Bid Increments in Second-Price Sealed Bid Auctions[1]

Abstract: This note concerns bidding in a hybrid first-price and second-price auction. The winning bidder sometimes pays his bid and sometimes pays an amount determined by the next highest bid. In internet auctions where bidders wait until the end of the auction to bid the auction reduces to a sealed-bid auction and the bid function we derive may be relevant in such cases.

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[1]Richard Guy Cox, University of Arkansas at Little Rock, Department of Economics and Finance, Little Rock, AR 72204, USA; email: rgcox@ualr.edu; ph: (501) 569-8875.

1. Motivation

Recently some economists have focused their attention on explaining the phenomenon of sniper bidding on eBay (see for example Ockenfels and Roth 2003, Roth and Ockenfels 2002, Wang 2003, and Bajari and Hortaçsu 2004). While I do not address the issue of sniper bidding here I note that when bidders wait until the final moments of an auction to bid, the ascending auction becomes a sealed-bid auction. Simply, when bidders snipe there are no further opportunities to respond to previous bids. This suggests that sniper bids may deviate from the bid-your-valuation bids suggested by eBay as the ideal strategy. However, to see this we must first establish that the equilibrium bids in a sealed-bid auction with eBay's price determination rules are below the bidders' valuations. I demonstrate this for an independent private values model.

In the pure second-price sealed-bid auction the bid-your-valuation strategy is an equilibrium. Under the rules at eBay sometimes the winner pays his bid and sometimes he pays an amount determined by the second highest bid. The actual amount depends on the specific bidding increment and how close the winning bid and next highest bid are to each other. Accordingly, the bid increment plays an important role in the result. In fact, I provide an example in which the equilibrium bids are below the bidders' corresponding valuations and that this distance is between the bids and valuations is greater than the bid increment. Accordingly, if bidders submit comparable early bids then there is the possibility that they may submit new bids which implies that waiting until the end of the auction to bid could be preferable for bidders when all bidders wait. Chwe (1989) addresses the question of bid increments in first-price auctions where bids are drawn from a set. I am not aware of results for the first- second-price hybrid case where bids are drawn from a continuum. The model and results follow.

2. Model

There is an auction for a single item. There are $n \geq 2$ risk neutral bidders. Let v_i be the valuation of bidder $i \in \{1, \ldots, n\}$. The valuations are independently and identically distributed and represent draws from the cumulative distribution function F with support $[\underline{v}, \overline{v}]$. The cdf F is twice differentiable and strictly increasing over its support without any atoms. The realized valuations are private knowledge while F is common knowledge.

The auctioneer establishes a reservation price or screening level r. Bidder i's bid, B_i must exceed r; otherwise, the bidder does not participate. All bidders simultaneously submit their bids. Without loss of generality we order the bids $B_n > B_{n-1} > \ldots > B_1$. If only one bidder participates then $B_{n-1} = r$. If we can find a monotonic bid function, B : supp $F \rightarrow$ supp F, then the probability that any 2 bids are equal when F has no atoms is 0 and I ignore the possibility of a tie. The high bidder, $i = n$, receives the object after the bids are submitted. Let $\Delta \in \mathbb{R}_+$ be the bidding increment. The payoff to bidder i is $v_i - \min\{B_n, B_{n-1} + \Delta\}$ if he wins, $i = n$, and 0 otherwise. In words, the high bidder will sometimes pay his own bid and will sometimes pay the second highest bid plus Δ. The transaction price depends on the bidder's bid with some positive probability. In this way the standard argument that in a second-price auction a bidder gains nothing by lowering his bid does not apply here.

3. Results

Building on the results of Riley and Samuelson (1981) we can derive a strictly increasing equilibrium bid function which is applicable to both first- and second-price auctions. For $r < \underline{v}$ we specify that a bidder with valuation $\underline{v}$ bids his valuation.

Riley and Samuelson (1981) show that in equilibrium the expected payment $E(v)$ must satisfy

$$E'(v_i) = v_i(n-1)F(v_i)^{n-1}F'(v_i).$$

The solution to the differential equation is

$$E(v_i) = v_i F(v_i)^{n-1} - \int_r^{v_i} F(x)^{n-1} dx,$$

where $v_i \geq r$.

