The Opportunity Cost of Duality

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Abstract

A dual representation of a technology, e.g. a cost function, may not contain all of the technological information, but it will contain all of the information about input vectors that would be chosen by a cost-minimizing firm. At least this much is clear for deterministic technologies. The main question addressed in this paper is whether the same can be said about stochastic technologies and their dual representations. Despite some pessimism expressed in the stochastic frontier literature on this question, we argue that there is no extra cost imposed in the stochastic case. Thus, the conclusion of this paper is: JUST DUAL IT!

The benefit or gain from duality theory is well known. It was expressed by Shephard ([16], page 28) as: "Statistical studies of cost functions are generally more accessible than corresponding empirical investigations of production functions, because economic data are most frequently in price and monetary terms." In this paper, we focus on the cost of duality theory.

The cost of using duality theory is the value of the information lost when passing from the primal to the dual. In the context of deterministic production and cost theory, this information loss is well known. Information about those regions of the production technology that are either nonmonotonic or nonconvex will be lost when we characterize technology via the cost function. On the other hand, the cost function *will* contain all of the information about input vectors

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that would be chosen by a cost-minimizing firm. Thus there are many who view the information loss as inconsequential.

The main question addressed in this paper is whether the information loss is any greater or more serious when we consider a stochastic model of production and cost. Our general conclusion is that the information loss is no different in the deterministic versus the stochastic model. This conclusion is somewhat more optimistic than those found in a 1982 paper by Roulon Pope, "To Dual or Not to Dual?" [15]

1. A Right Duality

In the literature on stochastic production frontiers, the stochastic model is specified either for the primal goods space or for the dual price space. One question that has engaged some researchers, see, e.g., Cornwell and Schmidt [2] and Greene [8], asks for the conditions under which the stochastic structure is preserved when going from the primal to the dual space. Cornwell and Schmidt [2] ask for a condition under which technical inefficiency defined with a production function is the same as technical inefficiency defined with a cost function. They observed that constant returns to scale is a sufficient condition.

To formalize this problem, let $x \in \mathsf{R}_+^N$ denote an N-dimensional vector of nonnegative inputs and let $w \in \mathsf{R}_+^N$ denote an N-dimensional vector of nonnegative input prices. The technology is given by $S = \{(x, y) : x \in \mathsf{R}_+^N, y \in \mathsf{R}_+^M, x$ can produce $y\}$, where y is the output vector. The cost function is defined by

$$C(y, w) = \inf_x \{wx : (x, y) \in S\}. \tag{1.1}$$

To render the above model stochastic a one-sided stochastic variable is added or rather a function of a stochastic variable, $g(v)$, is added. Thus we modify (1.1) to read as

$$\mathcal{C}(y, w, g(v)) = \inf_x \{wx : (x, y) \in S_{g(v)}\}. \tag{1.2}$$

Actually, $g(v)$ may be a composite of error terms. However, these details need not detain us. It is commonly assumed, and we shall do so here, that the cost function in (1.2) has a multiplicative form, i.e.

$$\mathcal{C}(y, w, g(v)) = C(y, w)g(v). \tag{1.3}$$

We will now show that the multiplicative stochastic structure in (1.3) will be preserved when going back to primal space if we choose the input distance function as the functional representation of primal space. To see this we refer to a result in Färe and Primont ([6], page 48, also, see Section 3) which states that the input distance function, $D_i(y,x)$, may be computed from the cost function by

$$D_i(y,x) = \inf_w \left\{ \frac{wx}{C(y,w)} \right\}. \tag{1.4}$$

If we apply this result to the cost function in (1.3) we find that

$$\begin{aligned} \mathcal{D}_i(y,x,g(v)) &= \inf_w \left\{ \frac{wx}{C(y,w)g(v)} \right\} \\ &= D_i(y,x)/g(v), \end{aligned} \tag{1.5}$$

i.e. $\mathcal{D}_i(y,x,g(v)) = D_i(y,x)/g(v) = D_i(y,x)h(v)$ where $h(v) = 1/g(v)$.

