Indeterminate Output Allocations in Collusive Equilibria and Multi-plant Firms

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Abstract

If n colluding oligopolists all have the cost function $C(q_i) = c\, q_i$, then it will not be possible to uniquely allocate among the firms the monopoly output that maximizes their *joint* profit. Similarly, if all plants of an n-plant firm have the cost function $C(q_i) = c\, q_i$, then it will not be possible to uniquely allocate the firm's optimal output among the n plants. This paper identifies the necessary and sufficient condition for such "allocative indeterminacies" to occur.

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1. Introduction

Introductory analysis of oligopoly usually involves a simple example of Cournot duopoly. The simple example frequently utilizes a linear market demand and cost functions that are characterized by identical and constant average and marginal costs. These assumptions simplify the algebra of solving for the Cournot equilibrium and demonstrating that by colluding—acting as a single production entity so as to maximize their joint profit—each of the two firms could increase its profit relative to the Cournot equilibrium. However, the nature of the equilibrium under collusion is such that it is not possible to uniquely allocate outputs (and, therefore, profits) to each of the two firms. In other words, as long as the *total* output is equal to the unique profit-maximizing monopoly output, *any* combination of outputs for the two firms summing to the monopoly output would be a feasible equilibrium when the firms collude. Obviously, this same "allocative indeterminacy" will also be faced by any single firm possessing two or more plants with identical and constant average and marginal costs for each plant, regardless of the structure of the market in which the multi-plant firm operates (competitive, monopolistically competitive, oligopolistic, or monopolistic). Recognizing that the linear market demand plays no role in the occurrence of the allocative indeterminacies, the purpose of this paper is to identify all functional forms of the cost functions that lead to this allocative indeterminacy for colluding firms in an oligopoly or the individual plants in a multi-plant firm. We will see that an allocative indeterminacy occurs *if and only if* each production unit—a firm that is part of a collusive oligopoly, or a plant within a multi-plant firm—has the same constant value for marginal cost.

2. The Cournot Model of Duopoly

The standard Cournot model of duopoly has two firms—call them "Firm 1" and "Firm 2"—producing a homogeneous product. There is neither entry of new firms, nor exit of existing firms (although an existing firm could choose to produce nothing). The firms are assumed to choose their output levels (rather than prices as in a differentiated goods model), and they choose their outputs simultaneously, so there does not exist the possibility of one firm gaining an advantage by delaying its output choice until its competitor has committed to a particular output. (For a discussion of the dynamics of a Cournot model, see Friedman, 1983.) The buyers of the homogeneous product have no transactions costs, and there are no differentials in transportation costs based on the locations of the sellers or the buyers. The single equilibrium price that must prevail in such a market is the price that equates the market quantity demanded and the [total] quantity supplied by the two firms. Further, although it does not affect the results of this paper in any way, the usual Cournot model of duopoly supposes that the market demand curve is linear. Let p denote the single price prevailing in the market at a particular time; let q_1 and q_2 denote the quantities produced by Firm 1 and Firm 2, respectively, and let Q denote the industry output produced, $Q \equiv q_1 + q_2$. Then the usual market demand can be described by the following inverse demand function,

$$p = p(Q) = a - bQ, \quad a, b > 0. \tag{1}$$

Finally, it is often assumed that each firm has the cost function $C(q) = cq$, so that we can define the cost function for Firm 1, C_1, and the cost function for Firm 2, C_2, with

$$C_i(q_i) = c\, q_i, \quad i = 1, 2. \tag{2}$$

To assure ourselves that a market will exist for the given product, the parameter c in (2) must be less than the parameter a in (1): $c < a$.

Before reviewing the Cournot equilibrium for this duopoly, recall the monopoly outcome. If there were a single firm with constant average and marginal cost equal to c, we can identify the market equilibrium quantity and price by equating marginal revenue and marginal cost. Letting Q_m denote the equilibrium market quantity under monopoly, it is a simple matter to show that $Q_m = (a - c)/2b$. Letting p_m denote the equilibrium price under monopoly, from the inverse demand function we find that $p_m = (a + c)/2$. Then the monopoly case can be summarized by

$$p_m = \frac{a + c}{2}, \quad Q_m = \frac{a - c}{2b}.$$

We now consider the Cournot equilibrium for the market described by eqs. (1) and (2). Each firm is assumed to pursue maximum profit, and the profit of Firm i, denoted by π_i $(i = 1, 2)$, is given by

$$\begin{aligned} \pi_i &= p(Q) \cdot q_i - C_i(q_i), \quad i = 1, 2 \\ &= [a - b(q_1 + q_2)]q_i - cq_i, \quad i = 1, 2. \end{aligned}$$

