Bid Increments in Second-Price Sealed Bid Auctions[1]

Abstract: This note concerns bidding in a hybrid first-price and second-price auction. The winning bidder sometimes pays his bid and sometimes pays an amount determined by the next highest bid. In internet auctions where bidders wait until the end of the auction to bid the auction reduces to a sealed-bid auction and the bid function we derive may be relevant in such cases.

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1. Motivation

Recently some economists have focused their attention on explaining the phenomenon of sniper bidding on eBay (see for example Ockenfels and Roth 2003, Roth and Ockenfels 2002, Wang 2003, and Bajari and Hortaçsu 2004). While I do not address the issue of sniper bidding here I note that when bidders wait until the final moments of an auction to bid, the ascending auction becomes a sealed-bid auction. Simply, when bidders snipe there are no further opportunities to respond to previous bids. This suggests that sniper bids may deviate from the bid-your-valuation bids suggested by eBay as the ideal strategy. However, to see this we must first establish that the equilibrium bids in a sealed-bid auction with eBay's price determination rules are below the bidders' valuations. I demonstrate this for an independent private values model.

In the pure second-price sealed-bid auction the bid-your-valuation strategy is an equilibrium. Under the rules at eBay sometimes the winner pays his bid and sometimes he pays an amount determined by the second highest bid. The actual amount depends on the specific bidding increment and how close the winning bid and next highest bid are to each other. Accordingly, the bid increment plays an important role in the result. In fact I will show that equilibrium bids are not more than the bid increment below their corresponding valuations. Chwe (1989) addresses the question of bid increments in first-price auctions where bids are drawn from a set. I am not aware of results for the first- second-price hybrid case where bids are drawn from a continuum. The model and results follow.

2. Model

There is an auction for a single item. There are $n \geq 2$ risk neutral bidders. Let v_i be the valuation of bidder $i \in \{1, \dots, n\}$. The valuations are independently and identically distributed and represent draws from the cumulative distribution function F with support $[\underline{v}, \overline{v}]$. The cdf F is twice differentiable and strictly increasing over its support without any atoms. The realized valuations are private knowledge while F is common knowledge.

The auctioneer establishes a reservation price or screening level r. Bidder i's bid, B_i must exceed r; otherwise, the bidder does not participate. All bidders simultaneously submit their bids. Without loss of generality we order the bids $B_n > B_{n-1} > \dots > B_1$. If only one bidder participates then $B_{n-1} = r$. If we can find a monotonic bid function, B : supp $F \to$ supp F, then the probability that any 2 bids are equal when F has no atoms is 0 and I ignore the possibility of a tie. The high bidder, $i = n$, receives the object after the bids are submitted. Let $\Delta \in \mathbb{R}_+$ be the bidding increment. The payoff to bidder i is $v_i - \min\{B_n, B_{n-1} + \Delta\}$ if he wins, $i = n$, and 0 otherwise. In words, the high bidder will sometimes pay his own bid and will sometimes pay the second highest bid plus Δ. The transaction price depends on the bidder's bid with some positive probability. In this way the standard argument that in a second-price auction a bidder gains nothing by lowering his bid does not apply here.

3. Results

Building on the results of Riley and Samuelson (1981) we can derive a strictly increasing equilibrium bid function which is applicable to both first- and second-price auctions. For $r < \underline{v}$ we specify that a bidder with valuation $\underline{v}$ bids his valuation.

Riley and Samuelson (1981) show that in equilibrium the expected payment $E(v)$ must satisfy

$$E'(v_i) = v_i(n-1)F(v_i)^{n-1}F'(v_i).$$

The solution to the differential equation is

$$E(v_i) = v_i F(v_i)^{n-1} - \int_r^{v_i} F(x)^{n-1} dx,$$

where $v_i \geq r$.

In this auction we must calculate the expected payment in two separate cases. Let $\hat{v}$ satisfy $B(\hat{v}) = \max\{r + \Delta, \underline{v} + \Delta\}$. In the first case $v_i < \hat{v}$. This auction is a first-price auction from this bidder's point of view. The expected payment is

$$B_i G(B)^{n-1},$$

where, G is the distribution of equilibrium bids. Note that a strictly increasing bid function implies $G(B(v)) = F(v)$. In this case the bid function is just

$$B(v \leq \hat{v}) = v_i - \frac{1}{F(v_i)^{n-1}} \int_r^{v_i} F(x)^{n-1} dx.$$

The second case to consider is that of high signal bidders, where $v_i > \hat{v}$. In this case the expected payment can be written by recognizing what happens on two intervals. Let $B_{n-1:n}$ denote the the second highest bid when there are n bids. On the interval $B_{n-1:n} \in [r, B-\Delta]$ the bidder pays the expected value of the second highest bidder's bid plus the bid increment Δ, while on the interval $B_{n-1:n} \in (B-\Delta, B)$ the winner pays his bid. Thus we have

$$\begin{aligned}
E(v) =& BG(B)^{n-1}\left[1-\frac{G(B-\Delta)^{n-1}}{G(B)^{n-1}}\right] \\
&+\int_r^{B-\Delta}(x+\Delta)(n-1)G(x)^{n-2}G'(x)dx \\
=& BG(B)^{n-1} - BG(B-\Delta)^{n-1} \\
&+(B-\Delta)G(B-\Delta)^{n-1} - \int_r^{B-\Delta} G(x)^{n-1}dx + \Delta G(B-\Delta)^{n-1} \\
=& BG(B)^{n-1} - \int_r^{B-\Delta} G(x)^{n-1}dx.
\end{aligned}$$