On the other hand, suppose we first specify the multiplicative stochastic structure in the primal space, i.e. we assume that $D_i(y,x,g(v)) = D_i(y,x)h(v)$. Then, when the cost function is computed, we get (again using Färe and Primont ([6], page 48))

$$\begin{aligned} \mathcal{C}(y,w,h(v)) &= \inf_x \left\{ \frac{wx}{D_i(y,x)h(v)} \right\} \\ &= C(y,w)/h(v) \\ &= C(y,w)g(v). \end{aligned} \tag{1.6}$$

We have shown that when the primal space is represented by the input distance function and the price space is represented by the cost function then a stochastic specification with multiplicative structure in either space implies the same structure in the other space. As long as the technology satisfies the conditions required for duality (monotonicity and convexity) then no special assumptions are needed for this result.

Unfortunately, many researchers have, instead, represented the primal space by the production function and the price space by the cost function. A multiplicative stochastic structure for the production function requires that $\mathcal{F}(x,g(v)) = F(x)g(v)$. This corresponds to an *output-oriented* measure of technical efficiency. However, a multiplicative stochastic structure for the cost function corresponds to an *input-oriented* measure of technical efficiency. For these two stochastic structures to be equivalent requires that the *output-oriented* measure equal the reciprocal of the *input-oriented* measure. As shown by Färe and Lovell [5], Färe and Grosskopf [3], and Färe and Primont [6], this is true if and only if the technology exhibits constant returns to scale. In this case we have

$$D_o(x,y)g(v) = \inf_{\theta}\left\{\theta : \frac{D_i(y/\theta,x)}{g(v)} \geqq 1\right\} \tag{1.7}$$

and

$$\frac{D_i(y,x)}{g(v)} = \sup_{\lambda}\left\{\lambda : D_o(x/\lambda,y)g(v) \leqq 1\right\}. \tag{1.8}$$

We conclude this section as follows. If the right duality relation is employed then there is no cost in passing from the primal to the dual space. However, if the wrong duality relation is employed then one must pay the cost associated with assuming constant returns to scale.

2. Allocative Efficiency

In his seminal paper, Farrell [7] distinguishes between technical and allocative efficiency. Technical input efficiency is equal to

$$\begin{aligned} T_i(y,x) &= \frac{wx/D_i(y,x)}{wx} \\ &= \frac{1}{D_i(y,x)}, \end{aligned} \tag{2.1}$$

and overall efficiency is given by

$$O_i(y,x,w) = \frac{C(y,w)}{wx}. \tag{2.2}$$

Recall Mahler's Inequality given by

$$wx \geqq D_i(y,x)C(y,w), \forall w \in \mathsf{R}^N_+, x \in \mathsf{R}^N_+. \tag{2.3}$$

Using (2.1) and (2.2), (2.3) implies that

$$O_i(y,x,w) \leq T_i(y,x) \leq 1$$

for all feasible (x,y). When $O_i(y,x,w) < T_i(y,x)$ this efficiency gap is accounted for by the allocative input efficiency measure given by

$$\begin{aligned} A_i(y,x,w) &= \frac{O_i(y,x,w)}{T_i(y,x)} \qquad (2.4) \\ &= \frac{C(y,w)}{wx/D_i(y,x)}. \end{aligned}$$

In the stochastic frontier literature there has been some concern about how to estimate allocative efficiency, e.g. Cornwell and Schmidt [2] and Greene [9]. The latter states that "The truly difficult issue in this context remains how to estimate allocative inefficiency in the context of a properly specified, internally consistent model." To address this issue recall Mahler's Inequality given by (2.3). Following Färe and Grosskopf [4], allocative input efficiency can be computed as

$$A_i(y, x, w) = \min_{\lambda}\{\lambda : \lambda(wx) \geqq D_i(y, x)C(y, w)\}. \tag{2.5}$$

Thus, it is clear from both (2.4) and (2.5) that

$$A_i(y, x, w) = \frac{D_i(y, x)C(y, w)}{wx}. \tag{2.6}$$

Note that the allocative input efficiency measure in (2.6) depends on both primal and dual variables, i.e., x, and w. Of course, both primal and dual information are contained in both the cost function and the distance function. By Shephard's primal and dual lemmata

$$\frac{\partial C(y, w)}{\partial w_n} = x_n(y, w) \qquad \text{(primal information)} \tag{2.7}$$

$$\frac{\partial D_i(y, x)}{\partial x_n} = w_n(y, x) \qquad \text{(dual information).} \tag{2.8}$$

