EQUILIBRIA IN EXCHANGE ECONOMIES WITH FINANCIAL CONSTRAINTS: BEYOND THE CASS TRICK

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ABSTRACT. We consider an exchange economy under incomplete financial markets with purely financial securities and finitely many agents. When portfolios are not constrained, Cass [?], Duffie [?] and Florenzano–Gourdel [?] proved that arbitrage-free security prices fully characterize equilibrium security prices. This result is based on a trick initiated by Cass [?] in which one unconstrained agent behaves as if he were in complete markets. This approach is unsatisfactory since it is asymmetric and no more valid when every agent is subject to frictions. We propose a new and symmetric approach to prove that arbitrage-free security prices still fully characterize equilibrium security prices in the more realistic situation where the financial market is constrained by convex restrictions, provided that financial markets are *collectively frictionless*.

KEYWORDS: Exchange economies, incomplete financial markets, purely financial securities, nominal assets, constrained portfolios, collectively frictionless financial markets, equilibrium security prices, arbitrage-free security prices.

JEL CLASSIFICATION: C62, D52, G10.

1. INTRODUCTION

We consider an exchange economy under incomplete financial markets with purely financial securities (nominal assets) and finitely many agents. By definition, a security price is arbitrage-free if any (unconstrained) portfolio does not yield a positive non-zero income. For frictionless financial markets, Cass [?], Duffie [?] and Florenzano–Gourdel [?] proved that arbitrage-free security prices fully characterize equilibrium security prices, in the sense that each equilibrium security price is arbitrage-free and each arbitrage-free security price can be embedded as an equilibrium security price. This result comes from the ability of commodity prices to adjust themselves to clear both commodity and financial markets. The trick initiated by Cass [?] and exploited by Duffie–Shafer [?], Duffie [?], Magill–Shafer [?], Florenzano–Gourdel [?], Rahi [?], Magill–Quinzii [?] among others is based upon the fact that one agent (this agent needs to have an unconstrained portfolio set) behaves at equilibrium as if he were in complete markets.

The requirement of a frictionless financial market is a highly idealized condition. In our model, the financial market is subject to frictions in the sense that agents are constrained

Date: March 29, 2005.

by convex restrictions on possible portfolio holdings. Luttmer [?] and Elsinger–Summer [?] illustrate that these constraints on portfolio sets are suitable for describing market frictions such as short-selling constraints and buying floors, margin and collateral requirements, bid-ask spreads and taxes, and proportional transaction costs.

Technically the approach initiated by Cass is unsatisfactory since it is asymmetric: there must be an agent (called Arrow–Debreu agent) who is unconstrained and who is the only one to face a demand correspondence in complete markets. Moreover the Cass trick is no longer valid in markets with frictions since constrained portfolio sets preclude the presence of an Arrow-Debreu agent. This paper is dedicated to the following two linked questions raised by P. Courrège :

- Is it possible to provide a symmetric approach ?
- Under which conditions on frictions is it possible to fully characterize the set of equilibrium security prices by arbitrage-free security prices ?

We prove in this paper that the answer to the first question is yes. We propose a new and symmetric proof: an *ad hoc* commodity price dependent security is artificially introduced in the financial market. Each agent is allowed to purchase a small amount of this security. The introduction of this *ad hoc* security ensures that for any commodity price and at least for one agent, his budget set has a non-empty interior. In particular the aggregate excess demand correspondence explodes if commodity prices hit the boundary of the price simplex. At equilibrium, we prove that no agent actually purchases the *ad hoc* security.

We also provide in this paper an answer to the second question : it is possible to fully characterize the set of equilibrium security prices by arbitrage-free security prices provided that the financial market is *collectively frictionless* in the sense that for any payoff achieved by an unconstrained portfolio there exists an agent for which this payoff is achieved by a portfolio belonging to his portfolio set. In other words, the union of all payoff sets covers the whole space of unconstrained payoffs. It is straightforward to check that any equilibrium security price is arbitrage-free provided that the financial market is collectively frictionless. The main contribution of our work is to prove that the converse is true. In fact, we prove that any arbitrage-free security price can be embedded in an equilibrium, provided that the financial market is *locally collectively frictionless* in the sense that for any payoff achieved by a portfolio, there exists an agent for which a proportion of this payoff is achieved by a constrained portfolio.

There is an abundant literature on asset pricing with frictions, but only a few studies explore the existence issue: Werner [?, ?], Siconolfi [?], Balasko–Cass–Siconolfi [?], Benveniste–Ketterer [?], Polemarchakis–Siconolfi [?] and Cass–Siconolfi–Villanacci [?]. As far as we know, this work is the first one to investigate the validity, in markets with frictions, of the full characterization of equilibrium security prices by arbitrage-free security prices.

The paper is organized as follows. In the next section, a model of an exchange economy with *general* purely financial markets is described and existence results are enunciated. In Section **??** the existence results are expressed in terms of the two usual examples. The last section is devoted to proving the main result. Some proofs are referred to the appendix.

2. The General Model

We propose in this section a model of exchange economies with *general* purely financial markets in the sense that uncertainty is represented by a set of possible states of nature without specifying the intertemporal structure. The two-period intertemporal model (see Cass [**?**], Werner [**?**, **?**], Magill–Shafer [**?**], Magill–Quinzii [**?**] and Florenzano [**?**]) and the multi-period intertemporal model (see Duffie [**?**], Florenzano–Gourdel [**?**] and Magill–Quinzii [**?**]) are presented in Section **??** as special cases of our general model.

2.1. **Exchange economies with *general* purely financial markets.** We consider a triple (Σ, I, F) where Σ and I are finite sets and F is a finite dimensional vector space. Each $\sigma \in \Sigma$ represents a state (of nature), each $i \in I$ represents an agent and each $\theta \in F$ represents a portfolio.

A linear operator W from F to $\mathbb{R}^\Sigma$ is called a payoff operator. For each portfolio $\theta \in F$, $W\theta$ denotes the image of θ, which as an element of $\mathbb{R}^\Sigma$ is denoted by $(W\theta(\sigma), \sigma \in \Sigma)$, each $W\theta(\sigma)$ representing the payoff at state $\sigma \in \Sigma$. A portfolio structure is a family $\Theta^\bullet = (\Theta^i, i \in I)$ where for each $i \in I$, Θ^i is a subset of F: for each $i \in I$, the set Θ^i represents the portfolio (restriction) set for agent i and the payoff set $W\Theta^i = \{W\theta \ : \ \theta \in \Theta^i\}$ represents the set of payoffs available for agent i. A financial market is a pair $\mathcal{E}^f = (W, \Theta^\bullet)$ where W is a payoff operator and $\Theta^\bullet$ is a portfolio structure.

A consumption market is a triple $(E, X^\bullet, u^\bullet)$ where E is a finite dimensional vector space, $X^\bullet$ is a family $(X^i, i \in I)$ with X^i subset of E^Σ and $u^\bullet$ is a family $(u^i, i \in I)$ with u^i real function from X^i to $\mathbb{R}$. The space E represents the commodity space and the dual E' the price space. A vector in E represents a consumption bundle for an agent and a vector in E^Σ a consumption plan. A vector in E' represents a spot price and a vector in $(E')^\Sigma$ a commodity price (system). For each agent $i \in I$, the set X^i represents the consumption set and the function u^i the utility function. If $x \in X^i$ we denote by $P^i(x)$ the set of strictly preferred consumption plans by agent $i \in I$, i.e.

$$P^i(x) = \{y \in X^i \ : \ u^i(y) > u^i(x)\}.[1]$$

Definition 2.1. An (exchange) economy (with purely financial markets) is here a pair

$$\mathcal{E} = (\mathcal{E}^c, \mathcal{E}^f) = (E, X^\bullet, u^\bullet, W, \Theta^\bullet)\,,$$

[1] In this paper, preferences are represented by utility functions. This restriction is not due to the approach we propose. For a more general result with non-complete and non-transitive preferences, we refer to Martins-da-Rocha–Triki [**?**].

where $\mathcal{E}^c = (E, X^\bullet, u^\bullet)$ is a consumption market and $\mathcal{E}^f = (W, \Theta^\bullet)$ is a financial market.

