Can Rivalry Increase Prices?

Christian Roessler[1]
U Melbourne, Dept. Economics

February 5, 2006

[1] Address: Department of Economics, University of Melbourne, Parkville, Victoria 3010, Australia. E-mail: roessler@econ.unimelb.edu.au.

Abstract

Spatially differentiated duopolists set higher-than-monopoly prices at some distances. This phenomenon is shown to occur in any finite-dimensional space for a class of reservation prices that covers concavity and convexity in perceived distance from a design. But an upper bound on the equilibrium duopoly price converges monotonically and quickly to the monopoly price in dimensionality. If consumers care about sufficiently many features of the product (a very small number of criteria is enough), monopoly nearly leads to an extreme price.

JEL Codes: D40, D43, L10, L43, C72

Competition in a market with spatially dispersed tastes can lead to higher equilibrium prices when the varieties are closer, or better substitutes. The pathological case in which a firm acting as a differentiated Bertrand oligopolist (demand-constrained by competition) sets a strictly lower price than the same firm acting as a monopolist (ignoring competition) was recognized by Greenhut, Hwang, and Ohta [6] in 1975. But thirty years later, the discovery has not changed the way applied economists and regulators view monopolies and trusts. The principle that rivalry is price-reducing and welfare-improving remains canonical.

Exceptions exist quite generally. Sanner [11] recently showed that, in a Hotelling-style bounded line market (with elastic demand), entry more typically raises the equilibrium price of an incumbent monopolist than reduces it, at least if brand locations are non-strategic. (If they are profit-maximzing, the possibility persists.) We will demonstrate that there are "inferior duopolies" (pricing above monopoly) in all unbounded finite-dimensional Euclidean spaces (with the usual quasilinear preferences and uniform type density). The support for the phenomenon always has positive measure, and this motivates our central question: is inferior duopoly relevant to competition policy? Can it be substantially worse than monopoly?

Rather unexpectedly, we are able to rule out that the duopoly price exceeds the monopoly price substantially, regardless of whether reservation prices are convex or concave in personal distance from the brand. An upper bound on duopoly price converges monotonically and quickly to the monopoly price in the "refinement" of tastes. By this we mean the number of product features consumers distinguish, or the dimensionality of the space in which subjectively ideal bundles of characteristics are distributed. A small degree of refinement ensures that the highest possible duopoly price is very close to the monopoly price.

The first building block of the paper is the proof that inferior duopoly exists in finite-dimensional type spaces. First we give an intuitive geometric explanation in the plane. In Section 3 we set out assumptions and derive demands and elasticities. Only the standard quadratic distance cost model is sufficiently tractable, so we fully develop this case. A unique equilibrium is shown to exist in symmetric prices; in other models, it can be verified that first- and second-order conditions are satisfied at a symmetric "local" equilibrium; for global concavity we must refer to simulations. These technical points, leading to the inferior duopoly theorem, are the subject of Section 4. The theorem complements some related results in the literature for fixed spaces of one or two characteristics.

Although a highly differentiated duopoly *always* results in higher equilibrium prices than monopoly, the premium is extremely small when as few as five product features are choice criteria. This is implied by our central theorem in Section 5, monotonic convergence of maximal prices to the monopoly price (in refinement of the preference model). The flipside of quick convergence is that little dimensionality is needed to achieve a given bound on this price ratio. Hence realistic preferences almost certainly belong to a model where the worst duopoly is barely distinguishable from monopoly in prices.

This is not quite the same as ruling out large welfare improvements through monopoly, because brand distance is not a constant in the argument. When different pockets of the space are served in monopoly and maximal-price duopoly, a strong welfare criterion like Pareto's is not informative. Consumer and total surplus increase in duopoly with brand distance at any given prices; the maximal duopoly price is therefore not the surplus-minimizing price. We do not delve deeply into this issue, since the basic efficiency aspect of locations is a separate matter and sensitive to our neglect of entry. In practice, prices may be the most useful indicators of welfare properties.

The appendices contain many technical details of the analysis, in particular the construction of gradients and the proof of price symmetry. An annotated and menu-based Matlab program is available from the author to generate numerical solutions and plots relating to the paper.

1 Geometry of the Pricing Puzzle

Product designs can be viewed as points in infinite-dimensional space: unique combinations of strengths in characteristics, which are represented by the coordinates. Only n of the characteristics are regarded as distinguishing features by consumers; individually preferred designs ("types") are therefore points in an n-dimensional subspace. A type's perceived distance from an available design is constructed as the distance from the design's projection into n-space (thus, distance in "irrelevant" dimensions is treated by consumers as nil).

Distances from a design reflect varying willingness to pay. If the distribution of points in the space (types) is uniform and the metric is Euclidean, then the set of types willing to pay a given price for a design is an n-dimensional sphere, the design's "market." When two designs set prices such that their markets do not intersect, we refer to them as local monopolists; else they are local duopolists. A monopolist is not locally constrained by the presence of a rival; within some neighborhood, changes in location would make no difference to its pricing problem. A duopolist has consumers in its market who are indifferent between the varieties at prevailing prices and would switch to a cheaper one if the designs were marginally closer. One might then expect that a smaller distance between designs induces lower prices.

If this were generally true, duopoly prices could never exceed the monopoly price. The argument is, however, not quite correct. We mustn't confound the "sunk" reduction in demand, owing to encroachment, with what is attributable to pricing. Since a firm with a closer substitute has fewer interior customers (who strictly prefer the design, given prices), it *may* have fewer marginal consumers, even though individual costs of switching to the rival design are reduced. Figure 1 illustrates that a slight nearing between the

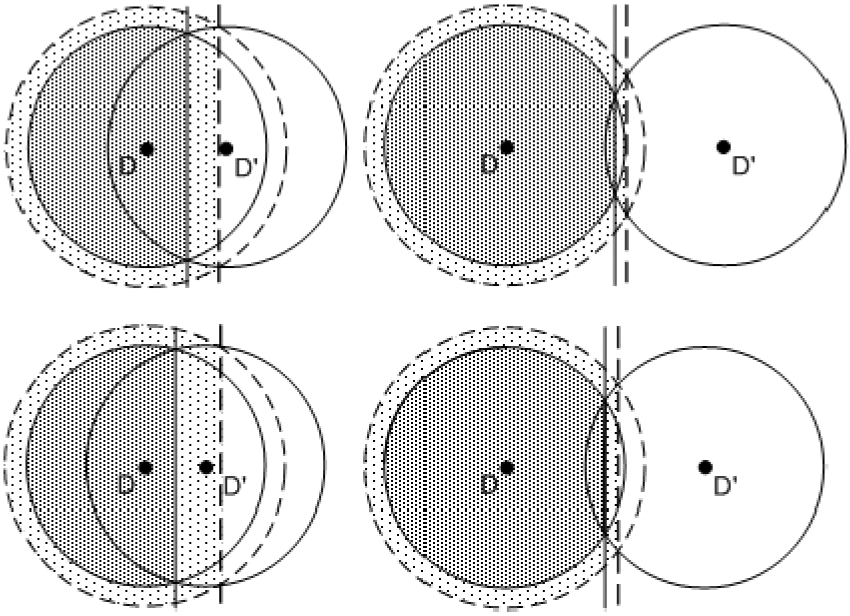

Figure 1: Price increase vs. decrease when distance shortens marginally

design locations (compare top with bottom) can lead to a larger marginal set (left) or a smaller marginal set (right).[1]

The optimal price resolves a tradeoff between extracting as much revenue as possible from consumers in the interior of the demand region and retaining the marginal consumers in the boundary. Closer competition among products that are relatively poor substitutes (right panel) can be favorable to a price increase when the margin thins sufficiently to offset the contraction of the interior, so that smaller revenue loss offsets smaller revenue gain. When the designs are close, the indifference hyperplane is sufficiently responsive for incentives to reduce price to dominate.

---

[1] Readers who are familiar with the geometry will recognize that the diagram depicts the quadratic distance cost model, where the set of consumers who are indifferent between the designs is a hyperplane. The hyperplane passes through all intersections of the sphere surfaces, as a simple consequence of transitivity: the surface intersections contain types who are indifferent between no purchase and either design, so they must be indifferent between designs.

Now consider local monopolies (at optimal prices) whose markets just touch. Prices rise in the transition to a duopoly regime because it is more expensive for a firm that faces competition to capture consumers at the boundary. While the monopolist bids against the numeraire, the duopolist bids against a rival design, which is necessarily more valuable to a type who is in the rival's market. The duopolist must therefore offer a larger price reduction to achieve a given expansion of its market area. Implicitly, it is suboptimal even to maintain the monopoly price, and price increases instead. This is the phenomenon we are concerned with, and which we will ascertain in a type space of arbitrary (finite) dimensionality.

These heuristic observations suggest that equilibrium prices increase at the transition from monopoly to duopoly, and beyond for some (smaller) distances. The maximal duopoly price lies somewhere in that interval. But our analysis is intended to make a counterpoint that is not apparent graphically: the maximal price is almost indistinguishable from the monopoly price for any preference model we consider, if it allows for a plausible number of choice criteria (dimensions of differentiation). The result is of some practical interest in itself, since it provides a subtly different rationale for antitrust policy. It also illustrates that theoretical exceptions to economic intuition are worth quantifying; in this case, intuition is essentially vindicated.

2 Demand Aggregation and Elasticity

As is customary in spatial models, we take the perceived distance (of an individual from a design in $\mathbb{R}^n$, where x_i is the distance in dimension i) to be a power k of the Euclidean metric,

$$d^k(x) = \sum_{i=1}^{n} x_i^{k/2};$$

types like x are uniformly distributed in the n-space. Reservation prices have the form

$$\rho(x) = v - d^k(x)\,t, \tag{1}$$

where $v > 0$ is the ideal value of the good (willingness to pay for a hypothetical design that matches x's type) and $t > 0$ ("transport cost") is the discount required per unit of perceived distance. Marginal utility of the good is zero after consuming the first unit. We shall consider $k \in (0, 2]$, i.e. concave and convex distance aversion; however, an analytical proof of global equilibrium is only feasible when $k = 2$.[2]

D's market is the set $\{x : \rho(x) - p \geq 0\}$, an n-sphere with radius $r = d(\bar{x})$ such that $\rho(\bar{x}) = p$. Since reservation prices decrease in distance from the brand, radius decreases in price. In local duopoly the set of "bimarginal" types, who are indifferent between designs D and D' at prices p and p' and belong to both markets, has positive measure; in monopoly it has measure zero. A duopoly regime is illustrated in Figure 2, where the design loci D and D' are separated by a distance δ.[3]

D's demand region $\{x : \rho(x) - p \geq 0,\ \rho(x) - p \geq \rho(x') - p'\}$ is the intersection of its market with the region where D is preferred to D'. This region is a halfspace, bounded by a hyperplane of indifferent types, if and only if $p = p'$ or the power of the distance function is 2.[4] There is little

---

[2] With k in this interval, d^k is submodular ($\partial^2 d^k / \partial x_i \partial x_j \leq 0$); larger distances in component features mostly determine perceived distance from a design. But the immediate reason for the restriction is that it will make payoffs quasiconcave. In the modular case $k = 2$, features are "independent" in that dissatisfaction with one feature does not affect the discount required for distance in other features. This is what makes the indifference set a hyperplane.