Notice that the expectation using the pdf for the truncated order statistic where $i = n-1$ times the probability of winning gives the same expression as the expectation using the pdf for the order statistic where $i = n$ which we use above. (The distribution of the order statistic is found in Arnold, Balakrishnan, and Nagaraja 1992).

Now we can solve for the bid function for the high valuation bidders and obtain

$$B(v > \hat{v}) = v - \frac{1}{F(v)^{n-1}}\left[\int_r^v F(x)^{n-1}dx - \int_r^{B-\Delta} G(x)^{n-1}dx\right], \qquad (1)$$

where we make use of the fact that $G(B) = F(v), \forall v \geq r$. It may be easier to see that the function is continuous and monotonic by observing that

$$\int_r^v F(x)^{n-1}dx = \int_r^{B(v)} G(x)^{n-1}dx.$$

Then notice that

$$\int_r^{B-\Delta} G(x)^{n-1}dx = \int_r^B G(x)^{n-1}dx - \int_{B-\Delta}^B G(x)^{n-1}dx.$$

Substituting in to 1 we obtain

$$\begin{aligned}B(v > \hat{v}) =& v - \frac{1}{F(v)^{n-1}} \int_r^v F(x)^{n-1} dx \\ &+ \frac{1}{F(v)^{n-1}} \left[\int_r^B G(x)^{n-1} dx - \int_{B-\Delta}^B G(x)^{n-1} dx. \right] \\ =& v - \frac{1}{F(v)^{n-1}} \left[\int_{B-\Delta}^B G(x)^{n-1} dx. \right].\end{aligned}$$

Because B is an element of a compact convex subset of $\mathbb{R}$ and the bid function is continuous then it has a fixed point. Actual solutions could be obtained by using a Newton-Raphson method and piece-wise evaluation of G beginning at the distribution for low signal bidders.

The above result leads us to the following observation.

Lemma 1 *in the auction described above the equilibrium bid* $B_i > v_i - \Delta$.

Proof: it suffices to show that

$$\frac{1}{F(v_i)^{n-1}} \int_{B-\Delta}^B G(x)^{n-1} dx < \Delta,$$

and

$$\frac{1}{F(v_i)^{n-1}} \int_r^{v_i} F(x)^{n-1} dx < \Delta.$$

The first part is true because on the interval Δ the area under the strictly increasing cdf $G(B)^{n-1}$ is strictly less than $\Delta F(v_i)^{n-1}$. To see that the second part is true we recall that $\hat{v}$ is the minimum value for which a bidder selects a bid above $\underline{v} + \Delta$. Then observe that

$$\int_r^{v_i} F^{n-1}(x) dx = \int_{\underline{v}}^{v_i} F^{n-1}(x) dx,$$

and

$$\int_{\underline{v}}^{\hat{v}} F^{n-1}(x) dx = \int_{\underline{v}}^{\underline{v}+\Delta} G^{n-1}(x) dx.$$

Then anywhere on the interval $[\underline{v}, \underline{v} + \Delta]$ the area under the strictly increasing cdf $G(B)^{n-1}$ is strictly less than $\Delta F(v_i)^{n-1}$. QED.

The result suggests that converting the ascending auction with a fixed closing time to a sealed-bid auction by waiting until the last moment to bid implies that equilibrium bids will fall slightly below the bidders' true valuations, where "slightly" is defined by the bidding increment. As such, equilibrium bids would be lower than when bidders apply eBay's suggested bid-your-valuation strategy. However, this does not imply that waiting until the last moment to bid is an equilibrium.

References

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This supplementary material is to make the formulation of the expected payment easier to see. Results concerning the order statistics are available in Arnold, Balakrishnan, and Nagaraja, (1992).

The order statistics are $X_{n:n} \geq X_{n-1:n} \geq \ldots \geq X_{1:n}$. The random variable has cdf F. The cdf of the order statistic $X_{i:n}$ is

$$F_{i:n} = \int_0^{F(x)} \frac{n!}{(i-1)!(n-i)!} t^{i-1}(1-t)^{n-i} dt.$$

(See page 13).

Additionally, the distribution of $X_{j:n}$ given $X_{i:n} = x_i$ for $j < i$ is the same as the distribution of the jth order statistic when the sample size is $i-1$ for the cdf F truncated on the right at x_i. Similarly for left truncation. (See page 23). Accordingly, the expectation of the second highest bid conditional on $B_i = B_{n:n}$ is

$$[G(B_i)]^{-(n-1)} \int_r^{B_i} (x)(n-1)G^{n-2}(x)G'(x)dx.$$