^{-h}(0) = 0$, it must be true that $W^h(\delta^h(0) - \theta Z^{i,h}(t)) - W^h(\delta^h(0)) < 0$. This result, as shown in Theorem 1, implies that

$$Z_0^{i,h}(t) + \sum_{\omega\in\Omega} \frac{W_\omega^h(\delta^h(0))}{W_0^h(\delta^h(0))} Z_\omega^{i,h}(t) > 0 \qquad (1')$$

and

$$Z_0^{(-h)}(t) + \sum_{\omega\in\Omega} \frac{W_\omega^h(\delta^h(0))}{W_0^h(\delta^h(0))} Z_\omega^{(-h)} > 0 \qquad (2')$$

(ii) Suppose $Z^{i,h}(t) \neq 0$ for $i=h$. Then

$$W^h(\delta^h(0) + \theta Z^{i,h}(t)) - W^h(\delta^h(0)) > 0$$

or

$$Z_0^{h}(\delta^h(0)) \;+\; \sum_{\omega\in\Omega} \frac{W_\omega^{h}(\delta^h(0))}{W_0^{h}(\delta^h(0))}\, Z^{h,h} \;>\; 0 \qquad (3')$$

(Note that for those i whose $Z^{i,h}(t) = 0$, $(2')$ and $(3')$ become equalities.)

Summing $(2')$ and $(3')$ as well as over $h=n, n+1$, we obtain

$$\sum_{\omega\in\Omega} \Big[Z_0^{(n)} \;+\; Z_0^{(n+1)} \;+\; \sum_{\omega\in\Omega} \frac{W_\omega^{n}(\delta^n(0))}{W_0^{n}(\delta^n(0))}\; Z_{\omega}$$

$$+\; \sum_{\omega\in\Omega} \frac{W_\omega^{n+1}(\delta^{n+1}(0))}{W_0^{n+1}(\delta^{n+1}(0))} \qquad (4')$$

Since $q^n(t)Z^{(n)}(t) + q^{n+1}(t)Z^{(n+1)}(t) = 0$, therefore,

$$Z_0^{(n)}(t) \;+\; Z_0^{(n+1)}(t) \;=\; -\sum_{\omega\in\Omega} q_\omega^{n}(t) Z_\omega^{(n)}(t) \;-\sum_{\omega\in\Omega} q_\omega^{n+1}(t) Z_\omega^{(n+1)}(t) \qquad (5')$$

Substitute $(5')$ into $(4')$, we obtain

$$\sum_{h=n}^{n+1} \; \sum_{\omega\in\Omega} \left[\frac{W_\omega^{h}(\delta^h(0))}{W_0^{h}(\delta^h(0))} \;-\; q_\omega^{h}(t) \right] Z_\omega^{(h)} \;>\; ($$

Now, observe that the time derivative of V(t) is

$$\dot{V}(t) \;=\; -2\sum_{h=n}^{n+1} \; \sum_{\omega\in\Omega} \left[\frac{W_\omega^{h}(\delta^h(0))}{W_0^{h}(\delta^h(0))} \;-\; q_\omega^{h}(t \right. \qquad (6')$$

According to (6′), we may conclude that V decreases monotonically in t along the solution path as long as $q^h(t)$, $h=n, n+1$, are not equilibrium price vectors. Since $V(t)$ decreases monotonically and is bounded from below by 0, $\lim_{t\to\infty} V(t) = \lim_{t\to\infty} \dot{V}(t) = 0$. However, $\dot{V}(t) = 0$ if and only if $Z^{i,n}(t) = Z^{i,n+1}(t) = 0$ for all i. Since $Z^{i,n}(t) = Z^{i,n+1}(t) = 0$ for all i requires $q^n(t) = q^{n+1}(t)$, we may conclude that $(q^n(t))$ and $(q^{n+1}(t))$ converge to a common vector q^* where $Z^{i,n}(q^*,q^*,\delta^i(0)) = Z^{i,n+1}(q^*,q^*,\delta^i(0)) = 0$ for all i. Moreover, $\delta(0)$ is Pareto optimal.

Case 2: $q(t) = q^n(t) = q^{n+1}(t)$ and $Z^{i,n}(t) \neq 0$ or $Z^{i,n+1}(t) \neq 0$ for some i.