Let $\mathcal{E} = (E, X^\bullet, u^\bullet, W, \Theta^\bullet)$ be an economy. For each commodity price $p \in (E')^\Sigma$ and each consumption plan $x \in E^\Sigma$, we define the vector $p \Box x \in \mathbb{R}^\Sigma$ by

$$p \Box x = (\langle p(\sigma), x(\sigma)\rangle, \sigma \in \Sigma) \in \mathbb{R}^\Sigma$$

where $\langle .,. \rangle : E' \times E \to \mathbb{R}$ is the natural duality. The vector $p \Box x$ represents the values following $\sigma \in \Sigma$ of the consumption plan x under the commodity price p. Given a commodity price $p \in (E')^\Sigma$, we say that a portfolio $\theta \in F$ finances a consumption plan $x \in E^\Sigma$ if $p \Box x \leqslant W\theta$, in the sense that

$$\forall \sigma \in \Sigma, \quad \langle p(\sigma), x(\sigma)\rangle \leqslant W\theta(\sigma).$$

A pair $(x, \theta) \in E^\Sigma \times F$ is called a budget feasible plan for agent i if the consumption plan x belongs to X^i, the portfolio θ belongs to Θ^i and finances x. Given a commodity price $p \in (E')^\Sigma$, the budget set $B^i(p)$ of agent i is the set of all budget feasible plans for i, i.e.

$$B^i(p) = \{(x, \theta) \in X^i \times \Theta^i \ : \ p \Box x \leqslant W\theta\}.$$

A consumption allocation $x^\bullet = (x^i, i \in I)$ is a family of consumption plans $x^i \in E^\Sigma$. A portfolio allocation $\theta^\bullet = (\theta^i, i \in I)$ is a family of portfolios $\theta^i \in F$. A budget feasible plan (x^i, θ^i) for agent i is optimal if there is no other budget feasible plan (y, η) for i such that y is strictly preferred to x^i, i.e. $[P^i(x^i) \times \Theta^i] \cap B^i(p) = \emptyset$.

Definition 2.2. A triple $(p, x^\bullet, \theta^\bullet)$ is an **equilibrium** for the economy $\mathcal{E}$ if $x^\bullet = (x^i, i \in I)$ is a consumption allocation, $\theta^\bullet = (\theta^i, i \in I)$ is a portfolio allocation and p is a commodity price such that

(i) for each $i \in I$, $(x^i, \theta^i) \in B^i(p)$ and $[P^i(x^i) \times \Theta^i] \cap B^i(p) = \emptyset$,
(ii) $\sum_{i \in I} x^i = 0$,
(iii) $\sum_{i \in I} \theta^i = 0$.

If condition (iii) is replaced by the following condition

(iii') $\sum_{i \in I} W\theta^i = 0$,

the family $(p, x^\bullet, \theta^\bullet)$ is then called a **weak equilibrium**.

Obviously an equilibrium is a weak equilibrium. The following Proposition **??** provides a condition for which the converse is true. If W is a payoff operator, then we denote by $\operatorname{Ker} W$ the kernel of W, i.e. $\operatorname{Ker} W = \{\theta \in F \ : \ W\theta = 0\}$. If Z is a closed convex subset of F then $\operatorname{As}(Z)$ denotes the asymptotic cone defined by $\operatorname{As}(Z) = \{v \in F \ : \ Z + \{v\} \subset Z\}$.

Proposition 2.1. *Let $\mathcal{E}$ be an economy.*

(a) *If the following condition is satisfied*

$$\operatorname{Ker} W \subset \bigcup_{i \in I} \operatorname{As}(\Theta^i), \tag{2.1}$$

then there exists an equilibrium as soon as there exists a weak equilibrium.

(b) *Condition* (**??**) *is satisfied if either* $\operatorname{Ker} W = \{0\}$ *or for every* $i \in I$*, the set* Θ^i *is a closed convex subset of* F *containing* 0 *such that* $\bigcup_{i\in I} \Theta^i = F$*.*

The proof of Proposition **??** is postponed to Appendix **??**. We refer to Won–Hahn [**?**] where it is shown how redundant assets contribute to risk sharing in a nontrivial manner.

2.2. **Assumptions.** Consider an economy $(E, X^\bullet, u^\bullet, W, \Theta^\bullet)$. Let $X = \prod_{i\in I} X^i$, let $\widehat{X}$ be the set of attainable consumption allocations, i.e.

$$\widehat{X} = \left\{ x^\bullet = (x^i, i \in I) \in X \;:\; \sum_{i\in I} x^i = 0 \right\}$$

and for each $i \in I$, let $\widehat{X}^i$ be the projection of $\widehat{X}$ on X^i.

Assumption (C)**.** For every agent $i \in I$:

C.1 the consumption set X^i is closed convex and $\widehat{X}^i$ is compact in E^Σ;
C.2 the utility function u^i is continuous and quasi-concave;
C.3 the vector 0 belongs to the interior of X^i;
C.4 for every attainable consumption allocation $x^\bullet \in \widehat{X}$, for every $\sigma \in \Sigma$, there exists $y \in X^i$, differing from x^i only at σ, such that $u^i(y) > u^i(x^i)$.

Assumption (F)**.** For every agent $i \in I$, the set $W\Theta^i$ is a closed convex subset of $\mathbb{R}^\Sigma$ containing 0.

If Θ is a subset of F, we let $\operatorname{As}(\Theta) = \{v \in F : \Theta + v \subset \Theta\}$. Note that if for each $i \in I$, Θ^i is closed convex containing 0, then Assumption F is satisfied if for each $i \in I$, either Θ^i is a finitely generated cone or $\operatorname{Ker} W \cap \operatorname{As}(\Theta^i) = \{0\}$.

Definition 2.3. A consumption market $(E, X^\bullet, u^\bullet)$ is said *standard* if Assumption C is satisfied. A financial market $(W, \Theta^\bullet)$ is said *standard* if Assumption F is satisfied . An economy $\mathcal{E} = (\mathcal{E}^c, \mathcal{E}^f)$ is said *standard* if the consumption market $\mathcal{E}^c = (E, X^\bullet, u^\bullet)$ and the financial market $\mathcal{E}^f = (W, \Theta^\bullet)$ are standard.

2.3. **The existence result.** Before presenting the existence results, we set the definition of collectively frictionless financial markets. If A is a subset of F then we denote by $\operatorname{cone} A$ the cone generated by A in the sense that $\operatorname{cone} A = \{\lambda a \;:\; a \in A \quad \text{and} \quad \lambda \geqslant 0\}$. If W is a payoff operator from F to $\mathbb{R}^\Sigma$, then $\operatorname{Im} W$ denotes the image of W in the sense that $\operatorname{Im} W = \{t \in \mathbb{R}^\Sigma \;:\; \exists \theta \in F,\; t = W\theta\}$.