[3] The perfect symmetry of the geometry implied by Euclidean distance allows us to treat the line through D and D' as parallel to one of the axes of the coordinate system; that is, the analysis is invariant to rotation. Hence the vertical distance between D and D' is zero without loss.

[4] For $k = 2$, this can be confirmed directly from the indifference condition $\rho(x) - p = \rho'(x) - p'$ (where ρ' is the reservation price function for design D' and p' is the actual price). For any k, the equality holds when prices are symmetric and $d(x) = d'(x)$ (d' is distance from D'); such points x form a hyperplane too.

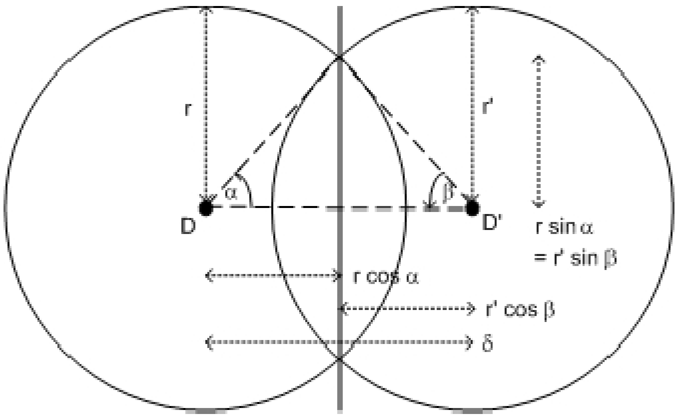

Figure 2: Notation

tractability when one cannot work with hyperplanes. For this reason we use the standard quadratic model ($k = 2$) to derive aggregate demand and present a fully functional example. We can show explicitly in this case that prices are symmetric in Nash equilibrium.

Although the hyperplane environment applies to any other power of the distance function in the neighborhood of symmetric prices, we cannot prove that such prices are global equilibria without constructing aggregate demand functions. Strictly speaking, we must employ a weaker equilibrium concept where strategies are only local best responses. We are willing to do this without behavioral justifications because our numerical investigations suggest that payoffs are always globally quasiconcave; thus local equilibria are also Nash equilibria. In our judgement, the premise that indifferent sets are hyperplanes, at prices relevant to our comparative statics, is safe whenever the metric is Euclidean.

In the quadratic case $k = 2$ then, or at symmetric prices, this hyperplane is orthogonal to $\overline{DD'}$ and passes through every point in the boundary of both market spheres (the statements are equivalent).[5] These observations give

---

[5] Orthogonality may be proven directly by reflecting a line segment between D and any point y in the indifference set over $\overline{DD'}$. The image of y must be in the indifference set

rise to a geometry in which we can calculate aggregate demand. We pause to summarize the economic assumptions underlying the geometry. (i) There are two designs located in bounded subsets of $\mathbb{R}^n$, $n \geq 2$. (ii) Ideal points form a continuum of "sufficient" expanse and uniform density (magnitude 1) in the neighborhood of each design.[6] Perceived distances are Euclidean. (iii) Every type values the ideal design identically and irrespectively of supplier (no quality difference). (iv) Marginal utility of the numeraire is constant and independent of type. In particular, reservation prices have the form (1), with $k = 2$ where we entertain asymmetric prices. (v) Individuals demand either one or zero units of the branded good, so as to maximize the (positive) difference between reservation price and selling price. (vi) Marginal cost of supply is constant (without further loss, zero). (vii) Pricing is nondiscriminatory and independently profit-maximizing.

Two sets of comments are in order to convince the reader that the assumptions are neither likely to suppress the pricing phenomenon from Section 2, nor are generally unreasonable. In order to bound the duopoly price, we should specify the model so that it is most conducive to a high price. The restrictions we impose are often relaxed in economic modeling such that (a) marginal utility of the good declines smoothly, not abruptly (hence consumers may buy more than one unit at a time), and (b) marginal utility of wealth is not constant, but declines. If we adopted (a), then the incentive for a price increase would always be smaller because interior clients reduce their purchases, so that the revenue gain is diminished. With (b), the margin widens as consumers are more averse to paying high prices; hence the revenue loss from a price increase is greater. Since we are interested in high duopoly prices, one-or-none individual demands and the absence of wealth effects involve no further loss.

- because the distance from D and D' is unchanged. The second claim derives from the transitivity of preferences; see footnote 1.

[6] Sufficient for the markets never to extend into a vacuum at reasonable prices: they are indeed spheres.

The restrictions on design locations within $\mathbb{R}^n$ are interpreted as technological constraints; as we have explained, n-dimensionality is owed to consumers' perceptions in a given preference model. We do not cover the line space ($n = 1$) because it poses special equilibrium existence problems. For this case, the price effect of competition, which can be positive as in higher-dimensional spaces, is fully discussed in Ohta [10], Greenhut, Norman, and Hung [7], and Sanner [11]. The uniform type distribution is common to most related literature; it introduces no arbitrary distortions. Since Hotelling [8] it is traditional to assume bounded type distributions. The image of the linear city, invoked by Hotelling, makes it plausible that the potential market comes somewhere to an end, and that sellers could hypothetically locate at these endpoints. But the literal spatial metaphor is exceptional in this respect. People are capable of imagining and desiring varieties that cannot currently be produced (and yet buy existing designs); so "local unboundedness" seems more natural.

A monopolist's demand is the volume of an n-sphere. The formula is derived in Appendix A:[7]

$$Q_n^m(p) \equiv V_n(r) = \frac{\left(\pi r^2\right)^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

(We will often omit making the arguments of a function explicit; e.g. radius is a function of price, therefore demand is.) From (1), radius is

$$r = \left(\frac{v-p}{t}\right)^{1/k}$$

since $\rho(r) = v - r^k t = p$.

Although k is assumed fixed at 2 for the purpose of demand aggregation in this section (which encompasses asymmetric prices), we carry k forward as a variable so that the relationship between models at symmetric prices is

---

[7] $\Gamma(\cdot)$ is the gamma (or generalized factorial) function.

transparent.

A duopolist loses the mass $L_n(r,r')$, the volume of a "spherical cap," to competition. The spherical cap may be thought of as a string of $(n-1)$-dimensional spheres with radii $\sqrt{r^2-s^2}$, $s\in[r\cos\alpha,r]$. Hence

$$\begin{aligned} L_n(p,p') &= \int_{r\cos\alpha}^{r} V_{n-1}\left(\sqrt{r^2-s^2}\right)ds = \int_0^{\alpha} V_{n-1}(r\sin\theta)\,r\sin\theta d\theta \\ &= rV_{n-1}(r)\int_0^{\alpha}\sin^n\theta d\theta. \end{aligned}$$

This integral has no closed form, but it can be calculated for $n=1$ and $n=2$, and there exists a recursive expression for n. It's shown in Appendix A that the loss is explicitly[8]

$$L_n(p,p') = V_n(r)\left(\frac{1}{2}-\frac{1}{2\pi}\sum_{j=0}^{(n-1)/2} B\left(\frac{1}{2},\frac{n}{2}-j\right)\sin^{n-2j-1}\alpha\cos\alpha\right)$$

if n is odd, and

$$L_n(p,p') = V_n(r)\left(\frac{\alpha}{\pi}-\frac{1}{2\pi}\sum_{j=0}^{n/2-1} B\left(\frac{1}{2},\frac{n}{2}-j\right)\sin^{n-2j-1}\alpha\cos\alpha\right)$$

if n is even. The bracketed terms vanish as $\cos\alpha\to1$ (hence $\alpha\to0$, $\sin\alpha\to0$), as one expects.[9]

The resulting expressions for the duopoly demand, $Q_n^d(r,r')=Q_n^m(r)-L_n(r,r')$, have identical price derivatives (easiest to find by differentiating

---

[8] $B(\cdot)$ is the beta function, a ratio of factorials.

[9] The function is not unambiguously defined for $\cos\alpha=1$ and n odd (both $\sin\alpha$ and its exponent $n-2j-1$ are zero when $j=(n-1)/2$). Hence our convention to treat $\cos\alpha=1$ as local monopoly.

$Q_n^d(p, p')$ for $n = 1, 2, \ldots$ and recognizing the pattern):

$$\frac{dQ_n^d}{dp} = \frac{dQ_n^d}{dr}\frac{\partial r}{\partial p} = \left(n\frac{Q_n^d}{r} - \frac{1}{B\left(\frac{1}{2}, \frac{n+1}{2}\right)} Q_n^m \sin^n \alpha \frac{\partial \alpha}{\partial r} \right) \frac{\partial r}{\partial p}, \tag{2}$$

where

$$\frac{\partial \alpha}{\partial r} = -\frac{r - \delta \cos \alpha}{\delta r \sin \alpha}. \tag{3}$$

(see Appendix B) and $\partial r / \partial p = -1/(kt)\, r^{1-k}$. A duopolist therefore has demand elasticity

$$E_n^d(p, p') \equiv -\frac{dQ_n^d}{dp}\frac{p}{Q_n^d} = \left(1 - \frac{1}{nB\left(\frac{1}{2}, \frac{n+1}{2}\right)} \frac{Q_n^m}{Q_n^d} r \sin^n \alpha \frac{\partial \alpha}{\partial r} \right) E_n^m(p) \tag{4}$$

where

$$E_n^m(p) = -n\frac{\partial r}{\partial p}\frac{p}{r} = \frac{n}{k}\frac{p}{v - p} \tag{5}$$

is the monopolist's elasticity function.

3 Inferior Duopoly

If $k = 2$, there exists a unique Nash equilibrium in symmetric prices in every preference model (that is, a type space of any finite dimensionality $n \geq 2$). Existence can be argued in standard fashion; symmetry is more difficult and proven in Appendix B. The duopoly regime is active whenever $\alpha \in (0, \pi)$. For $n \geq 2$, the elasticity function is continuous at the regime-switching points (and everywhere else) since $\sin \alpha \to 0$ as $\alpha \to 0$ or $\alpha \to \pi$ implies

$$\lim_{\alpha \to 0} E_n^d(p, p') = \lim_{\alpha \to \pi} E_n^d(p, p') = E_n^m(p).$$

Continuity of the elasticity function ensures existence of a price equilibrium because, by the Implicit Function Theorem, the solution $p^* (p')$ to

$$E_n (p, p') = \begin{cases} E_n^d (p, p') & \alpha \in (0, \pi) \\ E_n^m (p) & \alpha \in \{0, \pi\} \end{cases} = 1$$

(the best response) is continuous.