Using an argument similar to the one shown above, again we can show that $(q(t))$ converges to a price vector q^* and $\delta(0)$ is Pareto optimal. Q.E.D.

THEOREM 4 *Let $(q^{n+1}(0),q^n(0),\delta(0))$ be any given initial state such that $\delta^h(0) >> 0$ for $h=n,n+1$. Then, the equilibria of the trading process (**) are Pareto optimal.*

Proof. Similar to Theorem 2.

Thus far, we have shown that an increase in the number of arbitragers affects neither the stability of the market process nor the Pareto optimality of the resulting equilibrium. At this juncture, it is pertinent to raise the question about whether an increase in the number of arbitragers will (always) increase the consumer's relative share of the total welfare gains reaped from the creation of the derived securities. Unfortunately, the answer to this question is ambiguous. The nature of this ambiguity can best be illustrated by an example.

EXAMPLE 2: Assume the market data given in example 1 remains the same except that at time t there are now two arbitragers (agents 2 and 3).[9] Each agent possesses 50% of the existing primitive securities and hence 50% of the bundled securities D. Let the arbitragers select their prices independent of each other; say, $q^2_\Omega = (2, 4)$ and $q^3_\Omega = (3, 3)$. In addition, let $\bar{q} = (1, q^3_{\Omega_1}, q^2_{\Omega_2})$. Substituting the values of the relevant data into the following first order conditions:

$$\delta_0^{i*} = \frac{\bar{q}\delta^i}{2} + \frac{q^2_{\omega_1} e^i_{\omega_1}}{2d_{\omega_1}} + \frac{q^3_{\omega_2} e^i_{\omega_2}}{2d_{\omega_2}}$$

$$\delta_{\omega_1}^{i*} = \frac{\pi^i_{\omega_1}\bar{q}\delta^i}{2q^2_{\omega_1}} - \frac{(1+\pi^i_{\omega_2})e^i_{\omega_1}}{2d_{\omega_1}} + \frac{\pi^i_{\omega_1} e^i_{\omega_2} q^3_{\omega_2}}{2d_{\omega_2} q^2_{\omega_1}}$$

$$\delta_{\omega_2}^{i*} = \frac{\pi^i_{\omega_2}\bar{q}\delta^i}{2q^3_{\omega_2}} - \frac{(1+\pi^i_{\omega_1})e^i_{\omega_2}}{2d_{\omega_2}} + \frac{\pi^i_{\omega_2} e^i_{\omega_1} q^2_{\omega_1}}{2d_{\omega_1} q^3_{\omega_2}}$$

we obtain the agents' optimal choices:

$$\delta^{1*} = \left(6.85,\ \frac{97}{100},\ \frac{87}{100}\right)$$

$$\delta^{2*} = \left(4.025,\ \frac{20.75}{100},\ \frac{28.66}{100}\right)$$

$$\delta^{3*} = \left(4.025,\ \frac{20.75}{100},\ \frac{28.66}{100}\right)$$

Thus, $Z^1 = \left(-3.15,\ \frac{77}{100},\ \frac{67}{100}\right)$ and $Z^2 = Z^3 = \left(1.525,\ -\frac{19.25}{100},\ -\frac{11.74}{100}\right)$ while $Z^{1,2} = \left(-1.575,\ \frac{77}{100},\ 0\right)$ and $Z^{1,3} = \left(-1.575,\ 0,\ \frac{67}{100}\right)$. Note that $Z^{3,2} = Z^{2,3} = 0$. Since $Z^{1,2} + Z^{3,2} \neq -Z^2$ and $Z^{1,3} + Z^{2,3} \neq -Z^3$ rationing will take place. Arbitrager 2 will use the rationing coefficient $\lambda_1^2 = 0.54$ and apply it to $Z^{1,2}$ to arrange for himself a consumption plan of $c^{2'} = (3.35, 5, 6.5)$. Similarly, arbitrager 3 will use the rationing coefficient $\lambda_1^3 = 0.24$ and apply it to $Z^{1,3}$ to arrange for himself a consumption plan $c^{3'} = (2.88, 7, 4.9)$. After trading, the consumer's consumption plan becomes $c^{1'} = (8.81, 3, 7)$. These consumption plans yield for each agent the following utility level: $U^1(c^{1'}) = 4.1186$, $U^2(c^{2'}) = 2.9233$ and $U^3(c^{3'}) = 2.861$.