Definition 2.4. A financial market $(W, \Theta^\bullet)$ is said

(i) **frictionless** if

$$\forall i \in I, \quad W\Theta^i = \operatorname{Im} W;$$

(ii) **collectively frictionless** if

$$\bigcup_{i\in I} W\Theta^i = \operatorname{Im} W;$$

(iii) **locally collectively frictionless** if

$$\operatorname{cone} \bigcup_{i\in I} W\Theta^i = \operatorname{Im} W.$$

In other words, a financial market $(W, \Theta^\bullet)$ is collectively frictionless if for any payoff t in $\operatorname{Im} W$, there exists an agent $i \in I$ for which t belongs to his payoff set $W\Theta^i$; and it is locally collectively frictionless if for any payoff t in $\operatorname{Im} W$, there exist an agent $k \in I$ and $\lambda > 0$ such that λt belongs to $W\Theta^k$. Note that if $\Theta^\bullet$ is such that $\cup_{i\in I}\Theta^i = F$ then any financial market $(W, \Theta^\bullet)$ is collectively frictionless and if $\Theta^\bullet$ is such that 0 belongs to the interior of $\cup_{i\in I}\Theta^i$ then any financial market $(W, \Theta^\bullet)$ is locally collectively frictionless.

Definition 2.5. A standard financial market $\mathcal{E}^f$ is said *viable* if for every standard consumption market $\mathcal{E}^c$, there exists a weak equilibrium for the economy $\mathcal{E} = (\mathcal{E}^c, \mathcal{E}^f)$.

A vector $t = (t(\sigma), \sigma \in \Sigma)$ in $\mathbb{R}^\Sigma$ is said non-negative, denoted by $t \geqslant 0$ if for each $\sigma \in \Sigma$, $t(\sigma) \geqslant 0$. The set of non-negative vectors is denoted by $\mathbb{R}^\Sigma_+$. A vector $t \in \mathbb{R}^\Sigma$ is said positive, denoted by $t > 0$, if $t \neq 0$ and if it is non-negative.

Definition 2.6. A standard financial market $(W, \Theta^\bullet)$ *precludes arbitrage opportunities* if for each $i \in I$, there is no $t \in \operatorname{As}(W\Theta^i)$ such that t is positive, i.e.

$$\mathbb{R}^\Sigma_+ \cap \bigcup_{i\in I} \operatorname{As}(W\Theta^i) = \{0\}.$$

Proposition 2.2. *If a standard financial market is viable, then it precludes arbitrage opportunities.*

The proof of Proposition **??** is standard and postponed to Appendix **??**.

Definition 2.7. A payoff operator W is said arbitrage-free if there is no $t \in \operatorname{Im} W$ such that t is positive, i.e.

$$\mathbb{R}^\Sigma_+ \cap \operatorname{Im} W = \{0\}.$$

If W is an arbitrage-free payoff operator then any standard[2] financial market $(W, \Theta^\bullet)$ precludes arbitrage opportunities. Furthermore, if a financial market $(W, \Theta^\bullet)$ is frictionless and precludes arbitrage opportunities, then the payoff operator W is arbitrage-free. We prove in the following proposition that this equivalence is still valid if financial markets are collectively frictionless.

Proposition 2.3. *If a standard financial market $(W, \Theta^\bullet)$ is collectively frictionless and precludes arbitrage opportunities, then the payoff operator W is arbitrage-free.*

[2]In fact it is sufficient to assume that for each $i \in I$, $0 \in \Theta^i$.

The proof of Proposition **??** is in Appendix **??**. The relationship between the concept of arbitrage-free and viable financial markets are made explicit by Theorem **??** and Corollary **??**.

Theorem 2.1. *Let $(W, \Theta^\bullet)$ be a standard and locally collectively frictionless financial market. If W is arbitrage-free then $(W, \Theta^\bullet)$ is viable.*

Section **??** is dedicated to the proof of Theorem **??**. Combining Propositions **??**–**??** and Theorem **??**, we get a complete description of the set of viable financial markets.

Corollary 2.1. *Consider a standard financial market that is collectively frictionless, then the following assertions are equivalent :*

- *The financial market is viable.*
- *The financial market precludes arbitrage opportunities.*
- *The payoff operator is arbitrage-free.*

3. Application to intertemporal models

We first set some notations. If K is a finite then for every x, y in $\mathbb{R}^K$, $x \cdot y = \sum_{k \in K} x(k)y(k)$. If L is a finite set and z belongs to $\mathbb{R}^{K \times L}$ then for every $k \in K$, we let $z(k) = (z(k,\ell), \ell \in L) \in \mathbb{R}^L$ and for every $\ell \in L$, we let $z'(\ell) = (z(k,\ell), k \in K) \in \mathbb{R}^K$.

3.1. **The two-period intertemporal model.** The first example of our general model of exchange economies with purely financial markets (Section **??**) is the two-period intertemporal model studied by Cass [**?**], Werner [**?**, **?**], Magill–Shafer [**?**], Magill–Quinzii [**?**] and Florenzano [**?**]. In the two-period model the triple (Σ, I, F) and the payoff operator are specified as follows:

(a) $\Sigma = \{0\} \cup S$ where S is a finite set not containing 0.
(b) $F = \mathbb{R}^J$ where J is a finite set.
(c) $W = W(q, R)$ where $q \in \mathbb{R}^J$ and $R \in \mathbb{R}^{S \times J}$ is defined by

$$\forall \theta \in \mathbb{R}^J, \quad [W\theta](0) = -q \cdot \theta \quad \text{and} \quad \forall s \in S, \quad [W\theta](s) = R(s) \cdot \theta.$$

Furthermore, the corresponding consumption market $(E, X^\bullet, u^\bullet)$ is the general one adapted to the specification (a) of Σ and the corresponding portfolio structure $\Theta^\bullet$ is the general one adapted to the specification (b) of F.

This two-period intertemporal exchange economy extends over two time periods $t = 0$ and $t = 1$. Uncertainty at the second period is modelled by S. The set J is the set of financial nominal assets which are traded only in the first period ($t = 0$) and pay monetary returns in units of account in the second period. Asset $j \in J$ yields $R(s, j)$ units of account in state $\in S$ at date $t = 1$. The vector q represents the asset price at period $t = 0$ and the vector R represents the vector of returns. In this model, given a commodity price $p \in (E')^\Sigma$, a portfolio $\theta \in \mathbb{R}^J$ finances a consumption plan $x \in E^\Sigma$ if at time $t = 0$,

$$\langle p(0), x(0) \rangle + q \cdot \theta \leqslant 0$$

and at time $t = 1$, for each state $s \in S$,

$$\langle p(s), x(s) \rangle \leqslant R(s) \cdot \theta.$$

In the usual set-up of the literature the vector of returns R is given. The study concerns vectors of prices $q \in \mathbb{R}^J$ through the relation between the arbitrage-free condition and the existence of equilibrium.

Definition 3.1. A financial market $(W(q,R), \Theta^\bullet)$ (resp. an economy $(\mathcal{E}^c, W(q,R), \Theta^\bullet)$) satisfying the specifications (a), (b) and (c) is called a two-period intertemporal financial market (resp. a two-period intertemporal economy).