The best response is indeed well-defined as a function (single-valued everywhere). To see this, note that the duopoly payoff function is increasing and single-peaked (thus quasiconcave) in $-r \cos \alpha$, as one can solve $E_n^d (p, p') = 1$ explicitly for $-r \cos \alpha^*$ such that $-r \cos \alpha < -r \cos \alpha^*$ if and only if $E_n^d < 1$ (payoff increasing in price).[10] Because $-r \cos \alpha$ is linear in p,[11] and duopoly payoff is an increasing quasiconcave transformation, duopoly payoff is quasiconcave in p. It is easy to check that monopoly payoff is concave, so payoff is quasiconcave for all p.

Since $p^* (\tilde{p})$ is a continuous map from the compact set $[0, v]$ into itself (if $p > v$, no one buys), there exists a fixed point by Brouwer's theorem. If best responses were strictly monotonic, they would be well-defined mutual inverse functions, in a fully symmetric setting such as ours. Then price symmetry is immediate; in p-p' space, they could only intersect on the 45° degree line. Returning to the heuristics of Section 1, however, a price increase may be optimally met with a price decrease in duopoly. When p' increases, the indifference hyperplane shifts away from D, increasing Q_n^d (the interior set

---

[10] The details are, after substituting in (4) from (3) and for $\partial r / \partial p$:

$$E_n^d = 1 \iff -r \cos \alpha^* = kB \left(\frac{1}{2}, \frac{n+1}{2} \right) \frac{Q_n^d}{Q_n^m} r \sin^{1-n} \alpha \frac{v - p}{p} (1 - E_n^m) - \frac{r^2}{\delta}.$$

Then

$$E_n^d = E_n^m + \frac{1}{kB \left(\frac{1}{2}, \frac{n+1}{2} \right)} \frac{Q_n^m (p)}{Q_n^d (p, p')} \frac{1}{r} \sin^{n-1} \alpha \left(\frac{r^2}{\delta} - r \cos \alpha \right) \frac{p}{v - p}$$

is increasing in $-r \cos \alpha$, so $-r \cos \alpha < -r \cos \alpha^* \implies E_n^d (p, p') < 1$ and $-r \cos \alpha > -r \cos \alpha^* \implies E_n^d (p, p') > 1$.

[11] In Appendix B we show $\cos \alpha = \left(\delta^2 - r^2 + r'^2 \right) / (2 \delta r)$, so $\partial (r \cos \alpha) / \partial p = -1 / (2 \delta t)$.

measure) at all p. The surface of the demand region expands and dQ_n^d/dp (the marginal set measure) also increases; if the addition to the margin is relatively greater than to the interior, $p^* \left(p'\right)$ falls.

Despite the complications, equilibria turn out to be symmetric; we prove this part of Theorem 1 in Appendix B. Note that Lemma 2, in the text, is needed.

Theorem 1 *In the quadratic model ($k = 2$), the pricing game has a unique symmetric Nash equilibrium:* $p^* \left(p'^*\right) = p'^* \left(p^*\right)$.

As we have mentioned earlier, an analog of Theorem 1 cannot be given for other k, since it is essential to have an expression for aggregate demand at all p. But the argument for the tractable case $k = 2$ is only special in relying on indifference hyperplanes; the geometry is not abruptly different when there is curvature. In simulations, models in other k always seem to exhibit unique symmetric Nash equilibria. Yet, analytically we can only claim that these points are local equilibria (no payoff-improving marginal move is ignored); the reader should bear this aspect in mind when we refer to symmetric prices that satisfy first- and second-order conditions as equilibria.[12]

Lemma 2 introduces three facts about the relationship between α and β that are critical to all results in this section, including the price symmetry proof in Appendix B. The argument is geometric and therefore independent of k.

Lemma 2 ***(i)*** $r\left(\partial\alpha/\partial r\right) = r'\left(\partial\beta/\partial r'\right)$.
(ii) $\partial\alpha/\partial r > 0 \iff \alpha + \beta$ *is acute;* $\partial\alpha/\partial r < 0 \iff \alpha + \beta$ *is obtuse.*
(iii) $\partial\alpha/\partial r > 0 \iff \partial\beta/\partial p > 0$*;* $\partial\alpha/\partial r < 0 \iff \partial\beta/\partial r < 0$.

Proof. In a two-dimensional slice of the space, denote the tangent to D's market boundary at an intersection with D''s market boundary by T. As

---

[12] The quasiconcavity argument is valid for $k \in (0, 2)$ at symmetric prices, since the geometry is locally identical to $k = 2$ (indifferent set is a hyperplane) and the elasticity function (4) applies.

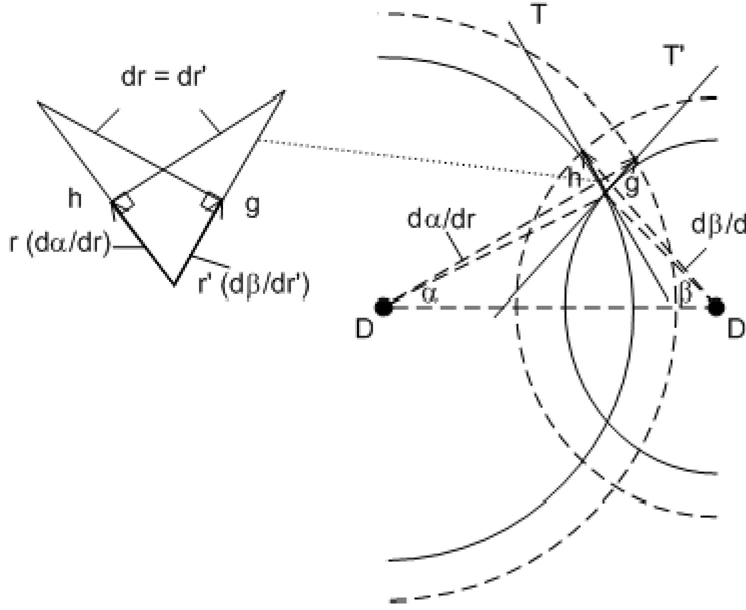

Figure 3: $r\,(\partial\alpha/\partial r) = r'\,(\partial\beta/\partial r')$

illustrated in Figure 3, the limiting approximation of $r\,(\partial\alpha/\partial r)$ is a vector in the direction of T. Similarly, $r'\,(\partial\beta/\partial r')$ is locally a vector in the direction of T'. By construction, each of $r\,(\partial\alpha/\partial r)$ and $r'\,(\partial\beta/\partial r')$ is the inner product of vectors g and h (refer to left panel); hence they are equal:

$$r\frac{\partial\alpha}{\partial r} = g \cdot h = \bar{h} \cdot \bar{g} = r'\frac{\partial\beta}{\partial r'},$$

where $\bar{h}$ and $\bar{g}$ are transposes of h and g. This verifies (i).[13]

A sequence of angle-sum arguments (see Figure 4) shows that the angle formed by g and h is $\alpha + \beta$.

---

[13] Equivalently, observe that the two triangles on the left are similar because they have one equal (right) angle and one shared angle. In addition, a side adjacent to the right angle has the same length in both triangles since radii r and r' are varied by equal increments. This makes the triangles identical (one can be obtained by reflecting and rotating the other).

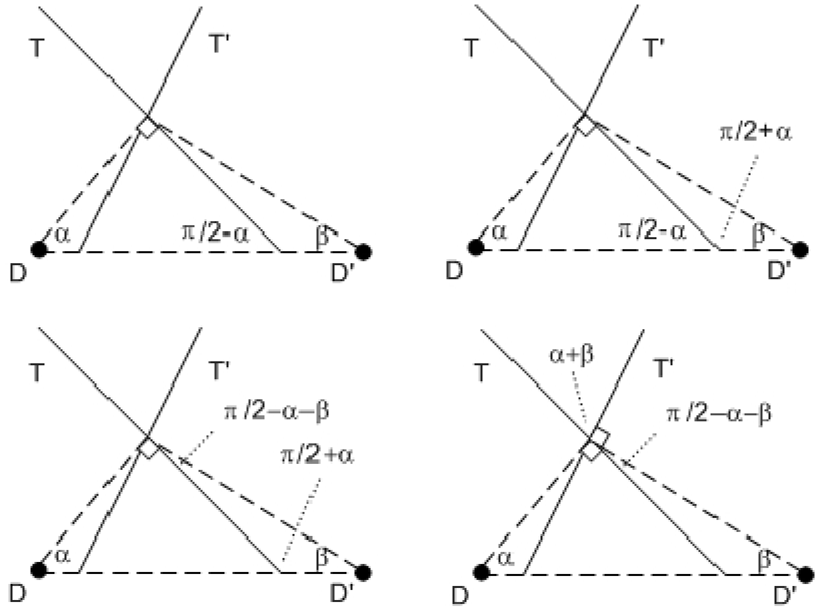

Figure 4: Tangents form angle $\alpha + \beta$

Since

$$g \cdot h \equiv \|g\| \, \|h\| \cos(\alpha + \beta) > 0 \iff 0 < \alpha + \beta < \frac{\pi}{2},$$

($\alpha + \beta$ is acute) and

$$g \cdot h \equiv \|g\| \, \|h\| \cos(\alpha + \beta) < 0 \iff \frac{\pi}{2} < \alpha + \beta < \pi$$

($\alpha+\beta$ is obtuse), we have established that $r\,(\partial\alpha/\partial r) = g \cdot h > 0$ if and only if $\alpha+\beta$ is acute. The same statement clearly applies to $r'\,(\partial\beta/\partial r') = r\,(\partial\alpha/\partial r)$; dividing the inequalities respectively by r and r' gives (iii).

■

We are now in a position to argue the existence of inferior duopolies. Prices are symmetric from this point on, hence we cover all k. Our strategy is to find a duopoly situation such that D sets the monopoly price P in response to $p' = P$. There will be a p' neighborhood of P in which $p^* (p') > P$ but the regime continues to be duopoly. It is apparent from the duopolist's elasticity function (4) that a sufficient condition is $\partial\alpha/\partial r = 0$ with $\alpha > 0$. For by Lemma 2 (iii), this implies $\partial\beta/\partial r = 0$; hence $E_n^d (p, p') = E_n'^d (p', p) = E_n^m (p)$, so that both set their prices to the optimal monopoly price P. In addition, by Lemma 2 (ii) $\alpha + \beta$ is then a right angle; with α and β strictly positive, we are indeed in the interior of the duopoly regime.

The equilibrium where duopoly price equals monopoly price is associated with the distance $\Delta = r (P) \cos\alpha + r' (P) \cos\beta$. Namely, since $\alpha = \beta = \pi/4$ by price symmetry and $\alpha + \beta = \pi/2$,

$$\Delta = 2r (P) \cos (\pi/4) = \sqrt{2} r (P) .$$

If the designs move marginally apart, α and β unambiguously decrease. Since $\alpha + \beta$ becomes acute, $\partial\alpha/\partial r > 0$ and $\partial\beta/\partial r > 0$, and both elasticities are now smaller than $E_n^m (p)$ at $p = p' = P$. Equilibrium prices must increase. We therefore state:

Theorem 3 *There exists a region of δs (brand distances) such that the symmetric equilibrium price exceeds the monopoly price and induces the duopoly regime.*

It is perhaps instructive to replicate this result without reference to the constructed elasticities. Fixing α at 45° so that it is locally invariant to price changes, consider an arbitrary angle θ in the n-sphere centered at D. Say that a ray extending from D at angle θ is a constrained direction if it intersects the indifference hyperplane before it intersects the market boundary, and is an unconstrained direction otherwise. In any unconstrained direction, the duopolist captures demand mass r, like a monopolist, and the rate of change with respect to a price change is $\partial r/\partial p$ for both.