Comparing the above utility levels with those obtained in the single arbitrager case given in example 1, we see that in this round of trading the consumer is worse off while the arbitragers collectively are better off. For two reasons, this result does not necessarily imply that in equilibrium the consumer will be made worse off by the presence of the second arbitrager. First, the reduction in the consumer's utility in this round of trading is due to the fact that the presence of the second arbitrager leads to two sets of security prices. Because the agents will buy from (sell to) the low (high) priced arbitrager, each arbitrager will experience a large net excess demand for one of the securities but none for the other (as shown in $Z^{2,3}$ and $Z^{3,2}$). In addition, because each arbitrager possesses only 50% of the bundled security D, the asymmetrical distribution of net excess demand puts a severe burden on each arbitrager's capacity to meet the consumer demand as well as his own portfolio requirement. It becomes necessary for the arbitrager to ration severely. More severe rationing practices retard the speed of adjustment towards an equilibrium in the security market and reduce the utility gain in each round of trading to the consumer as well as to the arbitragers.

Thus said, it becomes puzzling as to why the example actually shows that the arbitragers are collectively better off in the duopoly case as compared to the monopoly case. This anomaly stems from a bias introduced by the drastic change in the initial conditions--the initial endowments of the bundled securities--between the single and two arbitrager cases. To see this, we first observe that the arbitrager's utility gain from trade is the product of his net trade and his marginal utility of income. Since the marginal utility of income for the arbitrager increases with a reduction in his endowment of the bundled security D, despite the fact that each arbitrager's net trade decreases under the duopoly case as compared to the monopoly case, the collective utility of the two arbitragers has increased. This increase in the utility index does not reflect that the arbitragers are actually made better off in this round of trade but rather reflects that the index itself has been artificially inflated.

In order to drive this point home, let us modify the example slightly by assuming that now the two arbitragers collude with each other and charge identical prices for their securities; say, $q^2_\Omega = q^3_\Omega = (2,4)$. The reader can easily verify that the first round of trading leads to the following outcomes:

$$\delta^{1*} = \left(7,\ 1,\ \frac{11}{20}\right), \quad \delta^{2*} = \delta^{3*} = \left(3.95,\ \frac{18.5}{100},\ \frac{14.5}{100}\right),$$

$$Z^1 = \left(-3,\ \frac{4}{5},\ \frac{7}{20}\right), \quad Z^2 = Z^3 = \left(1.45,\ -\frac{21.5}{100},\ -\frac{25.5}{100}\right), \text{ and}$$

$$Z^{1,2} = Z^{1,3} = \left(-1.5,\ \frac{2}{5},\ \frac{17.5}{100}\right).$$ Under these circumstances, both arbitragers will adopt an identical rationing coefficient $\lambda = 0.96$ and thus gives rise to the following configurations of consumptions and utility levels: $U^1(7.12, 6.89, 7.16) = 4.1348$, $U^2(3.94, 5.8, 4.82) = U^3(3.94, 5.8, 4.82) = 3.055$. Comparing these results with those obtained in the monopoly case, we see that while the consumer's utilities are the same as they should be, the utilities for the arbitragers in the duopoly case are collectively higher than those derived in the monopoly case. But this result cannot be true for it is impossible for the colluding duopolists to outperform the monopolist! The increase in the collective utility in this case can be attributed solely to the fact that distributing the bundled securities to the two arbitragers increased the marginal utility of income to the arbitragers and thus increased the utility weights to their net trades. As a consequence, the arbitragers' utility indexes are artificially inflated.

Based upon the foregoing examples it is clear that the presence of the "endowment effect" prevents us from making any meaningful assessment on whether an increase in the number of arbitragers will raise the consumer's relative share of the welfare gain obtained from the introduction of the derived securities. In order to make any valid assessment, the "endowment effect" must be eliminated. We can accomplish this feat only by extending the present model to include simultaneous trading of both the derived and the primitive securities. As we have stated earlier, this extension must be postponed until a later date. Here, we only wish to observe that the above examples also suggest that once the "endowment effect" is eliminated, the relative share of the above mentioned welfare gain depends on the security prices quoted by the arbitragers because these prices determine the adjustment path. If an increase in the number of arbitragers leads to price competition or at least to differences in the prices quoted by the arbitragers, then an increase in the number of arbitragers will indeed raise the consumer's share of the welfare gain reaped from the introduction of the derived securities.

