Constructive Utility Functions on Banach spaces †

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Abstract

In this paper we prove the existence of continuous order-preserving functions on subsets of ordered Banach spaces using a constructive approach.

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1. Introduction

The object of this paper is to prove the existence of continuous real-valued order-preserving functions (also called utility functions) on subsets of Banach spaces using the "distance method" pioneered by Wold (1943-44) and Arrow-Hahn (1971). Apart from the intrinsic interest of such results, the infinite-dimensional method of proof, with its topological subtleties and complications, that is used in the paper will also lead to a better understanding of the finite-dimensional Wold and Arrow-Hahn theorems because in the well-known words of Debreu by "forcing one to greater generality, it brings out with greater clarity and simplicity the basic concepts of the analysis and its logical structure". In addition, as we will see, such results are also important from the applied point of view because infinite-dimensional spaces of the kind studied in this paper are now widely used in the literature in economics and related fields.

There are many approaches that can be used to prove the existence of continuous utility functions and the literature on the subject is now very vast. The reader is referred to Bridges and Mehta (1995) and Mehta (1998) and the references cited there for a detailed discussion of these different approaches. In this paper we concentrate on the "distance approach" and start by briefly describing the salient ideas that are involved. The "distance method" is first used in a classical paper of Wold (1943-44). A *very special* case of Wold's ideas may be illustrated by assuming that the preference relation is a continuous, monotonic total preorder $\precsim$ on the non-negative orthant X of $\mathbb{R}^n$. Wold defines the utility of any point $x \in X$ as the Euclidean distance from the origin to the unique point $d(x)$ (which exists because the preorder is continuous and monotone) on the diagonal D of X which is equivalent to x. This function may be proved to be continuous. [1] A related approach based on the concept of distance is used in Arrow-Hahn (1971, pp. 82-87) who prove the existence of a continuous utility function on a convex subset X of $\mathbb{R}^n$ for a continuous preference relation satisfying a local non-satiation condition. The main idea

[1] It is very important to observe that this is only a *special* case of Wold's argument. A brief sketch of a proof of a much more general result using the assumption of *weak monotonicity* is given by Wold in an appendix; see Beardon and Mehta (1994) for a rigorous proof of Wold's subtle theorem and analysis of its implications for utility theory.

of the Arrow-Hahn proof is to take an arbitrary point $x_0 \in X$ and then to define the utility u of any point x as the Euclidean distance from the point x_0 to the upper section $U(x) = \{y \in X : x \preccurlyeq y\}$ of x. The next step is to prove that this function is continuous. Finally, an extension procedure is used to extend this function to the whole space X. Utility functions of the Wold or Arrow-Hahn type may also be called *constructive* or *metric* utility functions in contrast to other utility functions (e.g. of the type proved by Debreu, 1954, 1964) where the emphasis is primarily on the *existence* problem. [2] The constructive approach used in this paper has the advantage that it enables one to write down an explicit formula for the utility function. [3]

Both the Wold and Arrow-Hahn approaches have been studied extensively in the literature. We refer the reader to Mehta (1998) for further information about this vast and, often technical, literature. In this paper we shall deal primarily with only two developments that have arisen from the theorem of Arrow-Hahn. We first observe that the Arrow-Hahn theorem is stated and proved for convex subsets of $\mathbb{R}^n$. We emphasize that the proof given by Arrow-Hahn *is not valid* in infinite-dimensional linear spaces because it depends crucially upon the fact that $\mathbb{R}^n$ has the Heine-Borel property so that any closed bounded set is compact. But this property does not hold, in general, in an infinite-dimensional space because, as is well-known, the unit ball in an infinite-dimensional normed space (or even a Hausdorff topological vector space) E is compact if and only if E is finite-dimensional (Narici and Beckenstein, 1985, p. 92). Therefore, a natural question to ask is if the Arrow-Hahn theorem can be generalized to infinite-dimensional spaces. Such a generalization is also important for economic applications because infinite-dimensional commodity spaces are now widely used in the economic theory literature. This question is addressed in Mehta (1989) where generalizations and extensions of the Arrow-Hahn theorem and method are proved in Banach spaces E with the strong, *weak** and Mackey topologies. In that paper, an attempt is made to retain most of the features of the con-

[2]It is interesting to observe that there are remarkable relationships between utility representation theorems of the Debreu-type and metrization theorems in topology (Mehta, 1988, and Herden and Mehta, 1995).