In a constrained direction, the duopolist captures mass μ (refer to left panel of Figure 5). By Pythagoras' theorem,[14]

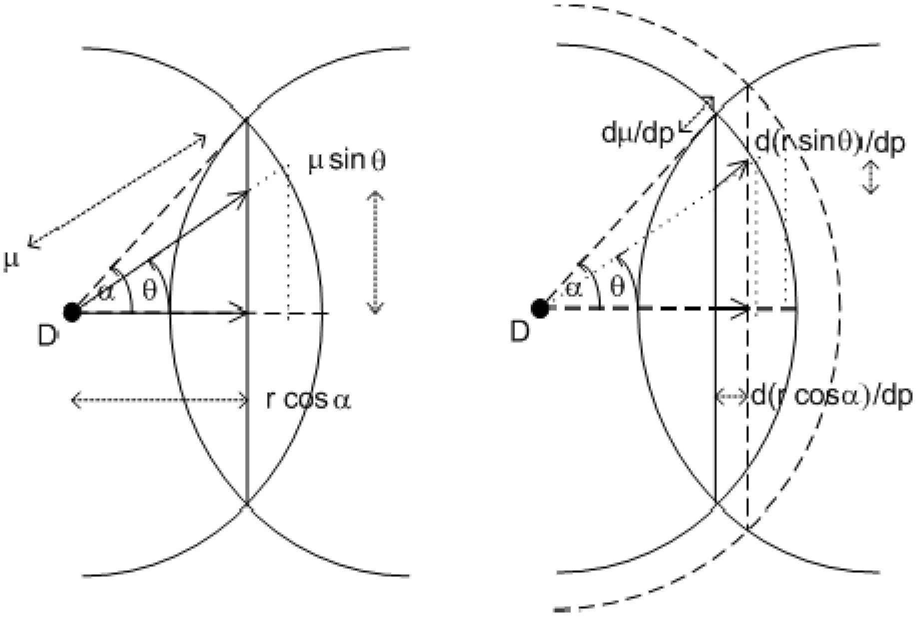

Figure 5: Monopoly price in duopoly

$$\mu^2 = r^2\cos^2\alpha + \mu^2\sin^2\theta \implies \mu = r\frac{\cos\alpha}{\cos\theta}.$$

The change in demand with respect to a marginal price change is $\partial\mu/\partial p$ in a constrained direction (see right panel of Figure 5). Another application of the Pythagoras theorem gives:

$$\begin{aligned}\left(\frac{d\mu}{dp}\right)^2 &= \left(\frac{\partial r}{\partial p}\cos\alpha - \sin\alpha\frac{\partial\alpha}{\partial p}\right)^2 + \left(\frac{d\mu}{dp}\right)^2\sin^2\theta \\ &= \left(\frac{\partial r}{\partial p}\right)^2\cos^2\alpha + \left(\frac{d\mu}{dp}\right)^2\sin^2\theta \implies \frac{d\mu}{dp} = \frac{\partial r}{\partial p}\frac{\cos\alpha}{\cos\theta}.\end{aligned}$$

Because α is constant, the set of constrained directions is unchanged from before. Therefore, $\mu/r = (\partial\mu/\partial p)\,/\,(\partial r/\partial p)$ for all directions and, integrating

---

[14] Rearrange to $\mu^2\left(1-\sin^2\theta\right) = r^2\cos^2\alpha$ and simplify.

over the sphere,

$$\frac{Q_n^d\left(p,p'\right)}{Q_n^m\left(p\right)}=\left.\frac{dQ_n^d\left(p,p'\right)/dp}{dQ_n^m\left(p\right)/dp}\right|_{d\alpha/dp=0}.$$

It follows that $E_n^d\left(p,p'\right)=\left.E_n^m\left(p\right)\right|_{d\alpha/dp=0}$, hence $\left.p^*\left(p'\right)\right|_{d\alpha/dp=0}=P$. Now again since $E_n^d\left(P,p'\right)=1$, demand becomes inelastic as $\partial\alpha/\partial p$ turns negative and $p^*\left(p'\right)$ necessarily exceeds P.

Alternatively the case can be made starting from the regime-switching point where $\alpha=\beta=0$ and both set the monopoly price. As distance shortens and we enter duopoly with symmetric prices, $\alpha+\beta$ is initially small, hence $\partial\alpha/\partial p<0$ and demand is inelastic at $p=P$ (which entails $E_n^m\left(P\right)=1$). The regime change is therefore associated with a price increase: neighboring duopoly games are always inferior to monopoly; they must result in a higher price. This explanation is in fact simpler, but the interior duopoly point that induces the monopoly price plays an important role in our subsequent analysis (it's the limit of the maximal price path) and is worth understanding.

The "inferior duopoly" phenomenon is, of course, not novel, although Theorem 3 represents a considerable generalization, both to spaces of dimension $n>2$ and to a wider range of convexities in distance. Greenhut, Hwang, and Ohta [6] first commented on the possibility of a price increase when many symmetrically arranged, identically behaved rivals move closer in the plane. Capozza and Van Order ([2] and [3]) confirmed this in the context long-run monopolistic competition, where the number of designs is determined by entry until profits are zero.[15] A systematic and accessible treatment for the line space is found in Sanner [11], who also considers the role of endogenous location choice.

The cited literature allows for continuous individual demand functions;[16]

---

[15] Strictly speaking, these treatments do not cover the duopoly case, since the demand region is taken to be a regular polygon (hence every design has more than two local rivals, prices are symmetric and sufficiently low).

[16] Although these models are set in specific spaces, one might suspect that our unit-

explanations for counterintuitive price increases have therefore emphasized the "dispensible" margins, populated with consumers who purchase the least because they are the most distant. Our setting does not admit this interpretation and accordingly offers a different type of intuition. The extensions we have made are of special interest because they enable us to derive the convergence property of the next section, and show that its quantitative implications are robust within a large class of models.

4 Maximal Price Equilibria

The duopoly price path in a space of dimension n satisfies $E_n^d(p,p)=1$ and has as solutions pairings of distances δ with the symmetric equilibrium price. The maximal price $\tilde{P}$ in this space is a duopoly price by Theorem 3; from $E_n^d\left(\tilde{P},\tilde{P}\right)=1$,

$$\tilde{P}=-\left(1-\frac{1}{nB\left(\frac{1}{2},\frac{n+1}{2}\right)}\frac{Q_n^m\left(\tilde{P}\right)}{Q_n^d\left(\tilde{P},\tilde{P}\right)}r\sin^n\alpha\frac{\partial\alpha}{\partial r}\right)^{-1}\frac{1}{n}\frac{r\left(\tilde{P}\right)}{\partial r\left(\tilde{P}\right)/\partial p}. \tag{6}$$

We may also consider a duopoly equilibrium where the symmetric price is the monopoly price P, hence $E_n^d(P,P)=E_n^m(P)=1 \implies$

$$P=-\frac{1}{n}\frac{r(P)}{\partial r(P)/\partial p}=\frac{k}{k+n}v. \tag{7}$$

From (6) and (7) follows that the maximal to monopoly price ratio can

---

demand model in n dimensions formally corresponds to a model in two dimensions with some particular individual demand function. This is however not the case, except trivially when $n=2$ and the individual demand function is constant. The argument, based on a duopoly version of the Greenhut et al. model, is available on request.

be expressed as

$$\frac{\tilde{P}}{P}=\frac{1}{\eta\left(P,\tilde{P}\right)}\frac{\partial r\left(P\right)/\partial p}{\partial r\left(\tilde{P}\right)/\partial p}\frac{r\left(\tilde{P}\right)}{r\left(P\right)} \tag{8}$$

with

$$\eta\left(P,\tilde{P}\right)\equiv 1+\frac{1}{B\left(\frac{1}{2},\frac{n+1}{2}\right)}\frac{Q_n^m\left(\tilde{P}\right)}{Q_n^d\left(\tilde{P},\tilde{P}\right)}\sin^{n-1}\alpha\frac{1-2\cos^2\alpha}{2n\cos\alpha}, \tag{9}$$

by substitution for $\partial\alpha/\partial r$ from (3) at $p=p' \iff \cos\alpha=\delta/\left(2r\right)$. Because

$$\frac{E_n^m\left(P\right)}{E_n^m\left(\tilde{P}\right)}=\frac{\partial r\left(P\right)/\partial p}{\partial r\left(\tilde{P}\right)/\partial p}\frac{r\left(\tilde{P}\right)}{r\left(P\right)}\frac{P}{\tilde{P}}=\frac{v-\tilde{P}}{v-P}\frac{P}{\tilde{P}}, \tag{10}$$

from definition (5), $\eta\left(P,\tilde{P}\right)$ equals the free elasticity ratio $E_n^m\left(P\right)/E_n^m\left(\tilde{P}\right)$ at the two prices, and $\eta\left(P,\tilde{P}\right)<1 \iff P<\tilde{P}$.

By Theorem 3, this price ordering is satisfied in every preference model, hence $\eta\left(P,\tilde{P}\right)<1$. Moreover, (10) implies that $\eta\left(P,\tilde{P}\right)$ increases monotonically as $\tilde{P}$ declines toward P. Convergence to P is therefore equivalent to $\eta\left(P,\tilde{P}\right)\rightarrow 1$. Monotonic convergence in n obtains if $\eta\left(P,\tilde{P}\right)$ increases whenever n increases. This maps out the proof of our main theorem.