Note that Firm 1's profit depends on Firm 2's output and Firm 2's profit depends on Firm 1's output, since the outputs of both firms contribute to the determination of the market price through the market quantity Q in the inverse demand function $p(Q)$. Each firm attempts to maximize its own profit through the choice of its own output. To identify each firm's optimal output, we differentiate π_i with respect to q_i $(i = 1, 2)$:

$$\begin{aligned} \frac{\partial \pi_i}{\partial q_i} &= p(Q) + q_i \frac{\partial p}{\partial q_i} - \frac{dC_i}{dq_i}, \quad i = 1, 2 \\ &= a - b(q_1 + q_2) + q_i(-b) - c, \quad i = 1, 2 \\ &= a - 2bq_i - bq_j - c, \quad j \neq i,\ i = 1, 2. \end{aligned} \tag{3}$$

Embodied in (3) is the "Cournot assumption": each firm naively assumes that if it changes its own output, the firm's competitor will not change the output it is producing; i.e. for Firm i, $\partial q_j / \partial q_i = 0$ $(j \neq i, i = 1, 2)$. In other words, the "conjectural variation" for each firm is zero. Solving $\partial \pi_i / \partial q_i = 0$ for q_i in terms of q_j $(j \neq i)$, the output of Firm i's competitor, we find

$$q_i^* = r_i(q_j) = \frac{a - c}{2b} - \frac{1}{2} q_j, \quad j \neq i,\ i = 1, 2. \tag{4}$$

The function $r_i(q_j)$, which expresses Firm i's profit maximizing output in terms of its competitor's output q_j, is Firm i's "reaction function," and each firm has a reaction function giving its optimal output in terms of its competitor's output.

A "Cournot equilibrium" occurs when neither firm has an incentive to change its output. Suppose that Firm 1 is producing ${q_1}^C$; then Firm 2 will maximize its profit by producing $r_2({q_1}^C)$. If Firm 2 is producing $r_2({q_1}^C)$, then Firm 1 will maximize its profit by producing $r_1[r_2({q_1}^C)]$. Firm 1 will have no incentive to change its output if $r_1[r_2({q_1}^C)]$ is equal to what Firm 1 is already producing, namely ${q_1}^C$. Then Firm 1's output in a Cournot equilibrium is found by solving the following equation for ${q_1}^C$:

$$r_1[r_2({q_1}^C)] = {q_1}^C. \tag{5}$$

Solving (5) using (4) for the forms of the reaction functions for both Firm 1 and Firm 2 yields Firm 1's output in a Cournot equilibrium: ${q_1}^C = (a - c)/3b$. Substituting Firm 1's Cournot equilibrium output into Firm 2's reaction function reveals that Firm 2 produces the same quantity; i.e. the Cournot equilibrium is symmetric with respect to the two firms' outputs in this model:

${q_2}^C = (a-c)/3b$. If we let Q_C denote the market output in a Cournot equilibrium, it is obvious that $Q_C = 2(a-c)/3b$. Substituting this into (1) we find that the Cournot equilibrium price in this model, denoted by p_C, is $p_C = (a+2c)/3$. Summarizing the Cournot equilibrium,

$$p_C = \frac{a+2c}{3}, \quad Q_C = \frac{2(a-c)}{3b}.$$

3. Collusion and the Indeterminacy of Individual Outputs

Once the Cournot equilibrium has been described and found, it is inevitably pointed out that each of the two firms could increase its profit by colluding. That is to say that if the firms maximized their *joint* profit, the outcome could mean a higher profit for *each* firm. To demonstrate this, consider the profits earned by the monopolist and the Cournot duopolists in the previous section. In the monopoly case, the single firm earns a profit of

$$\pi_m = (p_m - c)Q_m = \left(\frac{a+c}{2} - c\right)\left(\frac{a-c}{2b}\right) = \frac{(a-c)^2}{4b}, \tag{6}$$

while in the Cournot equilibrium each firm earns a profit of

$${\pi_i}^C = (p_C - c){q_i}^C = \left(\frac{a+2c}{3} - c\right)\left(\frac{a-c}{3b}\right) = \frac{(a-c)^2}{9b}, \quad i = 1, 2,$$

so that industry profits, Π^C, are twice that, but still less than the monopoly profit:

$$\Pi^C = \frac{2(a-c)^2}{9b} < \frac{(a-c)^2}{4b} = \pi_m. \tag{7}$$

Since each duopolist in the Cournot equilibrium has the same cost function, and it is the same as that of the monopolist whose profits are given in eq. (6), the duopolists *could* earn the monopoly profit by jointly producing the monopoly output of $Q_m = (a-c)/2b$. Therefore, a feasible collusive equilibrium is one in which each firm produces $q_i = (a-c)/4b$ $(i = 1, 2)$, which is half the monopoly output, and each firm earns a profit of $\pi_i = (a-c)^2/8b$ $(i = 1, 2)$, which is half the monopoly profit. Aside from the fact that each firm will be tempted to ignore the agreement and increase its profit by moving to its reaction curve, a more fundamental problem lies in initially allocating the monopoly output between the two duopolists; since the firms are technologically identical in every way and possess identical and constant average and marginal costs, there does not exist a unique way to allocate the monopoly output between them. In maximizing their joint profit, the firms want to solve the following problem,

$$\max_{q_1,q_2} (\pi_1 + \pi_2) = p(Q) \cdot q_1 - c \cdot q_1 + p(Q) \cdot q_2 - c \cdot q_2.$$

Unfortunately, however, the above problem can be written in such a way as to make the individual q_i's indistinguishable:

$$\begin{aligned} \max_{q_1,q_2} (\pi_1 + \pi_2) &= p(Q) \cdot (q_1 + q_2) - c \cdot (q_1 + q_2) \\ &= p(Q) \cdot Q - c \cdot Q. \end{aligned} \tag{8}$$

Clearly, while the above problem may have a unique solution for a single choice variable Q, and we would call that solution value the monopoly output Q_m, the problem will not have a unique solution for the two choice variables q_1 and q_2; there would be an infinite number of solutions, all of which can be described by $q_1 + q_2 = Q_m$. This indeterminacy raises an obvious question: since the form of $p(Q)$ obviously plays no significant role here, for what *cost functions* will this indeterminacy occur? Is it *only* when the two firms have identical cost functions of the form given in (2), or are there other instances as well?

4. The Indeterminate Cases

The Cournot model of eqs. (1) and (2) does not admit a unique allocation of outputs between the two firms when they collude because of the way in which their cost functions combine in the expression of their joint profit. Consider the joint profit of two duopolists when they have the same cost function, $C(q)$, but it is not necessarily given by the linear form in eq. (2),

$$\Pi^C = [p(Q) \cdot q_1 - C(q_1)] + [p(Q) \cdot q_2 - C(q_2)] \tag{9}$$

$$= p(Q) \cdot Q - [C(q_1) + C(q_2)]. \tag{10}$$

Note that eqs. (9) and (10) are perfectly general expressions of profit for two different situations: colluding duopolists producing a homogeneous product, and a two-plant firm producing one good for sale in one market. In each situation the firms, or firm, face demand represented by the inverse demand function $p(Q)$. It should be pointed out that we longer assume the inverse demand function is linear; the results of this paper in no way depend on linear demand. Two duopolists with identical cost functions as in (2) will be unable to uniquely allocate the monopoly output between them, and one firm with two plants possessing identical cost functions as in (2) will be unable to uniquely allocate the firm's optimal output. These allocative indeterminacies arise because the sum of the two relevant cost functions, $C(q_1) + C(q_2)$, can also be written in the form $C(q_1 + q_2) = C(Q)$, which utilizes the *same* cost function C but depends only on the total output of the two firms or plants:

$$C(q_1) + C(q_2) = c\,q_1 + c\,q_2 = c\,(q_1 + q_2) = C(q_1 + q_2) = C(Q).$$

This makes the variables q_1 and q_2 indistinguishable, as can be seen in eq. (8) as well. Then the condition that characterizes an allocative indeterminacy in the collusion and multi-plant problems is the following:

$$C(q_1) + C(q_2) = C(q_1 + q_2). \tag{11}$$

We would like to know what functions C satisfy eq. (11), which is known as the "Cauchy equation" in the study of functional equations. The study of functional equations attempts to solve equations for the form of unknown *functions*, much as the study of algebra attempts to solve equations for the values of unknown *variables*. The functional equation in (11) was solved in 1821 by A.L. Cauchy and the equation is known to be satisfied [for all real values of q_1 and q_2] by *only one* functional form (see Aczél, 1966, p. 34), namely

$$C(q) = c\,q,$$

which is exactly the form we considered in eq. (2). Although in the previous section we used a linear market demand, it would not be difficult to show that a cost function with the form given in eq. (2) is sufficient to lead to an allocative indeterminacy; we now see that the cost function in eq. (2) is also *necessary* to lead to an allocative indeterminacy. Furthermore, this result generalizes in a very straightforward manner to the case of n firms or plants, each possessing the same cost function. In this case, the condition that leads to an allocative indeterminacy in the collusion and multi-plant problems is just an obvious generalization of the Cauchy equation (11),

$$C(q_1) + C(q_2) + \cdots + C(q_n) = C(q_1 + q_2 + \cdots + q_n),$$

whose solution form for C is still that of eq. (2), $C(q) = c\,q$; see Aczél, (1966, p. 31). What we have seen to this point can now be formally stated.