If we introduce the error component, $g(v)$, from the previous section we see that allocative efficiency is independent of it since

$$\begin{aligned} A_i(y, x, w) &= \frac{D_i(y, x)}{g(v)} \frac{C(y, w)g(v)}{wx} \\ &= \frac{D_i(y, x)C(y, w)}{wx}. \end{aligned} \tag{2.9}$$

Equation (2.6) can be applied to generalize some results in a recent paper by Kumbhakar [12]. Specifically, we show that Kumbhakar's equation (17) that he derived for a translog cost function holds for any cost function. Rearrange (2.6) to get

$$wx = \frac{D_i(y, x)C(y, w)}{A_i(y, x, w)} \tag{2.10}$$

Take the natural log of both sides of (2.10) to get

$$\ln wx = \ln D_i(y, x) + \ln C(y, w) - \ln A_i(y, x, w) \tag{2.11}$$

If one sets $D_i(y,x)$ equal to one and adds a disturbance term, v, then the first half of Kumbhakar's equation (17) results. Next, differentiate both sides of (2.10) with respect to w_n to get

$$x_n = \frac{\partial C(y,w)}{\partial w_n}\frac{1}{A_i(y,x,w)} - \frac{C(y,w)}{(A_i(y,x,w))^2}\frac{\partial A_i(y,x,w)}{\partial w_n} \tag{2.12}$$

Multiply both sides of (2.12) by $w_n A_i(y,x,w)/C(y,w)$ to get

$$\frac{w_n x_n A_i(y,x,w)}{C(y,w)} = \frac{\partial C(y,w)}{\partial w_n}\frac{w_n}{C(y,w)} - \frac{\partial A_i(y,x,w)}{\partial w_n}\frac{w_n}{A_i(y,x,w)} \tag{2.13}$$

Finally, substitute (2.6) into the lefthand side of (2.13) to get

$$\frac{w_n x_n}{wx} = \frac{\partial C(y,w)}{\partial w_n}\frac{w_n}{C(y,w)} - \frac{\partial A_i(y,x,w)}{\partial w_n}\frac{w_n}{A_i(y,x,w)} \tag{2.14}$$

Equation (2.14) is the second half of equation (17) in Kumbhakar's paper.

3. Input Duality

In his 1953 book, *Cost and Production Functions* [16], R.W. Shephard established the duality between the cost structure and the production structure. This duality is summarized by a pair of optimization problems given by

$$\begin{aligned} C(y,w) &= \inf_x\{wx : D_i(y,x) \geqq 1\} \\ &\Leftrightarrow \\ D_i(y,x) &= \inf_w\{wx : C(y,w) \geqq 1\}, \end{aligned} \tag{3.1}$$

where the input distance function, $D_i(y,x)$, was originally defined (in 1953) from the production function by

$$D_i(y,x) = \sup_\lambda\{\lambda : F(x/\lambda) \geqq y\}.$$

However, a multi-output production technology can be defined by the input requirement sets

$$L(y) = \{x \in \mathsf{R}^N_+ : x \text{ can produce } y\}, y \in \mathsf{R}^M_+ \tag{3.2}$$

and the distance function may be defined from the input requirement sets by

$$D_i(y,x) = \sup_{\lambda}\{\lambda : (x/\lambda) \in L(y)\}. \tag{3.3}$$

If the technology satisfies weak disposability of inputs, an assumption that is implied by those needed for the duality in (3.1), then $D_i(y,x) \geqq 1 \Leftrightarrow x \in L(y)$. Thus the input duality theorem in (3.1) is valid for both the single output and the multioutput case, as was made clear in Shephard [17].

The duality theorem summarized in (3.1) requires the assumption that the input requirement set, $L(y)$, is nonempty, closed, and convex for each $y \in \mathsf{R}_+^N$. Moreover, if input prices are nonnegative, i.e. $w \in \mathsf{R}_+^N$, then the duality theorem requires that inputs are strongly disposable, i.e., $x' \geqq x \in L(y) \Rightarrow x' \in L(y)$. This assumption can be relaxed to weak disposability of inputs, i.e., $x \in L(y), \lambda \geqq 1 \Rightarrow \lambda x \in L(y)$, if some of the input prices can take on nonpositive values.