Theorem 4 *The maximal price $\tilde{P}$ converges monotonically to the monopoly price P in the dimension n of the type space.*

Proof. On the maximal price path in a space of fixed dimension, $\partial\tilde{P}/\partial\delta=0$; differentiating with $\tilde{P}$, $r\left(\tilde{P}\right)$, $\partial r\left(\tilde{P}\right)/\partial p$ constants and using $E_n^d\left(\tilde{P},\tilde{P}\right)=1$ to simplify,

$$\frac{\partial\tilde{P}}{\partial\delta}=\left(n\frac{\cos\alpha}{\sin\alpha}+\frac{\partial^2\alpha/\partial r\partial\delta}{\left(\partial\alpha/\partial r\right)\left(\partial\alpha/\partial\delta\right)}-\frac{\partial Q_n^d\left(\tilde{P},\tilde{P}\right)/\partial\alpha}{Q_n^d\left(\tilde{P},\tilde{P}\right)}\right)\left(E_n^m\left(\tilde{P}\right)-1\right)\frac{\partial\alpha}{\partial\delta}\tilde{P}.$$

Since $\tilde{P} > P$ by Theorem 3, $E_n^m\left(\tilde{P}\right) > 1$, so $\partial \tilde{P}/\partial \delta = 0$ implies

$$n\frac{\cos\alpha}{\sin\alpha} + \frac{\partial^2\alpha/\partial r\partial\delta}{\left(\partial\alpha/\partial r\right)\left(\partial\alpha/\partial\delta\right)} - \frac{\partial Q_n^d\left(\tilde{P},\tilde{P}\right)/\partial\alpha}{Q_n^d\left(\tilde{P},\tilde{P}\right)} = 0. \tag{11}$$

Substituting for $\partial Q_n^d\left(\tilde{P},\tilde{P}\right)/\partial\alpha = -Q_n^m \sin^n\alpha / B\left(\frac{1}{2},\frac{n+1}{2}\right)$ and the angle gradients, still with $\tilde{P}$ invariant (details in Appendix B), we have

$$\frac{1}{B\left(\frac{1}{2},\frac{n+1}{2}\right)}\frac{Q_n^m\left(\tilde{P}\right)}{Q_n^d\left(\tilde{P},\tilde{P}\right)}\sin^n\alpha = -\frac{\sin\alpha}{\cos\alpha}\frac{1+2\cos^2\alpha}{1-2\cos^2\alpha} - (n-1)\frac{\cos\alpha}{\sin\alpha}, \tag{12}$$

which allows (9) to be simplified to

$$\eta\left(P,\tilde{P}\right) = \left(1-\frac{1}{n}\right)\frac{1}{2\sin^2\alpha} - \frac{1}{n}\frac{1}{2\cos^2\alpha}. \tag{13}$$

Since $Q_n^m\left(\tilde{P}\right)$ can be factored out of $Q_n^d\left(\tilde{P},\tilde{P}\right)$, (12) is just a relationship between α and n. On $\alpha \in (0,\pi/2)$, the feasible duopoly scenarios given symmetric prices, the RHS has a single discontinuity at $\alpha = \pi/4$ and is increasing while $\alpha < \pi/4$. The LHS is always continuous and increasing when $\alpha < \pi/4$. As $\alpha \to 0$, $LHS \to 0$ and $RHS \to -\infty$; as $\alpha \to \pi/4$, LHS approaches a finite value, whereas $RHS \to \infty$. It follows that (12) has a single solution on $(0,\pi/4)$; but we know that any angle on the maximal price path belongs to this interval (at $\alpha = \pi/4$, a duopolist sets the monopoly price), so the angle on the maximal price path, $\tilde{A}(n)$, is uniquely determined. See the left panel of Figure 6.

Because $\alpha \to \pi/4$ as $n \to \infty$, $\lim_{n\to\infty}\eta\left(P,\tilde{P}\right) = 1/\left(2\sin^2\alpha\right) = 1$, which proves simple convergence. The sequence $\tilde{A}(n)$ implicitly relates $\eta\left(P,\tilde{P}\right)$ to n in (13). As is clear from the right panel of Figure 6, the function is monotonically increasing, concave, and convergent to 1, so that convergence

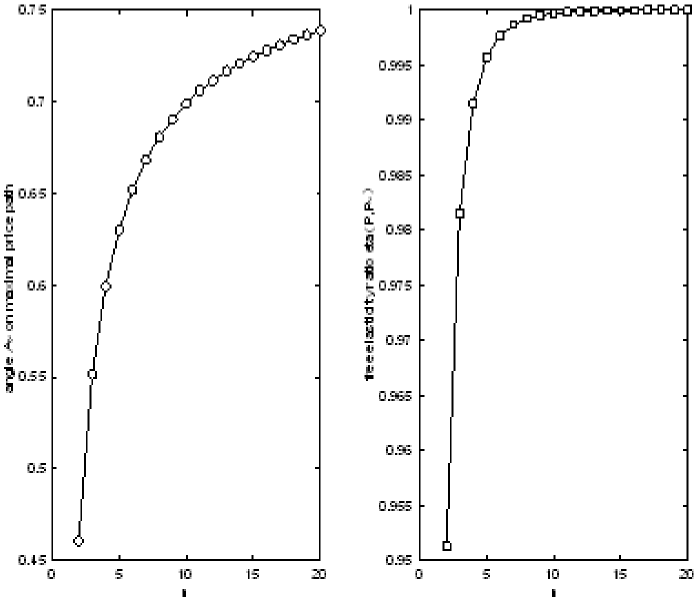

Figure 6: Convergence of α and $\eta\left(P,\tilde{P}\right)$

is also monontonic.[17]

■

The substantial aspect of Theorem 4 is monotonicity; convergence is a simple consequence of the "vanishing" property of hyperspheres (observable in the volume formula), which forces all prices to zero in the limit. To gain an intuitive understanding of this aspect, it is useful to know why hyperspheres diminish toward zero mass. Consider expanding a one-dimensional sphere (a line segment) of fixed radius 1/2 to a disk. As dimension is stepped up, the volume measure changes: the unit square replaces the unit interval as the basic mass element. The change of measure means that the population of consumer types spreads out uniformly into the plane: members of unit intervals reassemble in unit squares, maximally dispersed.

---

[17] One could make lengthy calculus arguments about the curvatures of (12) and (13), but these are continuous univariate functions, invariant to k and parameters (v and t), so the numerical inspection is conclusive.

Since the definition of mass is altered, a two-dimensional set maintains its one-dimensional mass only if it acquires full height (is stretched to a square). The disk is inscribed into the unit square, hence the mass of the market diminishes. In a particular dimension, this tendency is sensitive to the choice of radius;[18] but for any radius, the decline occurs at some finite n. The hypersphere eventually becomes arbitrarily small relative to the hypercube of the same radius, but this hypercube has finite mass, even in the limit (it can be packed with a finite number of unit hypercubes, which have mass 1 by definition). It follows that every infinite-dimensional hypersphere has zero measure.

Because demand regions are subsets of hyperspheres, their interior mass vanishes as $n \to \infty$, and no positive price can ultimately be an equilibrium. This may seem strange from an economic point of view, but it reflects that the uniform "redistribution" implicit in increasing dimensionality inflates the types' personal distances from any particular point.[19] In the limit, consumers "differentiate" themselves so much from each other and any available design that it is impossible to attract a set that has full dimension, hence positive measure, in the space.

Monotonicity does not follow from this argument because hypersphere volumes increase in dimensionality for small n, which is the region of the greatest practical interest. Again it's helpful to think about changes to interior and marginal sets as n increases. The volume of the monopolist's marginal set, $\partial Q_n^m(p)/\partial p$, is closely related to the surface area of the demand region, $\partial Q_n^m(p)/\partial r$. A straightforward calculation shows that the volume-to-surface-area ratio is r/n. Hence the marginal set expands relative to the interior set, and therefore price diminishes in n. When n is increased, the

---

[18] If $r \neq 1/2$, then the mass equivalent of the line segment market is a square of radius $\sqrt{2r}$ (generally, $(2r)^{n/(n+1)}$). For $r = 1$, for example, the disk of radius 1 contains the mass-equivalent square, hence mass increases.

[19] References to increases in n and changes in measure are only heuristics. Our analysis compares "fixed" preference models; we do not claim that there is an economic mechanism for preferences to evolve spatially along these lines.

type space expands orthogonally away from the axis along which D and D' are positioned, so that duopoly competition does not constrain the addition of depth to the bimarginal set. Because the marginal sets of a duopolist and a monopolist are similarly affected by greater dimensionality, the interior-to-marginal set ratios become more similar, and this induces more similar prices.

Theorem 4 has a compelling quantitative implication: one can expect $\tilde{P}$ to be very close to the monopoly price when consumers care about just a few product features. To get a sense of the strength of this claim, we pursue some further calculations. From (8), the price ratio may be written as

$$\frac{\tilde{P}}{P} = \frac{1}{\eta\left(P, \tilde{P}\right)} \frac{v - \tilde{P}}{v - P} = \left(1 + \eta\left(P, \tilde{P}\right) \frac{v - P}{P}\right)^{-1} \frac{v}{P} = \frac{k + n}{k + n\eta\left(P, \tilde{P}\right)} \quad (14)$$

by substituting for radii (first equality) and the monopoly price (7) (third equality). Since $\eta\left(P, \tilde{P}\right) < 1$, the denominator grow faster with n than the numerator, leading to convergence. Numerically, we can see in Figure 7 that convergence is monotonic and rapid. Even at $n = 3$, the price ratio is less than 2% greater than 1 in any model; at $n = 5$, less than half a percent. We emphasize that the price ratio path does not depend on the parameters v and t; this prediction is robust for reservation prices of the form (1).

That the price ratio declines in the convexity parameter k (see the right panel of Figure 7) is counterintuitive. Consistent with pure Bertrand competition, $\tilde{P} \to 0$ as $k \to 0$, as can be checked in (14).[20] But the monopoly price (7) also tends to zero with k, an artefact of inelastic individual demands and the unbounded the type space. (When reservation prices are insensitive

---

[20] Multipling by, and substituting for, P leads to:

$$\tilde{P} = \frac{k}{k + n\eta\left(P, \tilde{P}\right) v} v.$$

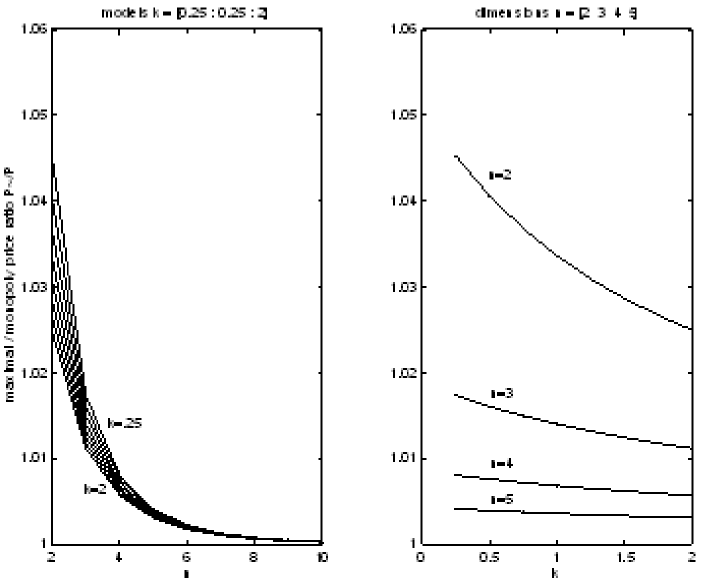

Figure 7: $\tilde{P}/P$ on maximal price path

to distance, aggregate demand is infinitely sensitive to price.) This is an unrealistic feature, and makes model comparisons by convexity somewhat uninteresting; the substantive insight is that convergence occurs quickly in all models.

Distances $\tilde{\Delta}$ on the maximal price path can be characterized analytically by substituting into $r\cos\alpha = \tilde{\Delta}/2$ from the definitions of radius for each model, and then for the maximal price. This leads to $\tilde{\Delta} = 2\cos\alpha\left(\left(v - \tilde{P}\right)/t\right)^{1/k}$ Because α goes to $\pi/4$ as $n \to \infty$, and $\tilde{P}$ to zero, the maximal-price distances converge to $\lim_{n\to\infty}\tilde{\Delta} = \sqrt{2}\,(v/t)^{1/k}$, which decreases in k. It seems reasonable to conjecture that large brand distances in a profitable (and uniformly populated) market are likely to attract entry. With small k, the maximal price equilibrium would then rarely be a stable arrangement.