Proposition 1 *If n colluding firms or plants of a multi-plant firm $(2 \leq n < \infty)$ have identical cost functions, then an allocative indeterminacy will arise if and only if the cost function for each firm or plant is $C(q) = c\,q$.*

Unfortunately, the result of Proposition 1 is not very interesting because the assumption of identical cost functions is potentially quite restrictive. Therefore, we now consider the problem without restricting the cost functions to be identical.

Suppose that $C_1(q_1)$ is Firm or Plant 1's cost function and $C_2(q_2)$ is Firm or Plant 2's cost function. The joint profit for the two firms, or the profit of the two-plant firm, is

$$\Pi = p(Q) \cdot Q - [C_1(q_1) + C_2(q_2)].$$

We will have an allocative indeterminacy if $C_1(q_1) + C_2(q_2)$ can be written equivalently as some third function C^* depending only on the total output $Q = q_1 + q_2$,

$$C_1(q_1) + C_2(q_2) = C^*(q_1 + q_2). \tag{12}$$

Eq. (12) is a generalization of the Cauchy equation (11) known as the "Pexider equation." From Aczél (1966, p. 142), the most general system of solutions of Pexider's equation—i.e. the solution functional forms for C_1, C_2, and C^*—with C^* continuous, is given by the following

$$C_1(q_1) = c\,q_1 + k_1, \quad C_2(q_2) = c\,q_2 + k_2, \quad C^*(Q) = c\,Q + k_1 + k_2,$$

where k_1 and k_2 are arbitrary constants. This result also generalizes very easily to n firms or plants, each possessing a *different* cost function. In such a case, the Pexider equation in (12) becomes,

$$C_1(q_1) + C_2(q_2) + \cdots + C_n(q_n) = C^*(q_1 + q_2 + \cdots + q_n),$$

and the solution functional forms for all of the cost functions is given by

$$\begin{aligned} C_i(q_i) &= c\,q_i + k_i, \quad i = 1, 2, \ldots, n, \\ C^*(Q) &= c\,Q + \sum_{i=1}^{n} k_i. \end{aligned} \tag{13}$$

This result can be formally stated as

Proposition 2 *If n colluding firms or plants of a multi-plant firm $(2 \leq n < \infty)$ all have different cost functions, then an allocative indeterminacy will arise if and only if the cost function for Firm or Plant i is $C_i(q_i) = c\,q_i + k_i$, for all $i = 1, 2, \ldots, n$.*

The interpretation of Proposition 2 in terms of economics is obvious: The n different cost functions all possess the same, constant marginal cost, given by the parameter c, and the only way in which these "different" cost functions differ is in the value of the parameter k_i, which has the obvious interpretation of Firm or Plant i's fixed cost $(i = 1, 2, \ldots, n)$.

Conclusion

What is plain to see from Propositions 1 and 2 is a very strong result: an allocative indeterminacy can occur among n colluding oligopolists or in an n-plant firm ($2 \leq n < \infty$) *if and only if* all n production units have the same constant value for marginal cost. There may be fixed costs present if the cost functions may differ, but *only* with respect to the units' fixed costs may their cost functions differ in order for an allocative indeterminacy to occur. This result suggests that we look separately at the short- and long-run cases. If we do that, then Proposition 2 has a very strong result in each case.

In short run analysis, we will find an allocative indeterminacy *if and only if* each of the n production units $(2 \leq n < \infty)$ has a cost function of the form given in eq. (13): $C_i(q_i) = c\,q_i + k_i$, where k_i is unit i's fixed cost $(i = 1, 2, \ldots, n)$.

In long run analysis, we will face an allocative indeterminacy *if and only if* each of the n production units $(2 \leq n < \infty)$ has a cost function of the form given in eq. (13). But long run cost functions generally do not include fixed costs, so we are forced to set $k_i = 0$ for all $i = 1, 2, \ldots, n$. This leaves us with identical cost functions for each firm or plant, and these identical cost functions are characterized by identical and constant average and marginal costs, as given in eq. (2): $C_i(q_i) = c\,q_i$ for all $i = 1, 2, \ldots, n$. Therefore, in long run analysis there is no case in which the firms or plants have different cost functions and an allocative indeterminacy occurs.

References

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York, 1966.

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