Weber-Fechner's Law, Demand Function and Related Topics

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Abstract

In our previous paper, we derive the demand function in the form $p = \dfrac{du(x)}{dx}$ and we apply the Weber-Fechner's law to the utility function and we obtain the demand function in the familiar form $p = \dfrac{A}{x}$. We compare our derivation of the demand function with the standard one. The differences are $i)$ different functional form of the utility function, $ii)$ different objective function to maximize, $iii)$ different treatment of the budget condition.

We also study how much quantity of goods we should distribute to N persons in n kinds of goods. By adding each person's demand function, we obtain the total demand function. By the market equilibrium, we obtain the only unique solution of how much quantity of goods we should distribute to each person. The quantity of goods is distributed according to each person's preference.

1 Introduction

Recently the new trend of the economics appears. In 2002, Kahnemann won the Nobel prize because of the contribution of applying the psychology to the economics[1][2]. His Nobel prize typically represents this new trend of the economics, trying to understand various phenomena in the economics more deeply from the psychological point of view.

The important law of the economics also comes from the knowledge of the behavioral psychology. In this context, we consider the economics as the application of the behavioral psychology. In micro economics, the utility function is one of the most important things, and the functional form of this utility function must be determined from the knowledge of the behavioral psychology. But the functional form of the utility function is still in the qualitative level as people use this or that functional form without any scientific reason. The most important law in the behavioral psychology is the Weber-Fechner's law[3][4][5], which

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represents the relation between the magnitude of physical stimulus and the magnitude of psychological sense in human being. Then, in our previous paper, we try to understand the functional form of the demand function from the point of view of the behavioral psychology and we apply this Weber-Fechner's law to the utility function, which gives the explicit functional form for the utility function and the demand function[6] [7].

In this paper, we compare our derivation of the demand function with the standard one and we clarify the differences. We also study the problem of how much quantity of goods we should distribute to each person in various kinds of goods.

2 Demand Function from the Weber-Fechner's Law: Review of Our Previous Result

We first review our previous result of deriving the demand function[6][7]. We consider n kinds of goods. We denote the utility function as $u(x_1, x_2, \cdots, x_n)$ where x_i is the quantity of the i-th goods. We assume that this utility function is of separable type

$$u(x_1, x_2, \cdots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n), \tag{1}$$

in the zero-th approximation, which is the quite natural assumption as each goods separately contributes to the total utility in the zero-th approximation[8]. Then this utility function of the i-th goods u_i is the function only of the quantity of the i-th goods x_i. Then the Gossen's law becomes in the form

$$\frac{du_1(x_1)/dx_1}{p_1} = \frac{du_2(x_2)/dx_2}{p_2} = \cdots = \frac{du_n(x_n)/dx_n}{p_n} = k, \tag{2}$$

where p_i is the price of the i-th goods and k becomes constant, independent of the kind of goods. By rescaling the price, we can choose $k = 1$. Then we can rewrite Eq.(2) in the form

$$p_i = \frac{du_i(x_i)}{dx_i}, \quad (i = 1, 2, \cdots, n), \tag{3}$$

and the demand function $p_i = D_i(x_i)$ is expressed in the form

$$p_i = D_i(x_i) = \frac{du_i(x_i)}{dx_i}, \quad (i = 1, 2, \cdots, n), \tag{4}$$

by using the utility function. Then we use the Weber-Fechner's law[3][4] for the utility function of the form

$$u_i(x_i) = \begin{cases} A_i \log\left(x_i/x_i^{(0)}\right) & \text{if } x_i \geq x_i^{(0)} \\ 0 & x_i < x_i^{(0)} \end{cases}, \tag{5}$$

where $x_i^{(0)}$ is the threshold of the quantity of goods. Hereafter, we consider only the case that the quantity of goods is greater than the threshold. Then we have the familiar form of the demand function in the form