Definition 3.2. Let $(W(q,R), \Theta^\bullet)$ be a standard two-period intertemporal financial market. The asset price $q \in \mathbb{R}^J$ is called an equilibrium asset price if $(W(q,R), \Theta^\bullet)$ is viable, i.e. for every standard consumption market $\mathcal{E}^c$, the two-period intertemporal economy $(\mathcal{E}^c, W(q,R), \Theta^\bullet)$ has a weak equilibrium.

Definition 3.3. Let $(W(q,R), \Theta^\bullet)$ be a standard two-period intertemporal financial market. The asset price $q \in \mathbb{R}^J$ is called arbitrage-free if $(W(q,R), \Theta^\bullet)$ is arbitrage free.

As usually done in the literature (see [**?**],[**?**] and [**?**]), we can invoke a strict separation theorem to prove that q is arbitrage-free if and only if there exists $\lambda \in \mathbb{R}^S$ such that $q = \sum_{s \in S} \lambda(s) R(s)$ and for each $s \in S$, $\lambda(s) > 0$. If we rephrase Corollary **??**, we get a complete characterization of equilibrium asset prices by means of arbitrage-free asset prices.

Corollary 3.1. *Consider a standard two-period intertemporal financial market which is collectively frictionless, then the asset price is an equilibrium if and only if it is arbitrage-free.*

This corollary generalizes the results in Cass [**?**], Werner [**?**, **?**], Magill–Shafer [**?**] and Florenzano [**?**] from frictionless financial markets to collectively frictionless financial markets.

3.2. **The multi-period intertemporal model.** The second example of our general model of exchange economies with purely financial markets (Section **??**) is the multi-period intertemporal model studied by Duffie [**?**], Florenzano–Gourdel [**?**] and Magill–Quinzii [**?**]. In the multi-period model the triple (Σ, I, F) and the payoff operator are specified as follows:

(a) Σ is an event tree of length $T \in \mathbb{N} \setminus \{0\}$,[3] the initial node of Σ is denoted by ξ and for each node $\sigma \neq \xi$ at date t, we denote by σ^- the unique node which immediately precedes σ at date $t-1$;

[3] We refer to Duffie [**?**], Florenzano–Gourdel [**?**] and Courrège–Lacroix–Matarasso [**?**] for precise definitions of an event tree.

(b) F is the subspace of $\mathbb{R}^{\Sigma\times J}$, where J is a finite set, defined by $\theta = (\theta(\sigma,j),(\sigma,j)\in \Sigma\times J)$ belongs to F if and only if $\theta(\sigma) = (\theta(\sigma,j), j\in J) = 0$ for each terminal node $\sigma\in\Sigma_T$.

(c) $W = W(S,D)$ where $S\in F$ and $D\in\mathbb{R}^{\Sigma\times J}$, is defined by

$$\forall\theta\in F,\quad \forall\sigma\in\Sigma,\quad [W\theta](\sigma) = \theta(\sigma^-)\cdot[S(\sigma)+D(\sigma)] - \theta(\sigma)\cdot S(\sigma).$$

with "$\theta(\xi^-)$" taken to be zero by convention.

Furthermore the corresponding consumption market $(E, X^\bullet, u^\bullet)$ is the general one adapted to the specification (a) of Σ and the corresponding portfolio structure $\Theta^\bullet$ is the general one adapted to the specification (b) of F.

This multi-period intertemporal exchange economy extends over $T+1$ time periods $t\in\{0,1,\dots,T\}$. The set J represents the set of purely financial securities. A security j is a claim to a dividend process $D'(j) = (D(\sigma,j),\sigma\in\Sigma)\in\mathbb{R}^\Sigma$ where $D(\sigma,j)\in\mathbb{R}$ represents the dividend paid by the security j at node σ. To each security j is associated the security price process $S'(j) = (S(\sigma,j),\sigma\in\Sigma)\in\mathbb{R}^\Sigma$ where $S(\sigma,j)$ represents the price of the security j, *ex dividend*, at node σ. That is, at each node σ, the security pays its dividend $D(\sigma,j)$ and is then available for trade at the price $S(\sigma,j)$ if $\sigma\notin\Sigma_T$. This convention implies that $D(\xi,j)$ and for every terminal node $\sigma\in\Sigma_T$, $S(\sigma,j)$ play no role. For convenience we pose $D(\xi,j)=0$ and for every $\sigma\in\Sigma_T$, $S(\sigma,j)=0$. The vectors $D = (D(\sigma,j),(\sigma,j)\in\Sigma\times J)$ and $S = (S(\sigma,j),(\sigma,j)\in\Sigma\times J)$ are called respectively the dividend process and the security price process.

In this model, given a commodity price $p\in(E')^\Sigma$, a portfolio $\theta\in F$ finances a consumption plan $x\in E^\Sigma$ if at each node $\sigma\in\Sigma$,

$$\langle p(\sigma), x(\sigma)\rangle + \theta(\sigma)\cdot S(\sigma) \leqslant \theta(\sigma^-)\cdot[S(\sigma)+D(\sigma)].$$

In the usual set-up of the literature the dividend process D is given. The study concerns security price processes $S\in F$ through the relation between the arbitrage-free condition and the existence of equilibrium.

Definition 3.4. A financial market $(W(S,D),\Theta^\bullet)$ (resp. an economy $(\mathcal{E}^c, W(S,D),\Theta^\bullet)$) satisfying the specifications (a), (b) and (c) is called a multi-period intertemporal financial market (resp. a multi-period intertemporal economy).

Definition 3.5. Let $(W(S,D),\Theta^\bullet)$ be standard two-period intertemporal financial market. The security price process $S\in F$ is called an equilibrium security price process if $(W(S,D),\Theta^\bullet)$ is viable, i.e. for every standard consumption market $\mathcal{E}^c$, the two-period intertemporal economy $(\mathcal{E}^c, W(S,D),\Theta^\bullet)$ has a weak equilibrium.

Definition 3.6. Let $(W(S,D),\Theta^\bullet)$ be standard multi-period intertemporal financial market. The security price process $S\in F$ is called an arbitrage-free security price process if $W(S,D)$ is arbitrage-free.

A usually done in the literature (see [?] and [?]), we can invoke a strict separation theorem to prove that S is arbitrage-free if and only if there exists $\lambda \in \mathbb{R}^\Sigma$ such that for each $\sigma \in \Sigma$, $\lambda(\sigma) > 0$ and

$$\forall \sigma \in \Sigma \setminus \Sigma_T, \quad \lambda(\sigma)S(\sigma) = \sum_{\sigma' \in \sigma^+} \lambda(\sigma')[S(\sigma') + D(\sigma')],$$

where σ^+ denotes the set of immediate successors of σ. Corollary **??** can be rephrased in terms of equilibrium security price processes.

Corollary 3.2. *Consider a standard multi-period intertemporal financial market which is collectively frictionless, then the security price process is an equilibrium if and only if it is arbitrage-free.*

This corollary generalizes the results in Duffie [?] and Florenzano–Gourdel [?] from frictionless financial markets to collectively frictionless financial markets.