But regarding welfare implications, the most immediate question is whether high-price equilibria are less efficient; since they imply a specific range of

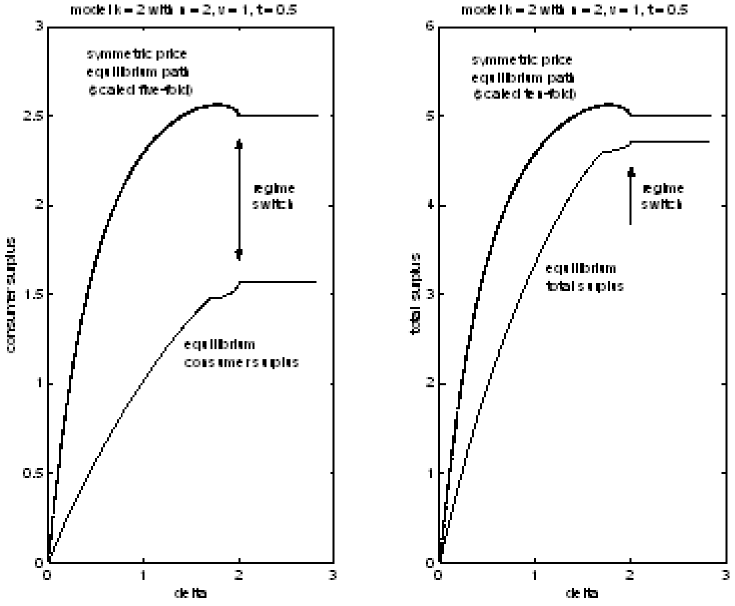

Figure 8: Consumer and total surplus vs. price

brand distances, this is by no means clear. Pareto improvement is too strong a criterion: a change in the identities of consumers served is implicit in most comparisons. Surplus measures face a similar problem: when D moves closer to D', there is a loss because some customers of D' experience an increase in the value of D but do not switch, so that the improvement is wasted on them. On the other hand, everyone who is made worse off by the change in locations bears the cost. As a result, surplus measures always favor equilibria in larger distances as can be seen in Figure 8.[21]

In our view, one should not dismiss the importance of price levels on these grounds. Uniform, dense distributions are a modeling choice made for analytical ease, but surplus is very sensitive to this stylization. Also,

---

[21] Equilibrium profit, the difference between total and consumer surplus, is always increasing in distance. This makes it trivial to endogenize location choice: firms would locate maximally apart in the feasible subset of the space. Hence local monopolies occur whenever possible; else inferior duopolies, if possible.

entrants may be expected to serve consumers that drop out of a market after a location change. One needs to bear in mind that, in a realistic policy scenario, relocating firms is usually not an option; the question is whether some form of entry is to be facilitated. Given that our setting is not rich enough to address these issues, surplus measures are probably not informative. It's plausible that, on balance, an appropriate welfare measure would closely mirror the behavior of prices.

5 Concluding Comments

Circumstantially one expects that tastes are, at least in a small degree, refined: people use multiple purchase criteria for most products. Our analysis suggests that a realistically low level of refinement suffices to compress the difference between monopoly and "inferior" duopoly into something very small. This is fortunate from the point of view of pragmatic regulation; a naive policy that indisciminately deters monopoly would incur no significant loss, if we leave the pure location effects on welfare aside.

Our treatment of product differentiation with an arbitrary number of features addresses pricing issues that could not be posed in the n-dimensional frameworks of Anderson, Palma, and Thisse [1], Feenstra and Levinsohn [5], and Irmen and Thisse [9], which were not designed to produce a sharp classification of competitive regimes. One may suspect that our definition of local monopoly, which depends on pricing rather than on there being a single firm, constructs the distinction from duopoly artificially, so that the terms do not have the usual meaning in this context.

The essential aspect of monopoly is that it does not have to compete, and when profit functions are well-behaved, this is economically equivalent to not competing on the margin. Whether a firm can attract custom away from "other" industries and encounter competition is ultimately a question of how low a price it is willing to set. At the optimal price, one local monopolist can disregard the other and behaves like a "traditional" monopolist. A regime switch to local duopoly creates a bilateral margin in which individual demand elasticity is escalated. It seems fair to interpret this as a transition to rivalry in the usual sense.

An interesting question arising from our convergence theorem is whether there are economic forces favoring (or precluding) taste refinement. Little has been said in economics about preference evolution, but it is conceivable that the dimensionality of the type space, where it relates to the features consumers perceive, changes through learning. To a firm, it may be a strategic variable that can be manipulated by design choices and advertising. Our comments on the convergence theorem suggest that a transition to another type space would have to involve a functional change in the distribution. Our analysis should be interpreted as comparative statics.

References

[1] ANDERSON, S.P., A. DE PALMA, J.F. THISSE: *Discrete Choice Theory of Product Differentiation.* Cambridge (Mass.): MIT Press, 1992.

[2] CAPOZZA, D.R., R. VAN ORDER: "Pricing under Spatial Competition and Spatial Monopoly," *Econometrica* 45 (1977), 1329-1338.

[3] CAPOZZA, D.R., R. VAN ORDER: "A Generalized Model of Spatial Competition," *American Economic Review* 68 (1978), 896-908.

[4] ECONOMIDES, N.: "Nash Equilibrium in Duopoly with Products Defined by Two Charateristics," *Rand Journal of Economics* 17 (1986), 431-439.

[5] FEENSTRA, R.C., J.A. LEVINSOHN: "Estimating Markups and Market Conduct with Multidimensional Product Attributes," *Review of Economic Studies* 62 (1995), 19-52.

[6] Greenhut, M.L., M. Hwang, L. Ohta: "Observations on the Shape and Relevance of the Spatial Demand Function," *Econometrica* 43 (1975), 669-682.

[7] Greenhut, M.L., G. Norman, C.S. Hung: *The Economics of Imperfect Competition.* Cambridge (UK): Cambridge University Press, 1987.

[8] Hotelling, H.: "Stability in Competition," *Economic Journal* 39 (1929), 41-57.

[9] Irmen, A., J.F. Thisse: "Competition in Multi-Characteristics Spaces: Hotelling Was Almost Right," *Journal of Economic Theory* 78 (1998), 76-102.

[10] Ohta, H.: "The Price Effects of Spatial Competition," *Review of Economic Studies* 48 (1981), 317-325.

[11] Sanner, H.: "Market Entry under Endogenous Product Choice May Raise Prices," *unpublished* (2005).

A Appendix: Aggregate Demand Construction

A.1 Volume of an n-Sphere

Denote the volume of an n-dimensional unit sphere as $V_n(1) = v_n$. For a general n-sphere, $V_n(r) = r^n v_n$. Since the cross-section of an n-sphere at distance s from the center is an $(n-1)$-sphere with radius $\sqrt{r^2 - s^2}$,

$$V_n(r) = \int_{-r}^{r} V_{n-1}\left(\sqrt{r^2 - s^2}\right) ds = v_{n-1} \int_{-r}^{r} \left(\sqrt{r^2 - s^2}\right)^{n-1} ds.$$

Substituting polar notation $s = r\cos\theta$,

$$V_n\left(r\right) = v_{n-1}\int_0^\pi r^n \sin^n\theta d\theta = rV_{n-1}\left(r\right)\int_0^\pi \sin^n\theta d\theta.$$

This integral is the beta function

$$B\left(\frac{n+1}{2},\frac{1}{2}\right) \equiv \sqrt{\pi}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}.$$

From $V_1\left(r\right) = 2r$, the series $\left\{V_n\left(r\right)\right\}$ may be constructed recursively; the general expression is:

$$V_n\left(r\right) = \frac{\left(\pi r^2\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

For odd n, this equals $V_n\left(r\right) = \left(2\pi r^2\right)^{(n+1)/2}/\left(\pi n!!\right)$; for even n, $V_n\left(r\right) = \left(2\pi r^2\right)^{n/2}/n!!$.

A.2 Loss to Competition

To express

$$L_n\left(p,p'\right) = rV_{n-1}\left(r\right)\int_0^\alpha \sin^n\theta d\theta$$

recursively, integrate by parts:

$$\begin{aligned}\int_0^\alpha \sin^n\theta d\theta &= (n-1)\int_0^\alpha \sin^{n-2}\theta\cos^2\theta d\theta - \sin^{n-1}\alpha\cos\alpha \\ &= (n-1)\int_0^\alpha \sin^{n-2}\theta d\theta - (n-1)\int_0^\alpha \sin^n\theta d\theta - \sin^{n-1}\alpha\cos\alpha.\end{aligned}$$

Collecting terms,[22]

$$
\begin{aligned}
L_n\left(p,p'\right) &= \frac{n-1}{n} r V_{n-1}\left(r\right) \int_0^{\alpha} \sin^{n-2}\theta d\theta - \frac{1}{n} r V_{n-1}\left(r\right) \sin^{n-1}\alpha\cos\alpha \\
&= \frac{2\pi}{n} r^2 L_{n-2}\left(p,p'\right) - \frac{1}{2\pi} B\left(\frac{1}{2},\frac{n}{2}\right) V_n\left(r\right) \sin^{n-1}\alpha\cos\alpha.
\end{aligned}
$$

The initial elements of the sequence are[23]

$$
\begin{aligned}
L_1\left(p,p'\right) &= r\int_0^{\alpha}\sin\theta d\theta = r\left(1-\cos\alpha\right) \\
L_2\left(p,p'\right) &= rV_1\left(r\right)\int_0^{\alpha}\sin^2\theta d\theta = r^2\left(\alpha-\sin\alpha\cos\alpha\right).
\end{aligned}
$$

Repeated substitution into (**??**) generates a sequence that can be expressed as:

$$
L_n\left(p,p'\right) = V_n\left(r\right)\left(\frac{1}{2} - \frac{1}{2\pi}\sum_{j=0}^{(n-1)/2} B\left(\frac{1}{2},\frac{n}{2}-j\right)\sin^{n-2j-1}\alpha\cos\alpha\right)
$$

---

[22] Applications of the volume formula give

$$
\frac{V_{n-1}\left(r\right)}{V_n\left(r\right)} = \frac{1}{\left(\pi r^2\right)^{1/2}}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \frac{n}{2\pi r}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)},
$$

since $\Gamma\left(1/2\right)=\sqrt{\pi}$. The last ratio is the beta function $B\left(1/2,n/2\right)$, so:

$$
V_{n-1}\left(r\right) = \frac{n}{2\pi r}B\left(\frac{1}{2},\frac{n}{2}\right)V_n\left(r\right).
$$

[23] For $n=2$, apply the double-angle formula $\sin^2\theta = \left(1-\cos 2\theta\right)/2$ to integrate

$$
\begin{aligned}
L_2\left(p,p'\right) &= \frac{1}{2}rV_1\left(r\right)\left(\alpha-\int_0^{\alpha}\cos 2\theta d\theta\right) \\
&= \frac{1}{2}rV_1\left(r\right)\left(\alpha-\frac{1}{2}\sin 2\alpha\right),
\end{aligned}
$$

and use another double-angle formula, $\sin 2\theta = 2\sin\theta\cos\theta$, to simplify.

for n odd;

$$L_n(p,p') = V_n(r)\left(\frac{\alpha}{\pi} - \frac{1}{2\pi}\sum_{j=0}^{n/2-1} B\left(\frac{1}{2},\frac{n}{2}-j\right)\sin^{n-2j-1}\alpha\cos\alpha\right)$$

for n even.