$$p_i = \frac{A_i}{x_i}. \tag{6}$$

Next, we consider the principle to determine the demand function from the point of view of the consumer's surplus. The demand function $p_i = D_i(x_i)$ is given by maximizing the consumer's surplus

$$(\text{consumer's surplus}) = \int_0^{x_i} D_i(x_i')dx_i' - p_i x_i, \tag{7}$$

for the i-th goods. Substituting the expression Eq.(4) into Eq.(7), we have

$$(\text{consumer's surplus}) = \int_0^{x_i} \frac{du_i(x_i')}{dx_i'}dx_i' - p_i x_i = u_i(x_i) - p_i x_i, \tag{8}$$

where we use $u_i(0) = 0$. Then we define the "consumer's profit" $\rho_i(x_i)$ as the utility minus expenditure, that is, $u_i(x_i) - p_i x_i$ and we have

$$(\text{consumer's surplus}) = (\text{"consumer's profit"}) = \rho_i(x_i) = u_i(x_i) - p_i x_i. \tag{9}$$

Then we can rewrite that the principle to determine the demand function is to maximize the "consumer's profit" $u_i(x_i) - p_i x_i$.

3 Comparison of the Derivation of the Demand Function

3.1 Standard derivation of the demand function

We first review the standard method to derive the demand function[8][9]. For simplicity, we assume that there is one person with two kinds of goods and each quantity of goods is given by x_1 and x_2. We denote each price of goods as p_1 and p_2 and the budget amount as m. We maximize the utility function of the Cobb-Douglas type

$$u(x_1, x_2) = Ax_1^{\alpha} x_2^{\beta}, \tag{10}$$

with the budget constraint $m = p_1x_1 + p_2x_2$. This problem is equivalent to maximize the following objective function

$$f(x_1, x_2, \lambda) = Ax_1^{\alpha} x_2^{\beta} - \lambda \left(p_1x_1 + p_2x_2 - m\right). \tag{11}$$

Maximizing this objective function, we have the following equations

$$\lambda p_1 = A\alpha x_1^{\alpha-1} x_2^{\beta}, \quad \lambda p_2 = A\beta x_1^{\alpha} x_2^{\beta-1}, \quad m = p_1x_1 + p_2x_2. \tag{12}$$

Then the demand quantities of goods x_1, x_2 and the value of the multiplier are determined and we have the demand function of the form

$$p_1 = \frac{\alpha}{(\alpha+\beta)} \frac{m}{x_1}, \quad p_2 = \frac{\beta}{(\alpha+\beta)} \frac{m}{x_2}, \quad \lambda = \frac{A\alpha^{\alpha}\beta^{\beta}(\alpha+\beta)^{1-\alpha-\beta}}{m^{1-\alpha-\beta} p_1^{\alpha} p_2^{\beta}}. \tag{13}$$

In this way, the expenditure of each goods p_ix_i $(i = 1, 2)$ is determined in such a way as the whole budget amount is relatively distribute into two parts according to the preference of goods, that is, $p_1x_1 = \dfrac{\alpha}{(\alpha+\beta)} m$, $p_2x_2 = \dfrac{\beta}{(\alpha+\beta)} m$.

3.2 Modified derivation of the demand function

Instead of the Cobb-Douglas type utility function, we use the separate type and Weber-Fechner's type utility function

$$u(x_1, x_2) = A_1 \log(x_1/x_1^{(0)}) + A_2 \log(x_2/x_2^{(0)}). \tag{14}$$

Assuming $x_1^{(0)} = x_2^{(0)}$, A_i $(i = 1, 2)$ represents the person's preference of the i-th goods. If $A_1 > A_2$, the person prefer the first goods to the second goods. Maximizing this utility function with the budget constraint $m = p_1x_1 + p_2x_2$, the demand quantities of goods x_1, x_2 and the value of the multiplier are determined and we have the demand function of the form

$$p_1 = \frac{A_1}{(A_1+A_2)} \frac{m}{x_1}, \quad p_2 = \frac{A_2}{(A_1+A_2)} \frac{m}{x_2}, \quad \lambda = \frac{(A_1+A_2)}{m}. \tag{15}$$

Making the correspondence $\alpha \leftrightarrow A_1$, $\beta \leftrightarrow A_2$, this demand function is in the same functional form as that in the standard derivation.

