THE MORISHIMA ELASTICITY OF SUBSTITUTION FOR THE PROFIT FUNCTION

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November 1994

*The authors would like to thank Thomas Mitchell and Daniel Primont for their helpful comments.

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In this note we derive expressions for the Morishima elasticity of substitution for a multiple inputs and outputs profit function. These expressions may have important applications to empirical studies.

I. Introduction

Hicks (1932) was the first one to introduce the concept of the elasticity of substitution for the case of two inputs. His idea has been generalized to the case of more than two inputs by Allen and Hicks (1934), Allen (1938) and Uzawa (1962) among others. As is well-known, the Allen-Uzawa partial elasticities of factor substitution (AES) have been widely used to classify pairs of inputs as substitutes or complements. Blackorby and Russell (1989) pointed out that the AES is not informative in the sense that it does not provide information about the relative factor shares, about curvature of the isoquant and moreover can not be interpreted in the spirit of the marginal rate of substitution. On the other hand, the Morishima Elasticity of Substitution (MES) discovered independently by Morishima (1967) and Blackorby and Russell (1975) does retain the characteristics of the Hicksian concept. Blackorby and Russell (1989, p. 883) notes that, "the Morishima elasticity of substitution (MES), (i) is a measure of curvature, or ease of substitution, (ii) is a sufficient statistic for assessing -quantitatively as well as qualitatively- the effects of change in price or quantity ratios on relative factor shares, and (iii) is a logarithmic derivative of a quantity ratio with respect to a marginal rate of substitution or a price ratio."

Chambers (1988) and Blackorby and Russell (1989) derived the expressions for the MES in the context of the cost function, where input prices and output quantities are given, and quantities of inputs and prices of outputs adjust. However, firm behavior is often modelled under the assumption of profit maximization. For example, Kohli (1993) used the variable profit function and estimated the U.S. demand for imports and supply of exports. It follows from Hotelling's lemma that the profit maximizing supplies of outputs and demands for inputs can be obtained by differentiation of the profit function with respect to output and input prices. In the system of the profit maximizing derived supplies of outputs and demands for inputs, the prices of outputs are exogenous but the quantities of output are endogenous. In empirical studies it is difficult, even sometimes impossible, to explicitly express the cost function from a flexible profit functional form. A problem arises naturally as how to calculate the MES from the profit maximizing derived supply and demand system.

In this note, expressions for the Morishima elasticity of substitution (MES) are derived for a multiple outputs and multiple inputs profit function. These expressions may have important applications to empirical studies. This note is organized as follows. The MES for the cost function is discussed in section 2. In section 3, we derive the MES for the profit function.

II. The Morishima Elasticity for the Cost Function

The notations here are similar to those in Blackorby and Russell (1989). The cost function can be defined as follows:

$$C(y,w) \equiv \min_x\{\Sigma w_i x_i: (y,x)\in T\} \text{ for } y\geq\mathbf{0} \text{ and } w\geq\mathbf{0},$$

where T denotes the production possibility; (y,x) is the vector of I-outputs and J-inputs; and w is the vector of input prices. It is assumed that C(y,w) is differentiable for positive input prices and output quantities.

Using Shephard's lemma, the cost minimizing derived demand is given by

$$x^c_i = C_i(y,w), \quad i = 1,...,J. \qquad (1)$$

Assume that the percentage change in the price ratio, w_j/w_i, is induced solely by changing the j^{th} price, then the Morishima elasticity of substitution is defined as follows (Chambers, 1988, p. 96; Blackorby and Russell, 1989):

$$M^c_{ij}(y,w) \equiv \partial\ln(x^c_i/x^c_j) \,/\, \partial\ln(w_j/w_i), \qquad i, j=1,...,J, \qquad (2)$$

where x^c_i/x^c_j represents the optimal quantity ratio. Note that the MES can be equivalently written as

$$M^c_{ij}(y,w) = \varepsilon^c_{ij}(y,w) - \varepsilon^c_{jj}(y,w), \; i, j = 1,...,J \qquad (3)$$

where $\varepsilon^c_{ij}(y,w)$ is the (constant-output) cross-price elasticity of demand. The MES can be equivalently defined for the production function in the single output case (Chambers, 1988, p.35). As is pointed out by Chambers (1988) and Blackorby and Russell (1989), the MES provides complete comparative statics information about relative factor shares. For a given increase in the price of the j^{th} input the relative share of the i^{th} input decreases if the Morishima elasticity of substitution is greater than one and increases if it is less than one.