In this auction we must calculate the expected payment in two separate cases. Let $\hat{v}$ satisfy $B(\hat{v}) = \max\{r + \Delta, \underline{v} + \Delta\}$. In the first case $v_i < \hat{v}$. This auction is a first-price auction from this bidder's point of view. The expected payment is

$$B_i G(B)^{n-1},$$

where, G is the distribution of equilibrium bids. Note that a strictly increasing bid function implies $G(B(v)) = F(v)$. In this case the bid function is just

$$B(v \leq \hat{v}) = v_i - \frac{1}{F(v_i)^{n-1}} \int_r^{v_i} F(x)^{n-1} dx.$$

The second case to consider is that of high signal bidders, where $v_i > \hat{v}$. In this case the expected payment can be written by recognizing what happens on two intervals. Let $B_{n-1:n}$ denote the the second highest bid when there are n bids. On the interval $B_{n-1:n} \in [r, B - \Delta]$ the bidder pays the expected value of the second highest bidder's bid plus the bid increment Δ, while on the interval $B_{n-1:n} \in (B - \Delta, B)$ the winner pays his bid. Thus we have

$$\begin{aligned}E(v) =&BG(B)^{n-1}\left[1 - \frac{G(B-\Delta)^{n-1}}{G(B)^{n-1}}\right] \\ &+ \int_r^{B-\Delta} (x+\Delta)(n-1)G(x)^{n-2}G'(x)dx \\ =&BG(B)^{n-1} - BG(B-\Delta)^{n-1} \\ &+ (B-\Delta)G(B-\Delta)^{n-1} - \int_r^{B-\Delta} G(x)^{n-1}dx + \Delta G(B-\Delta)^{n-1} \\ =&BG(B)^{n-1} - \int_r^{B-\Delta} G(x)^{n-1}dx.\end{aligned}$$

Notice that the expectation using the pdf for the truncated order statistic where $i = n - 1$ times the probability of winning gives the same expression as the expectation using the pdf for the order statistic where $i = n$ which we use above. (The distribution of the order statistic is found in Arnold, Balakrishnan, and Nagaraja 1992).

Now we can solve for the bid function for the high valuation bidders and obtain

$$B(v > \hat{v}) = v - \frac{1}{F(v)^{n-1}}\left[\int_r^v F(x)^{n-1}dx - \int_r^{B-\Delta} G(x)^{n-1}dx\right], \qquad (1)$$

where we make use of the fact that $G(B) = F(v), \forall v \geq r$.

Because B is an element of a compact convex subset of $\mathbb{R}$ and the bid function is continuous then it has a fixed point. Actual solutions could be obtained by using a Newton-Raphson method and piece-wise evaluation of G beginning at the distribution for low signal bidders. However, in general G is unknown and we must obtain solutions by construction.

The bid function itself is a straightforward extension of the results of Riley and Samuelson (1981). However, we can now specify how B changes as Δ changes and provide an example of bids that are "far" from bidders' true valuations as a relevant argument for sniper bidding.

By the mean value theorem we can rewrite (1) as

$$B = v - \frac{1}{F(v)^{n-1}} \int_r^v F(x)^{n-1} dx + G(c)^{n-1}(B - \Delta - r),$$

where $c \in [r, B-\Delta]$ is the solution to the mean value theorem. Then, rearranging we have,

$$B = \frac{1}{1-G(c)^{n-1}} \left[v - \frac{1}{F(v)^{n-1}} \int_r^v F(x)^{n-1} dx + G(c)^{n-1}(-\Delta - r) \right],$$

and the derivative is,

$$\frac{\partial B}{\partial \Delta} = -\frac{G(c)^{n-1}}{1-G(c)^{n-1}}.$$

Now consider a simple example where $n = 2$ and v is uniform on $[0,1]$. Table 1 provides bids for several bidder valuations and bid increments. We observe that some bids are less than the bidder's valuation by an amount greater than Δ. This suggests that if bidders do not wait until the end of the auction to bid, and bid according to (1), then they face the possibility of submitting larger bids later in the auction. If everyone waits until the end of the auction to bid, then equilibrium bids will be lower than otherwise.

Table 1. Bids

Δ	$v = .2$	$v = .25$	$v = .3$
0	.2000	.2500	.3000
.01	.1900	.2400	.2900
.05	.1355	.2078	.2566
.1	.1000	.1282	.1634
.15	.1000	.1250	.1500

Then in this simple setting the first and second order statistics for the bids when $\Delta = .1$ are .5690 and .1921 respectively. The expected revenue is .2921 while the sealed bid auction with $\Delta = 0$ has expected revenue of .3333. If an English auction proceeds with .1 unit jumps and bidders alternate then half the time the high valuation bidder starts the auction and half the time the low valuation bidder begins the auction and the expected revenue is .35. Because the distribution G is unknown, I cannot make more general observations on revenue. The result suggests that converting the ascending auction with a fixed closing time to a sealed-bid auction by waiting until the last moment to bid implies that equilibrium bids will fall below the bidders' true valuations without allowing the opportunity for new feasible bids. As such, equilibrium bids would be lower than when bidders apply eBay's suggested bid-your-valuation strategy. However, this does not imply that waiting until the last moment to bid is an equilibrium.

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