3.3 Our derivation of the demand function

According to our principle, we maximize the total "consumer's profit" for each goods. The total "consumer's profit" $\rho^{\text{total}}(x_1, x_2)$, which is the sum of the "consumer's profit" for each goods $\rho_1(x_1)$ and $\rho_2(x_2)$, is given by

$$\rho^{\text{total}}(x_1, x_2) = A_1 \log(x_1/x_1^{(0)}) + A_2 \log(x_2/x_2^{(0)}) - p_1 x_1 - p_2 x_2. \tag{16}$$

Then the demand quantities of goods x_1, x_2 are determined and we have the demand function becomes of the form

$$p_1 = \frac{A_1}{x_1}, \quad p_2 = \frac{A_2}{x_2}. \tag{17}$$

In our derivation, each demand function is determined independently, which is the natural consequence because we behave according to some demand function even if there exists only one goods. The expenditure of each goods $p_i x_i$ $(i = 1, 2)$ is proportional to the person's preference of goods A_i $(i = 1, 2)$, but the expenditure is independent of the budget amount. Then if $p_1 x_1 + p_2 x_2 = A_1 + A_2 \leq m$, we purchase both types of goods with the above quantity x_1, x_2, but we do not use up all budget in general. While if $p_1 x_1 + p_2 x_2 = A_1 + A_2 > m$, we do not purchase any type of goods.

3.4 Comparison of various derivation of the demand function

The utility function of the Cobb-Douglas type is not natural as it is not the separable type, then the modified derivation will be more natural than the standard one.

Further, the demand function of the first goods is determined in a relative way with that of the second goods in the standard derivation. But the demand function must be determined even if there exist only one kind of goods. In this sense, our derivation will be more natural than the standard one.

Furthermore, our derivation gives the similar result compared with the standard and the modified derivations, but the treatment of the budget condition is quite different so that the economical consequence becomes quite different. In the standard derivation and in the modified derivation, we use up all budget for the given prices p_1, p_2 and the given budget m. While in our derivation, we purchase if and only if the expenditure is less or equal to the budget amount. In this sense, our derivation will give the more reasonable consequence in the actual human behavior.

4 Problem of Distributing the Goods to Each Person

Next we consider the problem to distribute the quantity of goods to each persons. We consider that there are N persons and n kinds of goods. We denote the distributed quantity of the i-th goods for the j-th person as $x_i^{(j)}$ $(i = 1, 2, \cdots, n)$, $(j = 1, 2, \cdots, N)$ and we take the j-th person's utility function of the i-th goods in the form $u_i^{(j)}(x_i^{(j)}) = A_i^{(j)} \log(x_i^{(j)}/x_i^{(j)(0)})$. We take the price p_i of i-th goods to be common for all persons because of the law of one price. Then we maximize j-th person's "consumer's profit" of the form

$$\rho^{(j)}(x_1^{(j)}, x_2^{(j)}, \cdots, x_n^{(j)}) = \sum_{i=1}^{n} A_i^{(j)} \log(x_i^{(j)}/x_i^{(j)(0)}) - \sum_{i=1}^{n} p_i x_i^{(j)}, \quad (j = 1, 2, \cdots, N), \tag{18}$$

which gives

$$p_i = \frac{A_i^{(j)}}{x_i^{(j)}}, \quad (i = 1, 2, \cdots, n), \quad (j = 1, 2, \cdots, N). \tag{19}$$