4. Proof of Theorem ??

Consider a standard economy $\mathcal{E} = (\mathcal{E}^c, \mathcal{E}^f) = (E, X^\bullet, u^\bullet, W, \Theta^\bullet)$ satisfying

$$\operatorname{cone} \bigcup_{i \in I} W\Theta^i = \operatorname{Im} W.$$

We recall that for each $\sigma \in \Sigma$, $W(\sigma)$ is the linear form on F defined by $W(\sigma)\theta = [W\theta](\sigma)$, for every $\theta \in F$. Suppose that W is arbitrage-free, then we can invoke a strict separation theorem to prove that there exists $\lambda \in \mathbb{R}^\Sigma$ such that for each $\sigma \in \Sigma$, $\lambda(\sigma) > 0$ and

$$\sum_{\sigma \in \Sigma} \lambda(\sigma) W(\sigma) = 0. \tag{4.1}$$

We denote by $\operatorname{Ker} \lambda$ the vector subspace of all vectors $t \in \mathbb{R}^\Sigma$ such that

$$\lambda \cdot t = \sum_{\sigma \in \Sigma} \lambda(\sigma) t(\sigma) = 0.$$

Note that from (**??**), $\operatorname{Im} W \subset \operatorname{Ker} \lambda$.

The proof will be done in three major steps. After some notations, we begin by truncating the consumption market in order to get compact consumption sets. The second step is the core of the paper: as usual, but here symmetrically, we modify the right side of the budget constraints leading to *well behaved* demand correspondences. The third and last step consists in proving the existence of a weak equilibrium by applying a fixed-point theorem in only $(p, x^\bullet)$ since the arbitrage-free condition allows here to find endogenously $\theta^\bullet$.

4.1. **Notations.** We endow the finite dimensional space E with a norm $\|.\|$. The dual norm on E' is also denoted by $\|.\|$. If $(H,\|.\|)$ is a normed vector space (for instance $(E,\|.\|)$, $(E',\|.\|)$ or $(\mathbb{R},|.|)$) then the closed ball of radius $r>0$ on H with center 0 is denoted $B(H,r)$. The space H^Σ is endowed with the counting norm still denoted $\|.\|$ and defined by

$$\forall h=(h(\sigma),\sigma\in\Sigma)\in H^\Sigma,\quad \|h\|=\sum_{\sigma\in\Sigma}\|h(\sigma)\|.$$

We denote $\langle .,.\rangle$ the duality on $((E')^\Sigma,E^\Sigma)$ defined by

$$\forall (p,x)\in (E')^\Sigma\times E^\Sigma,\quad \langle p,x\rangle=\sum_{\sigma\in\Sigma}\langle p(\sigma),x(\sigma)\rangle.$$

A vector $z=(z(\sigma),\sigma\in\Sigma)$ in $\mathbb{R}^\Sigma$ is said non-negative, denoted by $z\geqslant 0$ if for each $\sigma\in\Sigma$, $z(\sigma)\geqslant 0$; z is said positive, denoted by $z>0$ if $z\neq 0$ and if it is non-negative; z is said strictly positive, denoted by $z\gg 0$ if for each $\sigma\in\Sigma$, $z(\sigma)>0$. We denote by $\mathbb{R}^\Sigma_+$ (resp. $\mathbb{R}^\Sigma_{++}$) the set of all $z\in\mathbb{R}^\Sigma$ satisfying $z\geqslant 0$ (resp. $z\gg 0$). We recall that for every (p,x) in $\mathbb{R}^\Sigma\times\mathbb{R}^\Sigma$, we denote by $p\cdot x=\sum_{\sigma\in\Sigma}p(\sigma)x(\sigma)$. If H is a vector subspace of $\mathbb{R}^\Sigma$, then $H^\perp$ denotes the vector subspace of all vectors $x\in\mathbb{R}^\Sigma$ such that for every y in H, $x\cdot y=0$.

4.2. **Truncating the consumption market.** The following lemma establishes that in order to prove Theorem **??**, we can suppose without any loss of generality that consumption sets are compact.

Definition 4.1. For any $r>0$, we let $\mathcal{E}^c_r$ be the *truncated* consumption market defined by

$$\mathcal{E}^c_r=(E,X^\bullet_r,u^\bullet_r)$$

where for each $i\in I$, $X^i_r=X^i\cap B(E^\Sigma,r)$ and u^i_r is the restriction of u^i to X^i_r.

Lemma 4.1. *Let $\mathcal{E}^c$ be a standard consumption market.*

(a) *For every $r>0$, the truncated consumption market $\mathcal{E}^c_r$ is standard.*
(b) *There exists $r>0$ such that*

$$\forall i\in I,\quad \widehat{X}^i\subset \operatorname{int} B(E^\Sigma,r). \tag{4.2}$$

(c) *If $r>0$ is such that (**??**) is satisfied then for each standard financial market $\mathcal{E}^f$, any weak equilibrium of the economy $(\mathcal{E}^c_r,\mathcal{E}^f)$ is a weak equilibrium of the economy $(\mathcal{E}^c,\mathcal{E}^f)$.*

The proof of this lemma is in Appendix **??**. Following Lemma **??**, we can suppose without any loss of generality that for each $i\in I$, the set X^i is compact.

Let π be the mapping from $(E')^\Sigma$ to $(E')^\Sigma$ defined by

$$\forall p=(p(\sigma),\sigma\in\Sigma)\in (E')^\Sigma,\quad \pi(p)=(\lambda(\sigma)p(\sigma),\sigma\in\Sigma)\in (E')^\Sigma.$$

Since $\lambda \in \mathbb{R}^{\Sigma}_{++}$, the mapping π is bijective. We restrict the commodity prices in the set Π defined by

$$\Pi := \{p \in (E')^{\Sigma} \; : \; \|\pi(p)\| \leqslant 1\}.$$

4.3. **Modified budget sets: the symmetric approach.** Let $\mathcal{A}$ be the space of continuous mappings from Π to $\mathbb{R}^{\Sigma}$. If α belongs to $\mathcal{A}$, then for each $p \in \Pi$, $\alpha(p)$ denotes the tuple $(\alpha(p,\sigma), \sigma \in \Sigma)$ where $\alpha(p,\sigma) \in \mathbb{R}$. For each $p \in \Pi$, we let $\gamma(p) = (\gamma(p,\sigma), \sigma \in \Sigma)$ be the vector in $\mathbb{R}^{\Sigma}$ defined by $\gamma(p,\sigma) = 1 - \|\pi(p)\|$ for each $\sigma \in \Sigma$. For each $\alpha \in \mathcal{A}$, $i \in I$, $p \in \Pi$, let $B^i_\alpha(p)$, $\beta^i_\alpha(p)$ and $d^i_\alpha(p)$ be the sets defined by

$$B^i_\alpha(p) := \{x \in X^i \; : \; \exists \theta \in \Theta^i, \; \exists \tau \in [0,1], \; p \square x \leqslant W\theta + \alpha(p)\tau + \gamma(p)\},$$

$$\beta^i_\alpha(p) := \{x \in X^i \; : \; \exists \theta \in \Theta^i, \; \exists \tau \in [0,1], \; p \square x \ll W\theta + \alpha(p)\tau + \gamma(p)\},$$

$$d^i_\alpha(p) = \{x \in X^i \; : x \in B^i_\alpha(p) \quad \text{and} \quad B^i_\alpha(p) \cap P^i(x) = \emptyset\}.$$

It is to be noticed how the right side of budget constraints in $B^i_\alpha(p)$ and $\beta^i_\alpha(p)$ includes the sum of two terms $\alpha(p)\tau$ and $\gamma(p)$. The first one completes the basic term $W\theta$ and the second one is the usual term introduced by Bergstrom [**?**] to deal with possibly non monotone preferences. We provide hereafter the properties of the correspondences defined above which allows the application of Kakutani's fixed-point theorem, and this for each $\alpha \in \mathcal{A}$. Then α will be specified in order that the corresponding fixed-point be a weak equilibrium.