III. The Morishima Elasticity for the Profit Function

The profit function is defined as the solution to the following profit maximization problem:

$\Pi(p,w) \equiv \max_{(y,x)}\{\Sigma p_i y_i - \Sigma w_j x$

$$\begin{bmatrix} \frac{\partial(\ln y_1 \ldots \ln y_I)}{\partial(\ln p_1 \ldots \ln p_I)} & \frac{\partial(\ln y_1 \ldots \ln y_I)}{\partial(\ln w_1 \ldots \ln w_J)} \\ \frac{\partial(\ln x^p_1 \ldots \ln x^p_J)}{\partial(\ln p_1 \ldots \ln p_I)} & \frac{\partial(\ln x^p_1 \ldots \ln x^p_J)}{\partial(\ln w_1 \ldots \ln w_J)} \end{bmatrix} \qquad (7)$$

where $E^{yp}(p,w)$, $E^{yw}(p,w)$, $E^{xp}(p,w)$ and $E^{xw}(p,w)$ are IxI, IxJ, JxI and JxJ submatrices, respectively.

We further assume that the Jacobian matrix of the supplies of outputs (5) is nonsingular for all non-negative (p,w), that is, $\det(\partial(y_1.....y_I)/\partial(p_1.....p_I))\neq 0$. Then by the implicit function theorem, we can define from equation (5) the prices of outputs as differentiable functions of outputs, i.e.,

$$p_i=p_i(y,w), \qquad i=1,...,I \qquad (8)$$

and

$$\partial(p_1.....p_I)/\partial(y_1.....y_I)=[\partial(y_1.....y_I)/\partial(p_1.....p_I)]^{-1}. \qquad (9)$$

With the above notations the main results are stated in the following theorem.

Theorem: Assume that the cost function C(y,w) is twice continuously differentiable at (y,w) and $w_j>o$ for j = 1,...,J; that the profit function $\Pi(p,w)$ is well defined and twice continuously differentiable at (p,w) and p_i, $w_j>0$, i = 1,...,I and j=1,...,J and that the Jacobian matrix of the supply of outputs is nonsingular for positive (p,w), i. e. $\det(\partial(y_1.....y_I)/\partial(p_1.....p_I))\neq 0$. Then

(i) $\varepsilon^c_{jk}(y,w) = E^{xw}_{jk} - [E^{xp}_{j1} ..... E^{xp}_{jI}]\,[E^{yp}]^{-1}\,[E^{yw}_{1k} ..... E^{yw}_{Ik}]'$

$$\equiv E^{xw}_{jk}(p,w) - [\partial \ln x^{\pi}_j/\partial(\ln p_1.....\ln p_I)]\ [\partial(\ln y_1.....\ln y_I)/\partial(\ln p_1.....\ln p_I)]^{-1}$$

$$[\partial(\ln y_1.....\ln y_I)/\partial \ln w_k]',$$

for every (y,w) and j, k = 1,...,J, where p is defined by (8), and

(ii) $M^c_{jk}(y,w) = E^{xw}_{jk} - E^{xw}_{kk} - D_{jk} + D_{kk}$,

where $D_{jk} = [E^{xp}_{j1} \ldots E^{xp}_{jI}]\ [E^{yp}]^{-1}\ [E^{yw}_{1k} \ldots E^{yw}_{Ik}]'$, and j, k = 1,...,J.