Shephard's input duality theorem is expressed as a pair of constrained optimization problems in (3.1). Färe and Primont [6] have shown that these problems may also be formulated as a pair of unconstrained optimizations, i.e.,

$$\begin{aligned} C(y,w) &= \inf_x \left\{\frac{wx}{D_i(y,x)}\right\} \\ &\Leftrightarrow \\ D_i(y,x) &= \inf_w \left\{\frac{wx}{C(y,w)}\right\}. \end{aligned} \tag{3.4}$$

The primal and dual Shephard's lemmata follow directly from (3.4), namely,

$$\begin{aligned} \nabla_w C(y,w) &= x(y,w) \\ \nabla_x D_i(y,x) &= w(y,x). \end{aligned} \tag{3.5}$$

Since Mahler's Inequality (2.3) holds with equality for optimal pairs, $x(y,w)$ and $w(y,x)$, (3.5) implies that

$$C(y,w)D_i(y,x) = \nabla_w C(y,w)\nabla_x D_i(y,x). \tag{3.6}$$

If either the cost or distance function fail to be differentiable at some point (y^o, w^o) or (y^o, x^o), respectively, both functions will still be subdifferentiable since the cost and distance functions are concave in w and x respectively. In this case,

$$C(y^o,w^o)D_i(y^o,x^o) = wx \text{ for all } w \in \nabla_x D_i(y^o,x^o), x \in \nabla_w C(y,w), \tag{3.7}$$

where $\nabla_x D_i(y^o,x^o)$ and $\nabla_w C(y,w)$ are subgradient sets.

4. Stochastic Cost and Production

We have followed the tradition of the stochastic frontier analysts by simply adding error term(s) to deterministic models. Here are two examples:

1. "Let the production function with technical inefficiency be written as

   $$y = f(x)e^{\varepsilon},$$

   where y is output, x is a vector of n inputs, $f(\cdot)$ is the production technology, and ε is the error term which is composed of technical efficiency, τ, and statistical noise, υ." Kumbhakar and Hjalmarsson ([13], page 275).

2. "Suppose producers use inputs $x \in \mathsf{R}_+^n$ to produce scalar output $y \in \mathsf{R}_+$ with technology

   $$y_i = f(x_i; \beta) \exp\{v_i + u_i\},$$

   where β is a vector of technological parameters to be estimated and $i = 1, ..., I$ indexes producers. The random disturbance term v_i is intended to capture the effect of statistical noise and is assumed to be independently and identically distributed as $N(0, \sigma_y^2)$. The disturbance term u_i is assumed to be distributed independently of v_i and to satisfy $u_i \leqq 0$. The deterministic production frontier is $f(x; \beta)$ and the stochastic frontier is $[f(x_i; \beta) \exp\{v_i + u_i\}]$." Lovell ([14], page 20).

Henn and Krug [10] and Krug [11] provided a foundation for stochastic production models. In particular, let

$$(\Omega, B, \mu) \tag{4.1}$$

be a probability space where Ω is a nonempty set of the states of the world, B is a Borel field and μ is a probability measure. An output correspondence is then given by

$$\mathcal{P} : \mathsf{R}_+^N \times \Omega \to \mathcal{P}(x, \omega), \tag{4.2}$$

where $\mathcal{P}(x, \omega)$ denotes the set of outputs that can be produced by $x \in \mathsf{R}_+^N$ given the state of the world $\omega \in \Omega$.

We note that if $\mathcal{P}(x, \omega)$ is a compact, nonempty set for $(x, \omega) \in R_+^N \times \Omega$, then there is a stochastic production function defined by

$$\mathcal{F}(x, \omega) = \max\{y : y \in \mathcal{P}(x, \omega)\}. \tag{4.3}$$

Moreover, if the ouput y is freely disposable then

$$\mathcal{P}(x, \omega) = \{y : 0 \leqq y \leqq \mathcal{F}(x, w)\}. \tag{4.4}$$

Thus the usual relationship between the output set and the production function holds for the stochastic counterpart.