Lemma 4.2. *For each $\alpha \in \mathcal{A}$, $i \in I$,*

(i) *the correspondence B^i_α is upper semicontinuous on Π with compact convex values,*
(ii) *the correspondence B^i_α is lower semicontinuous on* $\operatorname{int}\Pi$*,*
(iii) *the correspondence d^i_α is upper semicontinuous on* $\operatorname{int}\Pi$ *with non-empty compact convex values.*

The proof of Lemma **??** is in Appendix **??**.

4.4. **Applications of Kakutani's fixed-point theorem.** For each integer $n \geqslant 1$, let Π_n be the compact convex subset of $\operatorname{int}\Pi$ defined by

$$\Pi_n := \{p \in (E')^{\Sigma} \; : \; \|\pi(p)\| \leqslant 1 - 1/n\}$$

and we let[4] F_n be the correspondence from $\Pi_n \times X$ to $\Pi_n \times X$ defined by

$$F_n(p, x^\bullet) := \phi_n(x^\bullet) \times \prod_{i \in I} d^i_\alpha(p)$$

where $\phi_n(x^\bullet) := \{p \in \Pi_n \; : \; \forall p' \in \Pi_n, \; \langle \pi(p'), \sum_{i\in I} x^i\rangle \leqslant \langle \pi(p), \sum_{i \in I} x^i\rangle\}$. The correspondence ϕ_n has a closed graph. ¿From Lemma **??** the correspondence F_n is upper

[4] Note that we should write $F_{\alpha,n}$.

semicontinuous with non-empty convex compact values. Applying Kakutani's fixed-point theorem, there exists[5] $(p_n, x_n^\bullet)$ in $\Pi_n \times X$ such that:

$$\forall p \in \Pi_n, \quad \langle \pi(p), \sum_{i\in I} x_n^i\rangle \leqslant \langle \pi(p_n), \sum_{i\in I} x_n^i\rangle \tag{4.3}$$

and

$$\forall i \in I, \quad x_n^i \in B_\alpha^i(p_n) \quad \text{and} \quad B_\alpha^i(p_n) \cap P^i(x_n^i) = \emptyset. \tag{4.4}$$

Since $\Pi \times X$ is compact, passing to a subsequence if necessary, we can suppose that the sequence $(p_n, x_n^\bullet)_n$ converges[6] to $(\bar{p}, \bar{x}^\bullet)$ in $\Pi \times X$ which satisfies the following properties.

Claim 4.1. *For each* $\alpha \in \mathcal{A}$*,*

$$\forall p \in \Pi, \quad \langle \pi(p), \sum_{i\in I} \bar{x}^i\rangle \leqslant \langle \pi(\bar{p}), \sum_{i\in I} \bar{x}^i\rangle \tag{4.5}$$

and for each $i \in I$*,*

$$\bar{x}^i \in B_\alpha^i(\bar{p}) \quad \textit{i.e.} \quad \exists(\bar{\theta}^i, \bar{\tau}^i) \in \Theta^i \times [0,1], \quad \bar{p}\Box \bar{x}^i \leqslant W\bar{\theta}^i + \alpha(\bar{p})\bar{\tau}^i + \gamma(\bar{p}) \tag{4.6}$$

and

$$\beta_\alpha^i(\bar{p}) \neq \emptyset \Longrightarrow \bar{x}^i \in d_\alpha^i(\bar{p}). \tag{4.7}$$

Proof. Passing to the limit in (**??**) we get (**??**). Property (**??**) follows from (**??**) and the upper semicontinuity of B_α^i on Π. Let us now prove (**??**). From (**??**), for each $i \in I$, $n \in \mathbb{N}$, $\beta_\alpha^i(p_n) \cap P^i(x_n^i) = \emptyset$. The correspondences $\beta_\alpha^i : \Pi \to X^i$ and $P^i : X^i \to X^i$ have open graphs. It follows that $\beta_\alpha^i(\bar{p}) \cap P^i(\bar{x}^i) = \emptyset$. Now if $\beta_\alpha^i(\bar{p}) \neq \emptyset$ then $B_\alpha^i(\bar{p})$ is the closure of $\beta_\alpha^i(\bar{p})$, and since $P^i(\bar{x}^i)$ is open, we have $B_\alpha^i(\bar{p}) \cap P^i(\bar{x}^i) = \emptyset$. ☐

If we let $(\operatorname{Im} W)^\perp := \{\delta \in \mathbb{R}^\Sigma : \ \delta \cdot t = 0, \ \forall t \in \operatorname{Im} W\}$ then $\mathbb{R}^\Sigma = \operatorname{Im} W + (\operatorname{Im} W)^\perp$.

Definition 4.2. We let

- $\mathcal{A}^\lambda$ be the subset of all mappings $\alpha \in \mathcal{A}$ satisfying
$$\forall p \in \Pi, \quad \alpha(p) \in \operatorname{Ker} \lambda.$$
- $\mathcal{A}^\beta$ be the subset of all mappings $\alpha \in \mathcal{A}$ satisfying
$$\forall p \in \Pi, \quad \bigcup_{i\in I} \beta_\alpha^i(p) \neq \emptyset.$$
- $\mathcal{A}^\perp$ be the subset of all mappings $\alpha \in \mathcal{A}$ satisfying
$$\forall p \in \Pi, \quad \alpha(p) \in (\operatorname{Im} W)^\perp.$$

Claim 4.2. *The following properties are satisfied.*

[5] Once again we should write $p_{\alpha,n}$ and $x_{\alpha,n}^\bullet$.

[6] We should write $(\bar{p}_\alpha, \bar{x}_\alpha^\bullet)$.

(i) *If* $\alpha \in \mathcal{A}^\lambda$ *then* $\bar{x}^\bullet \in \widehat{X}$, *i.e.* $\sum_{i\in I} \bar{x}^i = 0$.
(ii) *If* $\alpha \in \mathcal{A}^\lambda \cap \mathcal{A}^\beta$ *then for every* $\sigma \in \Sigma$, $\bar{p}(\sigma) \neq 0$ *and for every* $i \in I$, $\bar{x}^i \in d^i_\alpha(\bar{p})$.
(iii) *If* $\alpha \in \mathcal{A}^\lambda \cap \mathcal{A}^\beta \cap \mathcal{A}^\perp$ *then* $\gamma(\bar{p}) = 0$, $\sum_{i\in I} W\bar{\theta}^i = 0$ *and for every* $i \in I$, $\alpha(\bar{p})\bar{\tau}^i = 0$.