A.3 Consumer Surplus

A consumer at distance s from the purchased design attains surplus

$$\rho(s) - p = v - p - s^k t = \left(r^k - s^k\right)t.$$

The consumer surplus generated by a monopolist is then an integral over individual surplus contours whose mass is the surface area of an n-sphere with radius s:

$$\begin{aligned} C_n^m(p) &= t\int_{s=0}^{r}\frac{\partial V(s)}{\partial s}\left(r^k - s^k\right)ds = \frac{n\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}t\int_{s=0}^{r}\left(r^k s^{n-1} - s^{k+n-1}\right)ds \\ &= \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}\frac{k}{k+n}tr^{k+n}. \end{aligned}$$

Analogously to the aggregate demand derivation, a duopolist creates consumer surplus

$$\begin{aligned} C_n^d(p,p') &= C_n^m(p) - \int_{r\cos\alpha}^{r} C_{n-1}^m\left(\sqrt{r^2-s^2}\right)ds \\ &= C_n^m(p)\left(1 - \frac{1}{B\left(\frac{1}{2},\frac{n+1}{2}\right)}\frac{k+n}{k+n-1}\int_0^{\alpha}\sin^{k+n}\theta d\theta\right). \end{aligned}$$

It is convenient to restrict k to integer values (1 or 2) so that closed-form expressions are available for the integral as in the demand derivation. (A general solution could be given in terms of Gaussian hypergeometric functions.)

Then

$$C_n^d(p,p') = C_n^m(p)\left(1 - \frac{1}{B\left(\frac{1}{2},\frac{n+1}{2}\right)}\frac{k+n}{k+n-1}\frac{L_{k+n}(p,p')}{rV_{k+n-1}(r)}\right).$$

Specifically,

$$\begin{aligned} C_n^d(p,p')\big|_{k=1} &= C_n^m(p) - \frac{1}{2}\frac{\Gamma\left(\frac{n}{2}\right)}{B\left(\frac{1}{2},\frac{n+1}{2}\right)}L_{n+1}(p,p')\,t \\ C_n^d(p,p')\big|_{k=2} &= C_n^m(p) - \frac{1}{\pi}\Gamma\left(\frac{n}{2}+1\right)L_{n+2}(p,p')\,t, \end{aligned}$$

and total surplus is $S_n(p,p') = 2C_n(p,p') + 2Q_n^d(p,p')\,p$ at symmetric prices.

B Appendix: Angles and Price Symmetry

B.1 Angle Gradients

We collect here the constructions of gradients that appear throughout the paper. With reference to the left and right panels of Figure 9, the Pythagorean identity implies, respectively,

$$\left(-\frac{\partial\alpha}{\partial\delta}\right)^2 = \left(-\frac{\partial\sin\alpha}{\partial\delta}\right)^2 + \left(\frac{\partial\cos\alpha}{\partial\delta}\right)^2 \implies \frac{\partial\alpha}{\partial\delta} = -\frac{1}{\sin\alpha}\frac{\partial\cos\alpha}{\partial\delta}$$

and[24]

$$\left(\pm\frac{\partial\alpha}{\partial r}\right)^2 = \left(\pm\frac{\partial\sin\alpha}{\partial r}\right)^2 + \left(-\pm\frac{\partial\cos\alpha}{\partial r}\right)^2 \implies \frac{\partial\alpha}{\partial r} = -\frac{1}{\sin\alpha}\frac{\partial\cos\alpha}{\partial r}$$

where the ± in the second identity resolves to + if $\alpha+\beta<\pi/2$ and to − if $\alpha+\beta>\pi/2$. When $\alpha+\beta=\pi/2$, the angle is invariant to marginal changes

---

[24] To solve for $\partial\alpha/\partial\delta$, use $\partial\sin\alpha/\partial\alpha = \cos\alpha$ and $\sin^2\alpha+\cos^2\alpha=1$.

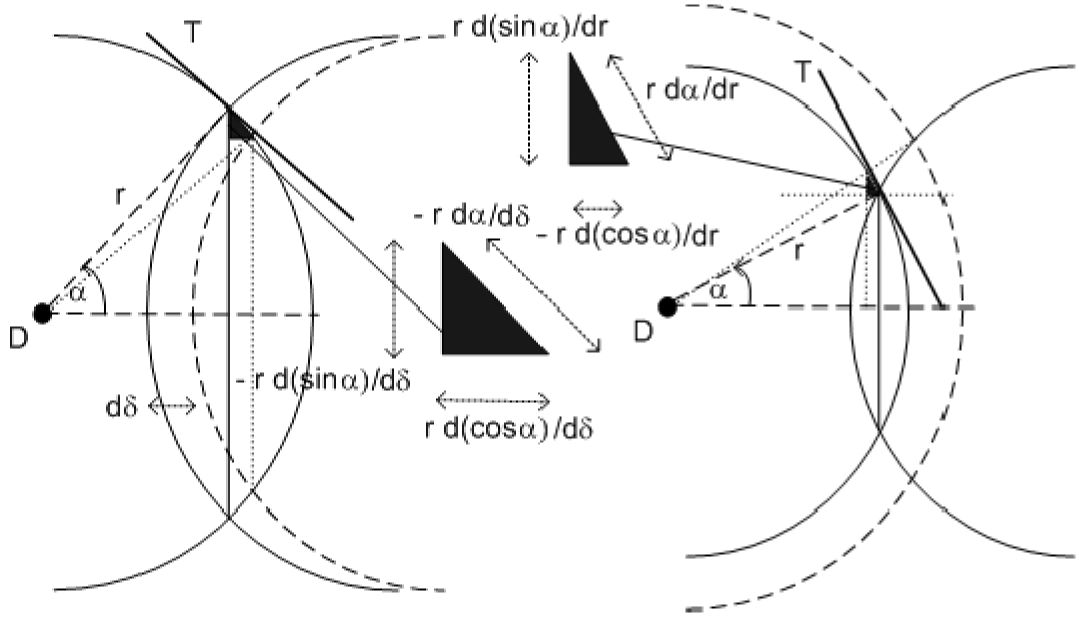

Figure 9: Construction of angle gradients

in radius, so that $\partial \cos\alpha / \partial r = 0$. The sign is determinate in

$$\left(\frac{\partial \alpha}{\partial r'}\right)^2 = \left(\frac{\partial \sin\alpha}{\partial r'}\right)^2 + \left(-\frac{\partial \cos\alpha}{\partial r'}\right)^2 \implies \frac{\partial \alpha}{\partial r'} = -\frac{1}{\sin\alpha}\frac{\partial \cos\alpha}{\partial r'}$$

(not drawn, but the construction is parallel to the right panel).

Per our graphic definition of α (in Figure 2), $r\cos\alpha$ is the distance along $\overline{DD'}$ between D and the indifference hyperplane. It's a subtle but important point to remember that we are in effect *defining* the cosine function *from our model*, and this definition is consistent with the usual one *only* if $k = 2$. For then, the indifferent type on $\overline{DD'}$ is characterized by

$$\begin{aligned} & v - p - (r\cos\alpha)^2\, t = v - p' - (r'\cos\beta)^2\, t \\ \iff \quad & r^2 - (r\cos\alpha)^2 = r'^2 - (r'\cos\beta)^2 \qquad (15) \end{aligned}$$

and, with $\sin^2\alpha+\cos^2\alpha=1$, the equation is equivalent to a law of Euclidean geometry: $r\sin\alpha = r\sin\beta$. The reader may verify that, for any k, the definition of the cosine implicit in Figure 2 satisfies the law of sines if and only if $\sin^k\alpha+\cos^k\alpha=1$.

To avoid having to keep track of different geometries, we always use the familiar cosine defined by (15).[25] Then angles are comparable, gradients (with respect to radius) are invariant to model, and (15) can be solved for

$$\cos\alpha = \frac{\delta^2+r^2-r'^2}{2\delta r}. \tag{16}$$

(because $r\cos\alpha + r'\cos\beta = \delta$). The difference between models is entirely reflected in the behavior of the radius function. Note that $r\cos\alpha = \delta/2$ if prices are equal (thus $r=r'$).

Differentiating (16) gives:

$$\begin{aligned}
\frac{\partial\cos\alpha}{\partial\delta} &= \frac{1}{r}-\frac{1}{\delta}\cos\alpha &\implies\quad \frac{\partial\alpha}{\partial\delta} &= -\frac{\delta - r\cos\alpha}{\delta r\sin\alpha}\\
\frac{\partial\cos\alpha}{\partial r} &= \frac{1}{\delta}-\frac{1}{r}\cos\alpha &\implies\quad \frac{\partial\alpha}{\partial r} &= -\frac{r-\delta\cos\alpha}{\delta r\sin\alpha}\\
\frac{\partial\cos\alpha}{\partial r'} &= -\frac{r'}{\delta r} &\implies\quad \frac{\partial\alpha}{\partial r'} &= \frac{r'}{\delta r\sin\alpha}.
\end{aligned}$$

Corresponding expressions for D' are found by straightforward substitutions. If $p=p'$, the above simplify to $\partial\alpha/\partial\delta = -1/\left(2r\sin\alpha\right)$ and $\partial\alpha/\partial r = -\left(1-2\cos^2\alpha\right)/\left(\delta\sin\alpha\right)$. When $\partial p/\partial\delta = 0$ (on the maximal price path),

$$\left.\frac{\partial^2\alpha}{\partial r\partial\delta}\right|_{p=p'} = \left(\frac{\sin\alpha}{\cos\alpha}\frac{1+2\cos^2\alpha}{1-2\cos^2\alpha}-\frac{\cos\alpha}{\sin\alpha}\right)\frac{\partial\alpha}{\partial r}\frac{\partial\alpha}{\partial\delta}$$

which identifies relationship (11) between α and n.

---

[25] As a result, if x is indifferent and belongs to $\overline{DD'}$ in model $k\neq 2$, then $d\left(x\right)\neq\cos\alpha$ in $r^k-d^k\left(x\right)=r'^k-d'^k\left(x\right)$ because $\cos\alpha$ satisfies (15) and is not redefined within model $k\neq 2$.

B.2 Proof of Symmetric Price Equilibrium, $k = 2$

Existence is argued in the text; here we rule out nonsymmetric price equilibria in the quadratic model. In local monopoly, both designs independently face the same optimization problem with a unique solution, hence choose the same price. We consider therefore the local duopoly case.