5. Discussion and Conclusion

Firms issue primitive securities in order to finance production. Because these securities promise streams of returns in the future under various states of nature, the consumer buys them to shift income over time and across these states of nature. As a rule, the consumer's ability to transfer income among the various states is limited. This is because the number of independent primitive securities typically is less than the number of the states of nature; only incomes falling within the return subspace spanned by the primitive securities can be attained. Fortunately, this restriction can be relaxed to some extent whenever the set of primitive securities is capable of identifying (resolving) more states than the dimensionality of the return subspace that they span. In this case, an appropriate number of derived securities can be created so as to augment the consumer's income shifting capability.

There are many ways to create derived securities. This paper calls the reader's attention to the simple device of unbundling and investigates the market mechanism for determining the prices of derived securities. In this setup, the market activities center around arbitragers who purchase primitive securities, unbundle them and then sell the derived securities to the consumers. The arbitrager knows that the consumers would demand derived securities because these securities enable each consumer to shift his income to where the primitive securities alone have failed to reach. The issue is in what manner will the securities be traded and at what prices.

Due to the fact that the market is intrinsically uncertain and a lack of information exists about the market demand for derived securities, the arbitrager is unable to choose once and for all the equilibrium prices for derived securities that will clear the market. He is, therefore, prevented from choosing prices by following a maximizing strategy. Instead, he will act as a satisficer and be contented with the prospect that trading will take place under disequilibrium situations. In this market environment, we have no choice but to examine the market adjustment process. The examination of the market process is important not only because it helps us to understand how the market attains an equilibrium but also because it enables us to gain insight on how a change in the adjustment rules affects the character of the market equilibrium.

In order to describe the determination of derived security prices, this paper uses a modified Edgeworth process, which consists of successive exchanges between the arbitrager and his customers guided by each trader's preferences and budget constraint. At each stage of this process, exchange takes place whenever both the buyer and the seller become better off by trading. The equilibrium market prices for the derived securities emerge at the end of the process where mutually beneficial trades are exhausted. This proposed process shares all common characteristics of the Edgeworth process described in the literature (see, for example, Uzawa

1962). Namely, it is stable and the resulting equilibrium is dependent on the traders' initial endowments and the path of adjustments. Hence, there exists multiplicity of equilibria. However, the proposed process does differ in a fundamental way from the Edgeworth process found in the literature. The traditional Edgeworth process assumes that the market functions autonomously and adjusts according to predetermined natural law. Specifically, the traditional Edgeworth process centers around a single neutral auctioneer who performs two functions: First, at any moment he selects prices at which commodities are traded. The auctioneer does not participate in any trading activity; the consumers trade among themselves. Second, the auctioneer only adjusts the market prices based upon the observed market excess demands according to a predetermined adjustment rule. In contrast, the Edgeworthian process that we advocate centers around arbitragers who use the market to pursue their own self-interests. Although each arbitrager performs the same functions as the auctioneer, the motive behind the arbitrager's actions is different. First, he selects prices which are deemed to be favorable to himself. Each arbitrager alone trades with the consumers; the consumers do not trade with each other. Second, each arbitrager also protects his interest by rationing the consumers whenever he finds that the prices set by him *ex ante* to be detrimental to himself *ex post*. In short, our departure from the tradition stems from the belief that the market activities center around the arbitragers and that the adjustment rules are determined consciously by them.

Since the arbitrager's interest is best served by setting prices (terms of trade) and by selecting rationing vectors in his own favor, the equilibrium outcome under the proposed process is, therefore, expected to be distributively more favorable to the arbitrager. It is important to observe at this juncture that, despite the monopoly position enjoyed by the arbitrager in the security market, the proposed Edgeworth process nonetheless leads the market to a Pareto efficient outcome. This is because, under the proposed Edgeworth process, the arbitrager does not exploit the consumer by setting monopoly prices;[10] instead, he extracts consumer surpluses through discriminatory quantity rationing. Thus, in the limit, even though the resources are allocated efficiently, the consumer is left in a inferior position from the distribution point of view. This observation should also imply that an increase in the number of arbitragers would lessen the distributive advantage accrued to the arbitrager. A formal proof for this proposition must be postponed until the present model is expanded to include tradings in both the derived and the primitive securities.