Proof of part (i). Suppose that $\sum_{i\in I} \bar{x}^i \neq 0$. It follows from (**??**) that

$$\|\pi(\bar{p})\| = 1 \quad \text{and} \quad \langle \pi(\bar{p}), \sum_{i\in I} \bar{x}^i \rangle > 0. \tag{4.8}$$

Hence $\gamma(\bar{p}) = 0$ and premultiplying by $\lambda(\sigma)$ the budget inequality (**??**) at state $\sigma \in \Sigma$, and summing among σ, we get

$$\langle \pi(\bar{p}), \bar{x}^i \rangle = \lambda \cdot (\bar{p} \Box \bar{x}^i) \leqslant \lambda \cdot (W\bar{\theta}^i) + [\lambda \cdot \alpha(\bar{p})]\bar{\tau}^i.$$

Since $\operatorname{Im} W \subset \operatorname{Ker} \lambda$, we have $\lambda \cdot (W\bar{\theta}^i) = 0$, and since $\alpha(\bar{p})$ belongs to $\operatorname{Ker}\lambda$, $\langle \pi(\bar{p}), \bar{x}^i \rangle \leqslant 0$. Summing among i, $\langle \pi(\bar{p}), \sum_{i\in I} \bar{x}^i \rangle \leqslant 0$, which yields a contradiction with (**??**). □

Proof of part (ii). Since $\alpha \in \mathcal{A}^\beta$ there exists an agent $k \in I$ for which $\beta^k_\alpha(\bar{p}) \neq \emptyset$. From (**??**) the vector $\bar{x}^k$ belongs to $d^k_\alpha(\bar{p})$, i.e. $B^k_\alpha(\bar{p}) \cap P^k(\bar{x}^k) = \emptyset$. ¿From Assumption C.4 we get that $\bar{p}(\sigma) \neq 0$ for each $\sigma \in \Sigma$. Therefore, in view of Assumption C.3, we deduce that for every $i \in I$, $\beta^i_\alpha(\bar{p}) \neq 0$. Once again from (**??**), $\bar{x}^i$ belongs to $d^i_\alpha(\bar{p})$, for every $i \in I$. □

Proof of part (iii). ¿From Claim **??**(ii), for every $i \in I$, $\bar{x}^i \in d^i_\alpha(\bar{p})$. Applying Assumption C.4, we get that

$$\forall i \in I, \quad \bar{p} \Box \bar{x}^i = W\bar{\theta}^i + \alpha(\bar{p})\bar{\tau}^i + \gamma(\bar{p}).$$

Premultiplying by $\lambda(\sigma)$ the budget inequality at state σ, and summing among $\sigma \in \Sigma$, we get

$$\langle \pi(\bar{p}), \bar{x}^i \rangle = \lambda \cdot (W\bar{\theta}^i) + [\lambda \cdot \alpha(\bar{p})]\bar{\tau}^i + \|\lambda\| \, (1 - \|\bar{p}\|).$$

Since $\operatorname{Im} W \subset \operatorname{Ker}\lambda$, we have $\lambda \cdot (W\bar{\theta}^i) = 0$. Since $\alpha(\bar{p})$ belongs to $\operatorname{Ker}\lambda$, we get that $\langle \pi(\bar{p}), \bar{x}^i \rangle = \|\lambda\| \, (1 - \|\bar{p}\|)$. Summing among i, we get $\gamma(\bar{p}) = 0$. It follows that for each $i \in I$,

$$\bar{p} \Box \bar{x}^i = W\bar{\theta}^i + \alpha(\bar{p})\bar{\tau}^i.$$

Summing among i, we get

$$\alpha(\bar{p})(\sum_{i\in I} \bar{\tau}^i) = W(-\sum_{i\in I} \bar{\theta}^i).$$

Since $\alpha \in \mathcal{A}^\perp$, it follows that for every $i \in I$, $\alpha(\bar{p})\bar{\tau}^i = 0$. □

It follows from Claim **??** that for every $\alpha \in \mathcal{A}^\lambda \cap \mathcal{A}^\beta \cap \mathcal{A}^\perp$, $(\bar{p}, \bar{x}^\bullet, \bar{\theta}^\bullet)$ is a weak equilibrium of $\mathcal{E}$. The proof is completed by the following lemma.

Lemma 4.3. *The set* $\mathcal{A}^\lambda \cap \mathcal{A}^\beta \cap \mathcal{A}^\perp$ *is non-empty.*

The proof of Lemma **??** based on the fact that the financial market is locally collectively frictionless, is postponed to Appendix **??**.

Appendix A.

A.1. **Proof of Proposition ??.**

Proof of part (a) of Proposition **??**. Let $(p, x^\bullet, \theta^\bullet)$ be a weak equilibrium. It follows that $\sum_{i\in I}\theta^i$ belongs to $\operatorname{Ker} W$, and thus from (**??**) there exists and agent $k \in I$ such that $-\sum_{i\in I}\theta^i$ belongs to $\operatorname{As}(\Theta^k)$. Consider the portfolio allocation $\eta^\bullet = (\eta^i, i \in I)$ defined by $\eta^i = \theta^i$ if $i \neq k$ and $\eta^k = \theta^k - \sum_{i\in I}\theta^i$. For each $i \in I$, η^i belongs to Θ^i and it is now routine to check that $(p, x^\bullet, \eta^\bullet)$ is an equilibrium. □

Remark A.1. In Proposition **??**, the condition (**??**) can be replaced by the weaker condition

$$[-\sum_{i\in I}\Theta^i] \cap \operatorname{Ker} W \subset \sum_{i\in I}[\operatorname{As}(\Theta^i) \cap \operatorname{Ker} W].$$

Claim A.1. *If for every $i \in I$, the set Θ^i is a closed convex subset of F containing 0 then*

$$\bigcup_{i\in I}\Theta^i = F \quad \iff \quad \bigcup_{i\in I}\operatorname{As}(\Theta^i) = F.$$

Proof. Since 0 belongs to Θ^i, the cone $\operatorname{As}(\Theta^i)$ is a subset of Θ^i. Hence if $\cup_{i\in I}\operatorname{As}(\Theta^i) = F$ then $\cup_{i\in I}\Theta^i = F$. Suppose now that $\cup_{i\in I}\Theta^i = F$ and let η in F. For each $k \in \mathbb{N}$, $k\eta$ belongs to $\cup_i\Theta^i$, thus there exists $i \in I$ and an increasing sequence $(k_n)_n$ of integers such that $k_n\eta$ belongs to Θ^i for each $n \in \mathbb{N}$. Hence, for every θ in Θ^i

$$\frac{1}{k_n}(k_n\eta) + (1 - \frac{1}{k_n})\theta \in \Theta^i.$$

Now since Θ^i is closed, passing to the limit we get $\theta + \eta$ belongs to Θ^i, i.e. η belongs to $\operatorname{As}(\Theta^i)$. □

Proof of part (b) of Proposition **??**. Part (b) is a direct consequence of Claim **??**. □

A.2. **Proof of Proposition ??.**

Proof. Let $\mathcal{E} = (\mathcal{E}^c, \mathcal{E}^f)$ be a standard economy. Let $(p, x^\bullet, \theta^\bullet)$ be a weak equilibrium and suppose that $\mathcal{E}^f$ does not preclude arbitrage opportunities. Then there exists $i \in I$ and $t \in \operatorname{As}(W\Theta^i)$ such that $t > 0$. It follows that $W\theta^i + t \in W\Theta^i$, i.e. there exists $z^i \in \Theta^i$ such that $Wz^i > W\theta^i$. But x^i satisfies the budget constraint $p\Box x^i \leqslant W\theta^i$, it follows that $p\Box x^i < Wz^i$. The economy $\mathcal{E}$ satisfies Assumption C.4, hence there exists y in $P^i(x)$ such that $p\Box y \leqslant Wz^i$. This is in contradiction with the optimality of (x^i, θ^i). □

A.3. **Proof of Proposition ??.**

Proof. Let $\mathcal{E}^f = (W, \Theta^\bullet)$ be a standard financial market that is collectively frictionless. Assume that W is not arbitrage-free, then there exists $t \in \operatorname{Im} W$ such that $t > 0$. For each $k \in \mathbb{N}$, kt belongs to $\operatorname{Im} W$. Since the financial market is collectively frictionless,