If the first-order conditions, $E_n^d(p, p') = E_n'^d(p', p) = 1$, are written as

$$-\frac{1}{nB\left(\frac{1}{2}, \frac{n+1}{2}\right)} Q_n^m r \sin^n \alpha \frac{\partial \alpha}{\partial r} = \frac{1 - E_n^m}{E_n^m} Q_n^d \tag{17}$$

and

$$-\frac{1}{nB\left(\frac{1}{2}, \frac{n+1}{2}\right)} Q_n'^m r' \sin^n \beta \frac{\partial \beta}{\partial r'} = \frac{1 - E_n'^m}{E_n'^m} Q_n'^d, \tag{18}$$

then the left sides are equal by Lemma 2 (i), and because r^n and r'^n can be factored out of Q_n^m and $Q_n'^m$, leaving equal constants and $r^n \sin^n \alpha = r'^n \sin^n \beta$.

The following is then an equilibrium identity:[26]

$$\left(1 - E_n^m\right) E_n'^m Q_n^d = \left(1 - E_n'^m\right) E_n^m Q_n'^d. \tag{19}$$

We will differentiate (19) with respect to p and p' and show a contradiction with $p \neq p'$. To understand this argument, consider that the set of fixed points, which (19) characterizes, is parameterized by the brand distance δ. The solutions to (19) and first-order conditions, only one of which is an additional restriction, are one-dimensional curves in three-dimensional parameter space; there is one degree of freedom to change either p, p' or δ, and

---

[26] At this point we could directly rule out asymmetric equilibria where $\alpha + \beta \geq \pi/2$. For then $\partial\alpha/\partial r$ and $\partial\beta/\partial r'$ are negative by Lemma 2 (ii) and (iii); this implies $E_n^m \leq 1$ and $E_n'^m \leq 1$ (else the left and right sides of (17) and (18) have opposite signs). Since E_n^m is increasing in price (and demand is decreasing in price), $p < p'$ makes the left side of (19) greater than the right side; conversely, $p > p'$ makes the right side greater than the left side. We are, however, mainly interested in the case $\alpha + \beta < \pi/2$ (where symmetric duopoly prices will exceed the monoply price).

satisfy (19) with optimized quantities and elasticities. Because duopoly best responses are continuous and differentiable in p, p' and δ, the fixed-point correspondence is continuous and differentiable (though not necessarily convex-valued) in any one of these variables. Our method of proof is then to show that continuity of the fixed-point correspondence is violated unless $p = p'$ everywhere.[27]

The derivatives of $f(p, p', \delta) = (1 - E_n^m) E_n'^m Q_n^d - (1 - E_n'^m) E_n^m Q_n'^d$ with respect to p and p' are:

$$\left.\frac{df}{dp}\right|_{E_n^d = E_n'^d = 1} = -\left(\frac{p}{E_n^m}\frac{dE_n^m}{dp} + (1 - E_n^m)\left(1 + \frac{p}{Q_n'^d}\frac{dQ_n'^d}{dp}\right)\right)\frac{Q_n^d E_n'^m}{p} = 0 \tag{20}$$

$$\left.\frac{df}{dp'}\right|_{E_n^d = E_n'^d = 1} = \left(\frac{p'}{E_n'^m}\frac{dE_n'^m}{dp'} + (1 - E_n'^m)\left(1 + \frac{p'}{Q_n^d}\frac{dQ_n^d}{dp'}\right)\right)\frac{Q_n'^d E_n^m}{p'} = 0, \tag{21}$$

where we use the equilibrium condition $E_n^d = 1 \iff \partial Q_n^d/\partial p = -Q_n^d/p$ to substitute for $\partial Q_n^d/\partial p$, (analogously for $\partial Q_n'^d/\partial p'$), as well as (19) to factor out demand functions.[28] The rightmost bracketed terms are the cross-price

---

[27] The usual strategy is to construct the derivative dp/dp' of the best response function (via the implicit function theorem) and bound it between -1 and 1. When best response functions are symmetric, any intersection is mirrored on the other side of the 45° line in orthogonal direction. To produce an asymmetric equilibrium, i.e. one with its mirror image not on the 45° line itself, the function must cross a line with slope -1 twice, hence it must be steeper than -1 somewhere. But this approach turns out to be very tedious here.

[28] For the purpose of differentiation, note that free elasticities are constant with respect to rival price.

elasticities[29]

$$\begin{aligned}\frac{p'}{Q_n^d}\frac{dQ_n^d}{dp'} &= -\left(1-E_n^m\right)\frac{E_n'^m}{E_n^m}\frac{\partial\alpha/\partial r'}{\partial\beta/\partial r'}\\ \frac{p}{Q_n'^d}\frac{dQ_n'^d}{dp} &= -\left(1-E_n'^m\right)\frac{E_n^m}{E_n'^m}\frac{\partial\beta/\partial r}{\partial\alpha/\partial r};\end{aligned}$$

the leftmost terms are "elasticities of free elasticity":[30]

$$\begin{aligned}\frac{dE_n^m}{dp} &= \left(1+\frac{k}{n}E_n^m\right)\frac{E_n^m}{p}\\ \frac{dE_n'^m}{dp'} &= \left(1+\frac{k}{n}E_n'^m\right)\frac{E_n'^m}{p'}.\end{aligned}$$

---

[29] Rival price affects demand only through the angle, so

$$\begin{aligned}\frac{dQ_n^d}{dp'} &= -\frac{1}{B\left(\frac{1}{2},\frac{n+1}{2}\right)}Q_n^m\sin^n\alpha\left(\partial\alpha/\partial r'\right)\left(\partial r'/\partial p'\right);\\ \frac{dQ_n^d}{dp} &= -\frac{Q_n'^m}{B\left(\frac{1}{2},\frac{n+1}{2}\right)}\sin^n\beta\left(\partial\beta/\partial r\right)\left(\partial r/\partial p\right).\end{aligned}$$

Multiplying these expressions respectively by $\left(\partial\beta/\partial r'\right)/\left(\partial\beta/\partial r'\right)$ and $\left(\partial\alpha/\partial r\right)/\left(\partial\alpha/\partial r\right)$, and substituting from $Q_n^m\sin^n\alpha=Q_n'^m\sin^n\beta$ and (17),(18), (19), and (5), one arrives at:

$$\begin{aligned}\frac{dQ_n^d}{dp'} &= -\left(1-E_n^m\right)\frac{E_n'^m}{E_n^m}\frac{\partial\alpha/\partial r'}{\partial\beta/\partial r'}\frac{Q_n^d}{p'}\\ \frac{dQ_n'^d}{dp} &= -\left(1-E_n'^m\right)\frac{E_n^m}{E_n'^m}\frac{\partial\beta/\partial r}{\partial\alpha/\partial r}\frac{Q_n'^d}{p}.\end{aligned}$$

[30] Directly from the definition (5) of free elasticity,

$$\begin{aligned}\frac{dE_n^m}{dp} &= \left(1+\frac{1}{n}E_n^m+\frac{\partial^2r/\partial p^2}{\partial r/\partial p}p\right)\frac{E_n^m}{p}\\ \frac{dE_n'^m}{dp'} &= \left(1+\frac{1}{n}E_n'^m+\frac{\partial^2r'/\partial p'^2}{\partial r'/\partial p'}p'\right)\frac{E_n'^m}{p'}.\end{aligned}$$

Substitute

$$\frac{\partial^2r/\partial p^2}{\partial r/\partial p}p=\frac{k-1}{k}\frac{p}{v-p}=\frac{k-1}{n}E_n^m,$$

and analogously for $\left(\partial^2r'/\partial p'^2\right)/\left(\partial r'/\partial p'\right)$.

Thus (20) and (21) separately imply

$$\begin{aligned} 2\frac{E'^m_n}{E^m_n} - \frac{n-k}{n} E'^m_n &= \left(1 - E^m_n\right)\left(1 - E'^m_n\right) \frac{\partial \beta / \partial r}{\partial \alpha / \partial r} \\ 2\frac{E^m_n}{E'^m_n} - \frac{n-k}{n} E^m_n &= \left(1 - E^m_n\right)\left(1 - E'^m_n\right) \frac{\partial \alpha / \partial r'}{\partial \beta / \partial r'}, \end{aligned}$$

and together:

$$\frac{2 - \frac{n-k}{n} E^m_n}{2 - \frac{n-k}{n} E'^m_n} \frac{\left(E'^m_n\right)^2}{\left(E^m_n\right)^2} = \frac{\partial \beta / \partial r}{\partial \alpha / \partial r} \frac{\partial \beta / \partial r'}{\partial \alpha / \partial r'}. \tag{22}$$

Since[31]

$$\frac{\partial \beta / \partial r}{\partial \alpha / \partial r} \frac{\partial \beta / \partial r'}{\partial \alpha / \partial r'} = \frac{r^2}{r'^2} = \frac{E'^m_n}{E^m_n} \frac{p}{p'},$$

(22) simplifies to:

$$\frac{2 - \frac{n-k}{n} E^m_n}{2 - \frac{n-k}{n} E'^m_n} \frac{E'^m_n}{E^m_n} = \frac{p}{p'}.$$

If $p < p$, then the left side must be smaller than 1. But this is not possible, given $E'^m_n > E^m_n$ and that $2 - \frac{n-2}{n} E'^m_n$ and $2 - \frac{n-2}{n} E^m_n > 0$ cannot have opposite signs, from nonnegativity of prices. Only $p = p'$ satisfies the reduced equilibrium condition.

■

---

[31] Using the expressions for $\partial \alpha / \partial r$ and $\partial \alpha / \partial r'$ in Appendix B,

$$\frac{\partial \alpha}{\partial r} = -\frac{r - \delta \cos \alpha}{\delta r \sin \alpha}, \ \frac{\partial \alpha}{\partial r'} = \frac{r'}{\delta r \sin \alpha}, \ \frac{\partial \beta}{\partial r'} = -\frac{r' - \delta \cos \beta}{\delta r' \sin \beta}, \ \frac{\partial \beta}{\partial r} = \frac{r}{\delta r' \sin \beta}.$$

Multiply

$$\frac{\partial \alpha / \partial r'}{\partial \beta / \partial r'} = -\frac{r'}{r' - \delta \cos \beta}, \ \frac{\partial \beta / \partial r}{\partial \alpha / \partial r} = -\frac{r}{r - \delta \cos \alpha}$$

(from $r \sin \alpha = r' \sin \beta$) respectively by r' and r; expand $r \cos \beta$ to $\delta - r \cos \alpha$, then substitute $\cos \alpha = \left(\delta^2 + r^2 - r'^2\right) / \left(2 \delta r\right)$ to get:

$$\frac{\partial \alpha / \partial r'}{\partial \beta / \partial r'} = \frac{2 r'^2}{\delta^2 - r^2 - r'^2}, \ \frac{\partial \beta / \partial r}{\partial \alpha / \partial r} = \frac{2 r^2}{\delta^2 - r^2 - r'^2}.$$