Proof (i) Since the profit-maximizing derived demands provide the least cost way to produce the profit-maximizing supplies, it follows that

$$x^c_j(y,w) = x^{\pi}_j(p(y,w),w) = x^{\pi}_j(p,w), \quad j = 1,\ldots,J, \tag{10}$$

We can differentiate (10) with respect to w_k and evaluate the derivative at (p,w) to get:

$$\partial x^c_j(y,w)/\partial w_k = \partial x^{\pi}_j(p,w)/\partial w_k + \Sigma_s\ (\partial x^{\pi}_j(p,w)/\partial p_s)(\partial p_s(y,w)/\partial w_k). \tag{11}$$

Multiplying both sides by $w_k/x^c_j(y,w)$ and using (10), we have

$$\varepsilon^c_{jk}(y,w) = E^{xw}_{jk}(p,w)$$

$$+ \Sigma_s\{(\partial x^{\pi}_j(p,w)/\partial p_s)(p_s/x^{\pi}_j(p,w))\}\{(\partial p_s(y,w)/\partial w_k)(w_k/p_s)\}$$

$$= E^{xw}_{jk}(p,w) + [E^{xp}_{j1} \ldots E^{xp}_{jI}]\ [\delta_{1k} \ldots\ldots \delta_{Ik}]', \tag{12}$$

where $\delta_{sk} \equiv \partial \ln p_s(y,w)/\partial \ln w_k$, s=1,...,I.

$\mathbf{0}_I = \partial(y_1...y_I)/\partial(p_1...p_I)\ [\partial(p_1...p_I)/\partial w_k] + [\partial(y_1...y_I)/\partial w_k]$.

Since we assume that $\partial(y_1...y_I)/\partial(p_1...p_I)$ is a nonsingular matrix,

$$[\partial(p_1...p_I)/\partial w_k] = - [\partial(y_1...y_I)/\partial(p_1...p_I)]^{-1}\ [\partial(y_1...y_I)/\partial w_k].$$

Therefore,

$$[\delta_{1k}......\ \delta_{Ik}]' \equiv P^{-1}\ [\partial(p_1...p_I)/\partial w_k]\ w_k$$

$$= - P^{-1}\ [\partial(y_1...y_I)/\partial(p_1...p_I)]^{-1}\ [\partial(y_1...y_I)/\partial w_k]\ w_k$$

$$= - [\partial(\ln y_1...\ln y_I)/\partial(\ln p_1...\ln p_I)]^{-1}\ [\partial(\ln y_1...\ln y_I)/\partial \ln w_k]$$

$$= - [E^{yp}]^{-1}\ [E^{yw}{}_{1k}\ ...\ E^{yw}{}_{Ik}]', \qquad (13)$$

where $P \equiv \text{diag}(p)$ is a diagonal matrix.

Finally (i) is obtained by substituting (13) into (12).

The $\varepsilon^c_{jk}(y,w)$ in (i) expresses equation (11) in terms of elasticities. The right-hand side of (i) can be obtained directly from (7). Note that the left-hand side of (11) is the compensated change in demand (holding y constant) in response to a change in w_k. Moreover, the right-hand side of (11) shows that the compensated change is equal to the change in demand holding the prices of outputs fixed plus the change in demand when the prices of outputs change times how much prices have to change to keep outputs constant. Equation (11) is analogous (though not exactly) to the well-known Slutsky decomposition in consumer theory.

(ii) The $M^c_{jk}(y,w)$ for the profit function can be obtained by the expressions in (i) and definition of the MES in (3). **Q. E. D.**

Corollary 1 In the single-output case, the MES is given by[1]

$$M^c_{jk}(y,w) = E^{xw}_{jk} - E^{xw}_{kk} - D_{jk} + D_{kk},$$

where

$$D_{jk} = E^{xp}_{j1}\, E^{yw}_{1k} \,/\, E^{yp}_{11} \,.$$

Corollary 2 In the two-output case, the MES is given by

$$M^c_{jk}(y,w) = E^{xw}_{jk} - E^{xw}_{kk} - D_{jk} + D_{kk},$$

where

$$D_{jk} = \frac{E^{xp}_{j1}E^{yp}_{22}E^{yw}_{1k} - E^{xp}_{j2}E^{yp}_{12}E^{yw}_{1k} - E^{xp}_{j1}E^{yp}_{21}E^{yw}_{2k} + E^{xp}_{j2}E^{yp}_{11}E^{yw}_{2k}}{E^{yp}_{11}\, E^{yp}_{22} - E^{yp}_{21}\, E^{yp}_{12}}.$$

[1] This result is a solution to an exercise in Chambers (1988, p.135).

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