$kt \in \cup_i W\Theta^i$. Thus there exists $i \in I$ and an increasing sequence $(k_n)_n$ of integers such that $k_n t$ belongs to $W\Theta^i$ for each $n \in \mathbb{N}$. Now let $t^i \in W\Theta^i$, then

$$(1 - \frac{1}{k_n})t^i + \frac{1}{k_n}k_n t \in W\Theta^i.$$

Now passing to the limit we get $t^i + t$ belongs to $W\Theta^i$, which means that t belongs to $\mathrm{As}\,(W\Theta^i)$. This implies that $\mathcal{E}^f$ does not preclude arbitrage opportunities. □

A.4. **Proof of Lemma ??.**

Proof. Let $\mathcal{E}^c$ be a standard consumption market. Part (a) is straightforward and part (b) follows from the compactness of $\widehat{X}^i$ for each $i \in I$. We now prove part (c). Let $\mathcal{E} = (\mathcal{E}^c, \mathcal{E}^f)$ be a standard economy and let $(\bar{p}, \bar{x}^\bullet, \bar{\theta}^\bullet)$ be a weak equilibrium of $\mathcal{E}_r = (\mathcal{E}^c_r, \mathcal{E}^f)$. Suppose that it is not a weak equilibrium of $\mathcal{E} = (\mathcal{E}^c, \mathcal{E}^f)$. Then for some i, there exists $(x^i, \theta^i) \in X^i \times \Theta^i$ such that $u^i(x^i) > u^i(\bar{x}^i)$ and (x^i, θ^i) belongs to $B^i(\bar{p})$. Recall that for each $i \in I$, $\bar{x}^i$ belongs to $\mathrm{int}\, B(E^\Sigma, r)$. Then it is easy to find $0 < \lambda \leqslant 1$ such that

$$\bar{x}^i + \lambda(x^i - \bar{x}^i) \in X^i_r$$

and satisfies the same budget constraints. From Assumption C.2, we also have

$$u^i(\bar{x}^i + \lambda(x^i - \bar{x}^i)) > u^i(x^i),$$

which yields a contradiction. □

A.5. **Proof of Lemma ??.**

Proof of part (i). Let $\alpha \in \mathcal{A}$, $i \in I$ and (x_n, p_n) be a sequence in $X^i \times \Pi$ converging to $(x, p) \in X^i \times \Pi$ and such that $x_n \in B^i_\alpha(p_n)$. For each $n \in \mathbb{N}$, there exists $\theta_n \in \Theta^i$ and $\tau_n \in [0, 1]$ such that

$$p_n \,\square\, x_n \leqslant W\theta_n + \alpha(p_n)\tau_n + \gamma(p_n). \tag{A.1}$$

Passing to a subsequence if necessary, we can suppose that there exists $\tau \in [0, 1]$ such that the sequence $(\tau_n)_n$ converges to τ. For each $n \in \mathbb{N}$, we let $t_n = W\theta_n \in \mathrm{Im}\, W$.

If the sequence (t_n) is not bounded then passing to a subsequence if necessary, we can suppose that $\lim_n \|t_n\| = +\infty$. Multiplying (**??**) by $1/\|t_n\|$ and passing to the limit, there exists $v \in \mathbb{R}^\Sigma_+ \cap \mathrm{Im}\, W$ with $\|v\| = 1$. This contradicts the fact that W is arbitrage-free.

It follows that the sequence (t_n) is bounded, then passing to a subsequence if necessary, we can suppose that there exists $t \in \mathbb{R}^\Sigma$ such that (t_n) converges to t. Since the financial market is standard, $W\Theta^i$ is closed and $t \in W\Theta^i$. In particular x belongs to $B^i_\alpha(p)$. □

Proof of part (ii). For every $p \in \mathrm{int}\,\Pi$, the set $\beta^i_\alpha(p)$ is non-empty (take $x = 0$, $\tau = 0$ and θ such that $W\theta = 0$) and then $B^i_\alpha(p)$ is the closure of $\beta^i_\alpha(p)$. Since the correspondence β^i_α has an open graph on $\mathrm{int}\,\Pi$, it is lower semicontinuous and then B^i_α is lower semicontinuous on $\mathrm{int}\,\Pi$. □

Proof of part (iii). Note that $d^i_\alpha(p)$ is the argmax of u^i on $B^i_\alpha(p)$. Since u^i is continuous and B^i_α is continuous on $\operatorname{int}\Pi$, it follows from the Berge's Maximum Theorem [**?**] that d^i_α is upper semicontinuous on $\operatorname{int}\Pi$ with non-empty values. The convexity of $d^i_\alpha(p)$ follows from the quasi-concavity of u^i. □

A.6. **Proof of Lemma ??.**

Proof. Since for each $i \in I$, 0 belongs to the interior of X^i, there exists $r > 0$ such that $U := B(E^\Sigma, r)$ satisfies for each $i \in I$, $U \subset X^i$. Let Δ be the set of all vectors $\delta \in \mathbb{R}^\Sigma$ such that $\delta \in \operatorname{Ker}\lambda \cap (\operatorname{Im} W)^\perp$ and $\|\delta\| \leqslant 1$. Let the correspondence Γ from Π to Δ be defined by

$$\Gamma(p) := \{\delta \in \Delta \ : \ \exists u \in U, \ \exists v \in F, \quad p \Box u - \gamma(p) \ll Wv + \delta\}$$

It is straightforward to check that the correspondence Γ is lower semicontinuous with convex values. In order to apply a continuous selection result (Florenzano [**?**, Proposition 1.5.3, p.31]), we prove that for every p in Π, $\Gamma(p)$ is non-empty. Let $p \in \Pi$, if $\gamma(p) > 0$ then 0 belongs to $\Gamma(p)$ (take $u = 0$ and $v = 0$). Suppose now that $\gamma(p) = 0$, then there exists $x \in U$ such that $p \Box x < 0$. We can thus find a vector t in $\operatorname{Ker}\lambda$ such that $p \Box x \ll t$. Since $\mathbb{R}^\Sigma = \operatorname{Im} W + (\operatorname{Im} W)^\perp$, it follows that there exist $v \in F$ and δ in $(\operatorname{Im} W)^\perp$ such that $t = Wv + \delta$. Note that δ belongs also to $\operatorname{Ker}\lambda$. Moreover we have $p \Box x \ll t = Wv + \delta$. For $\nu > 0$ small enough, νx belongs to U, $\|\nu\delta\| \leqslant 1$ and thus $\nu\delta$ belongs to $\Gamma(p)$. Applying Proposition 1.5.3 in Florenzano [**?**], there exists α a continuous selection of Γ. Following the definition of Δ, the mapping α belongs to $\mathcal{A}^\lambda \cap \mathcal{A}^\perp$.

We assert that $\alpha \in \mathcal{A}^\beta$. Indeed, let $p \in \Pi$ and assume that $\gamma(p) > 0$ then for each $i \in I$, $X^i \subset \beta^i_\alpha(p)$ (take $\tau = 0$ and take $\theta \in \Theta^i$ such that $W\theta = 0$). Assume now that $\gamma(p) = 0$, since $\alpha(p)$ belongs to $\Gamma(p)$, there exists $u \in U$ and $v \in F$ such that $p \Box u \ll Wv + \alpha(p)$. Since the financial market is locally frictionless, for $\tau > 0$ small enough, there exists an agent $k \in I$ such that $W(\tau v)$ belongs to $W(\Theta^k)$. It follows that τu belongs to $\beta^k_\alpha(p)$. □

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