Finally, because transactions in the security market take place in disequilibrium situations and the market adjusts incessantly, various derived securities are created to meet the consumers' specific demands at each time and then retired soon after. This constant creation and retirement of derived securities leaves the impression that an excessive number of these securities exists in the market place. This impression notwithstanding, when the market reaches an equilibrium, at most m derived securities remain. In our case there are $J + m$ securities altogether at equilibrium; namely, J primitive securities which are certificates of claim for dividends paid by the firms, and m derived securities which are instruments used to transport dividend incomes to the various states of nature.[11] In this situation, the payoff capability of the derived securities are fully supported by the dividends derived from the primitive securities. For example, in the single arbitrager case, should the state ω obtain, the arbitrager receives d_ω from D as dividends. He, in turn, pays out $\delta^i d_\omega$ to consumer i, i=1,2,...,n. Because $\Sigma^n_{i=1} \delta^i_\omega = 1$, all dividends are paid out. This example demonstrates that the derived securities generate no wealth of their own; they merely help to deliver the wealth generated by the primitive assets to the various states of nature as desired by the consumers.

The above remarks naturally lead to a future research project which would involve the expansion of the model to include trading of both primitive and derived securities. This extension requires the relaxation of the assumption that all primitive securities are held by the arbitragers for the purpose of supporting the creation of derived securities. This expanded model would serve two purposes: (1) as observed earlier, it would enable us to establish the proposition that an increase in the number of arbitragers increases the consumers' relative share of the benefits obtained from the creation of derived securities, and (2) it would force us to examine explicitly the relationship between the market prices of the primitive securities and those of the derived securities. Regarding the second issue, since derived securities are created by taking the primitive securities out of circulation, the price relationship between the primitive and derived securities must, therefore, satisfy a "no arbitraging profit" condition. The establishment of the "no arbitraging profit" condition not only would enable us to identify the above-mentioned price relationship but also would allow us to ascertain just what consitutes an optimal amount of the derived securities.

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Endnotes

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2. Let $\Omega = \{1,2,...,\omega,...,m\}$ be a finite set of mutually exclusive and exhaustive states of nature. There are m elementary securities. The ωth elementary security, $\omega \in \Omega$, is one which pays $1 if the state ω occurs and pays $0 otherwise.

3. Mutual funds and collateralized mortgage obligations (CMO's) are examples of bundled and unbundled securities.

4. Note that a basic security is a scalar multiple of the corresponding elementary security. In this paper, we choose the basic securities as the units of analysis.

5. Note that since unbundling is not unique, the derived securities $D'_1 = (1, 5, 0)$ and $D'_2 = (1, 2, 0)$ can serve the purpose at hand just as well.

6. For simplicity, discrete approximations of the solutions for the continuous processes are used in this and the next examples.

7. This model can be extended readily to include many arbitragers.

8. Since the existing arbitrager holds in his hands all primitive securities, it is fair to raise the question just how another arbitrager can come into existence. There are a number of ways that a second arbitrager can emerge under this circumstance. For example, the courts may enforce the anti-trust law and order the firm to divest, or the existing partnership may voluntarily split up its assets and form two independent entities. Thus, the security market gains another independent arbitrager.

9. In order to make meaningful comparison of the market outcomes between examples 1 and 2, we assume that the single arbitrager, a partnership, is ordered by the court to divest at the beginning of time t.

10. The arbitrager sets prices by adjusting them according to the market excess demand; this process is identical to the one postulated for the competitive market in the literature.

11. In the case where both primitive and derived securities are traded in the security market, only (m-J) derived securities and the J primitive securities will be needed to span the desired m-dimensional return space. However, in our case, the J primitive securities are assumed to be held by the arbitrager for unbundling purposes and, therefore, are out of circulation. Hence, m derived securities are needed to span the needed return space.