[3]The same feature is to be found in certain measure-theoretic approaches (Candeal and Induráin, 1993).

structive Arrow-Hahn *method*.

Another quite different motivation and advantage of proceeding in this manner is the fact that the Arrow-Hahn method of *extending* a continuous function from a closed subset to the whole space is directly related to the fundamental extension problem in set-theoretic, algebraic and differential topology (see, Nachbin 1965, Hu, 1959 and Guillemin and Pollack, 1974) so that powerful topological theorems can be brought to bear on the utility representation problem.

On the other hand, it should be noted that in Arrow-Hahn (1971) and Mehta (1989) a preliminary definition of a utility function u is given in terms of the Euclidean metric of $\mathbb{R}^n$ on a *subset* of the space. But then an extension procedure of some sort is required to get a utility function on the whole space (if the preorder does not have a $\precsim$-first element). Various extension procedures have been used in the literature for this purpose (see, e.g. Beardon 1997, Bridges, 1988, Mehta, 1981, 1991b, 1992 and Candeal, Induráin and Mehta, 1995, Theorem 3). Some of the extension procedures that have been used in the literature may vitiate the "distance approach" because then the utility of a point x is not defined *directly* in terms of Euclidean distance.

Therefore, a natural question to ask in this context is whether one can construct a continuous utility function of the Arrow-Hahn type by defining the utility of any point *directly* in terms of Euclidean distance on the whole space. This aspect of the Arrow-Hahn method was addressed in the papers of Alcantud-Manrique (2001) and Alcantud (2002). In these papers the following idea is used. Suppose that one wants to define the utility of any point x *directly* in terms of Euclidean distance without using any extension method. Then it is natural to consider a proper subset X of $\mathbb{R}^n$ and to take a point z outside the set X and then to define the utility of $x \in X$ as the Euclidean distance of z to the upper section $U(x)$ of x. In these papers it is shown that indeed one can prove the existence of a metric utility function of this kind directly on the whole space X without using any extension procedure. [4]

[4] It should be observed that, in contrast to the Arrow-Hahn approach, the distance methods used by Wold and others do enable one to define a (continuous) utility function defined in terms of Euclidean distance on the whole space without using any kind of extension procedure (see, e.g. Wold, 1943-44,

However, the papers of Alcantud-Manrique (2001) and Alcantud (2002) deal only with finite-dimensional spaces. As in the Arrow-Hahn case, essential use is made in these two papers of the Heine-Borel Theorem so that the proofs do not apply to infinite-dimensional spaces. Therefore, the main objective of this paper is to prove the existence of continuous utility functions on Banach spaces by combining the ideas in the papers of Mehta (1989) and Alcantud (2002). As a consequence, we will be able to generalize the ideas and results in Alcantud (2002) and Arrow-Hahn (1971) to infinite-dimensional spaces. Infinite-dimensional spaces of the kind studied in the paper are now extensively used in economics and related fields (see, e.g. Bewley, 1972, Brown and Lewis, 1981, Mas-Colell, 1975, and Toussaint, 1984).

For the very elementary mathematical ideas that we employ in this paper the reader is reader is referred to Fabin et. al. (2001), Holmes (1975), Jameson (1974) and Narici and Beckenstein (1985).