We also note that $\mathcal{P}(x, \omega)$ can be written as a stochastic linear model or, equivalently, as a stochastic DEA model, namely

$$\begin{aligned} \mathcal{P}(x, \omega) = \{y : \quad & \sum_{k=1}^{K} z_k y_{km}(\omega) \geqq y_m, m = 1, ..., M, \\ & \sum_{k=1}^{K} z_k x_{kn}(\omega) \leqq x_n, n = 1, ..., N, \\ & z_k \geqq 0, k = 1, ..., K\}, \end{aligned} \tag{4.5}$$

where $k = 1, ..., K$ indexes the observations or DMU's. Note that here we have only modelled the technology as stochastic. Clearly the inputs, $x_n, n = 1, ..., N$, may also be stochastic.[1]

Suppose that ω is mapped into a stochastic variable v such that

$$\mathfrak{f}(x, v) = \mathcal{F}(x, \omega). \tag{4.6}$$

The stochastic frontier analyst often imposes a multiplicative separability condition, namely

$$\mathfrak{f}(x, v) = f(x)g(v) \tag{4.7}$$

Clearly, this holds if and only if

$$\frac{\partial^2 \ln \mathfrak{f}(x, v)}{\partial x \partial v} = 0.$$

See Blackorby, Primont, and Russell [1] for the details.

To formulate a stochastic cost and production duality we first define the stochastic input distance function. Here, for convenience, we define it in terms of the stochastic output set $\mathcal{P}(x, \omega)$, i.e.,

$$\mathcal{D}_i(y, x, \omega) = \inf_{\lambda}\{\lambda : y \in \mathcal{P}(x/\lambda, \omega)\}. \tag{4.8}$$

[1]The stochastic DEA model and its solutions are addressed in a subsequent paper.

From arguments similar to those found in Färe and Primont [6], it is clear that whenever inputs are weakly disposable then

$$\mathcal{D}_i(y, x, \omega) \geq 1 \Leftrightarrow y \in \mathcal{P}(x, \omega), \tag{4.9}$$

i.e. the stochastic input distance fully describes the stochastic technology.

The stochastic cost function can be defined in terms of the above distance function as

$$\mathcal{C}(y, w, \omega) = \inf_x \{wx : \mathcal{D}_i(y, x, \omega) \geq 1\}. \tag{4.10}$$

The stochastic cost function depends on ω, the state of the world. We have not introduced the idea of expected cost, which we could have done, but chose not to.

Again, since the state of the world is treated as a parameter, the stochastic distance function can be recovered from the stochastic cost function in the same way as in the deterministic case. Thus we have the stochastic Shephard's input duality theorem

$$\begin{aligned} \mathcal{C}(y, w, \omega) &= \inf_x \{wx : \mathcal{D}_i(y, x, \omega) \geqq 1\} \\ &\Leftrightarrow \\ \mathcal{D}_i(y, x, \omega) &= \inf_w \{wx : \mathcal{C}(y, w, \omega) \geqq 1\}. \end{aligned} \tag{4.11}$$

It easy to show that

$$\begin{aligned} \mathcal{C}(y, w, \omega) &= C(y, w) g(\omega) \\ &\Leftrightarrow \\ \mathcal{D}_i(y, x, \omega) &= D_i(y, x)/g(\omega). \end{aligned}$$

Thus, if we want to impose the multiplicative form we can do so at no cost, i.e. without imposing any additional assumptions on the technology.

The above duality theorem may also be written in terms of two unconstrained optimization problems

$$\begin{aligned} \mathcal{C}(y, w, \omega) &= \inf_x \left\{ \frac{wx}{\mathcal{D}_i(y, x, \omega)} \right\} \\ &\Leftrightarrow \\ \mathcal{D}_i(y, x, \omega) &= \inf_w \left\{ \frac{wx}{\mathcal{C}(y, w, \omega)} \right\} \end{aligned} \tag{4.12}$$

It is now routine to obtain the two stochastic Shephard's lemmata given by

$$\frac{\partial \mathcal{C}(y,w,\omega)}{\partial w_n} = x_n(y,w,\omega) \tag{4.13}$$

and

$$\frac{\partial \mathcal{D}_i(y,x,\omega)}{\partial x_n} = w_n(y,x,\omega). \tag{4.14}$$

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