Then the total demand quantity of the i-th goods X_i is given by $X_i = \sum_{j=1}^{N} x_i^{(j)}$. Using $x_i^{(j)} = \frac{A_i^{(j)}}{p_i}$, we have the total demand function of the i-th goods $p_i = D_i^{\text{total}}(X_i)$ in the form

$$p_i = D_i^{\text{total}}(X_i) = \frac{A_i^{\text{total}}}{X_i}, \tag{20}$$

where $A_i^{\text{total}} = \sum_{j=1}^{N} A_i^{(j)}$. Combining this total demand function with the total supply function $p_i = S_i^{\text{total}}(X_i)$, we have the equilibrium quantity X_i^* and the equilibrium price $p_i^* = \frac{A_i^{\text{total}}}{X_i^*}$ for the i-th goods. Using $x_i^{*(j)} = \frac{A_i^{(j)}}{p_i^*}$, the j-th person's distribution of the quantity of the i-th goods is given by

$$x_i^{*(j)} = \frac{A_i^{(j)}}{A_i^{\text{total}}} X_i^*, \quad (i = 1, 2, \cdots, n), \quad (j = 1, 2, \cdots, N). \tag{21}$$

In this way, according to each person's preference of the i-th goods, the quantity of that goods is distributed, that is, if $A_i^{(j)} > A_i^{(k)}$, we have $x_i^{*(j)} > x_i^{*(k)}$. In the above, we assume that all persons have enough budget amount, that is, $\sum_{i=1}^{n} p_i^* x_i^{*(j)} \leq m^{(j)}$, $(j = 1, 2, \cdots, N)$.

While if there exist the M poor persons who have not enough budget in the form $\sum_{i=1}^{n} p_i^* x_i^{*(j)} > m^{(j)}$, $(j = 1, 2, \cdots, M)$, the above theoretical equilibrium is not realized. Then the optimum distribution is not realized. But if the sum of all person's budget amount is larger than the sum of all person's expenditure, that is, $\sum_{j=1}^{N}\sum_{i=1}^{n} p_i^* x_i^{*(j)} \leq \sum_{j=1}^{N} m^{(j)}$, we can realize the optimum state by taking the tax from the rich persons and giving the financial aid to the poor persons in such a way as all persons can purchase all kinds of goods, though such policy cause all persons to loose the incentive to work.

5 Summary and Discussion

In our previous paper, we derive the demand function by maximizing the "consumer's profit" defined by the utility minus expenditure and we obtain the demand function in the form $p = \frac{du(x)}{dx}$. We apply the Weber-Fechner's law to this utility function, which gives the demand function in the familiar form $p = \frac{A}{x}$.

In this paper, we compare our derivation of the demand function with that of the standard one. In our derivation, the demand function is derived i) by using the utility function which satisfies the Weber-Fechner's law, ii) by maximizing the "consumer's profit", iii) by making the choice that we purchase if and only if the expenditure is less or equal to the budget amount. While in the standard derivation, the demand function is derived i) by using the utility function of the Cobb-Douglas type, ii) by maximizing the utility function with the budget constraint, iii) by use up all budget.

We also study the problem to distribute the quantity of goods to each person. We find that we can distribute the quantity of goods according to each person's preference by maximizing each person's "consumer's profit". According to our method, the distributed quantity of goods is uniquely determined.

The standard way of distributing the quantity of goods to each person is realized by the Pareto optimum state. The key point of the Pareto optimum state is that i) the Pareto optimum state is realized by increasing each person's utility, *ii*) the quantity of goods is exchanged between person to person but not in the market, so that the total quantity of each kind of goods is fixed as there is no additional supply flow through the market, *iii*) there are infinitely many Pareto optimum states even if we start from the same Pareto non-optimum state. While, the key point of our method is that *i*) our optimum state is realized by maximizing each person's "consumer's profit", *ii*) our optimum state is realized through the market equilibrium, *iii*) our optimum state is uniquely determined as the market equilibrium state is uniquely determined. Nowadays, we obtain almost all kinds of goods through the market. Then our optimum mechanism is more realistic than the Pareto optimum mechanism.

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