2. Preliminaries

A *preorder* $\precsim$ on a set X is a reflexive and transitive binary relation on X. Each preordered set $(X, \precsim)$ gives rise to an equivalence relation $\sim$ on X by defining $x \sim y \iff [(x \precsim y) \wedge (y \precsim x)]$ for $x, y \in X$. The equivalence class of any element $x \in X$ is denoted by $[x]_\sim$ and the quotient set by $X/\sim$. A preorder on a set X is said to be *total* if for all $x, y \in X$ we have $(x \precsim y) \vee (y \precsim x)$. If $\precsim$ is a preorder on a set X then $x \prec y$ if and only if $x \precsim y$ and $\neg(y \precsim x)$. An *order* on a set X is an anti-symmetric preorder. A *chain* on a set X is an irreflexive, transitive and weakly connected binary relation on X. For further elaboration of these concepts see Bridges and Mehta (1995, Chapter 1) or Mehta (1998). In applications in economics and related fields a *preference relation* on a set X of alternatives is often defined as a total preorder on X.

For any totally preordered set $(X, \precsim)$ we denote by $t^{\precsim}$ the *order topology* associated with the preorder. A sub-base for this topology is given by the order-intervals of X of

Mehta, 1981, p. 117 and Mehta, 1998, pp. 12-13).

the form $\{a \in X : a \prec x\}$ and $\{a \in X : x \prec a\}$. A topology t on the set X is a *natural* topology if it is finer than the order topology.

Suppose now that $\precsim$ is a preorder on a topological space (X,t). Then, the *upper section* (*lower section*) associated with $x \in X$ is defined by $U(x) = \{y \in X : x \precsim y\}$ $(L(x) = \{y \in X : y \precsim x\})$.[5] For each $x \in X$ the strict upper (and lower) section is defined in the natural manner. We say that the relation $\precsim$ is *t-upper semicontinuous* if $U(x)$ is t-closed for each $x \in X$ and *t-lower semicontinuous* if $L(x)$ is t-closed for each $x \in X$. The preorder $\precsim$ is *t-continuous* if it is both upper and lower semicontinuous with respect to the topology t.

Let $\precsim$ be a total preorder on a topological space (X,t). Then $\precsim$ is *locally non-satiated* if for each $x \in X$ and each neighbourhood V of x there is $y \in V$ such that $x \prec y$. A point $z \in X$ is a *global satiation point* if $x \precsim z$ for all $x \in X$.

Let $(Y, \leqslant)$ and $(X, \precsim)$ be totally preordered sets. A function $f : Y \to X$ is said to be *order-preserving* or an *order-monomorphism* if $x \leqslant y \iff f(x) \precsim f(y)$ for al $x, y \in Y$. If a total preorder is interpreted as a preference relation $\precsim$ on a set X of alternatives then an order-preserving function $X \longrightarrow \mathbb{R}$ (where $\mathbb{R}$ is endowed with the natural order) is also called a *utility function*. A utility function is said to *represent* the preference relation $\precsim$ and the problem of the existence of such an order-preserving function is called a utility representation problem. The ordered set $(Y, \leqslant)$ is said to be *order-embeddable* into $(X, \precsim)$ if there exists an order-preserving function $f : (Y, \leqslant) \to (X, \precsim)$. It is said to be *continuously order-embeddable* if there is a continuous order-preserving function $f : (Y, \leqslant, t^{\leqslant}) \to (X, \precsim, t^{\precsim})$.

Let E be a Banach space. The open ball of radius r around the point $x \in E$ is denoted by $B(x,r)$. Let X be a subset of E and $x \in E$. Then the distance between x and X is denoted as usual by $d(x,X)$. The closure of X -relative to E- will be denoted by $cl_E(X)$.

Let E be a linear space equipped with a norm $||.||$. We say that this normed space is *uniformly convex* if, for any $a > 0$ there exists $b > 0$ such that for any two unit vectors

[5] An upper section (lower section) is also called an upper contour set (lower contour set).

$x, y \in X$, $1- \parallel \frac{x+y}{2} \parallel < b$ implies $\parallel x - y \parallel < a$. [6]

This means intuitively that if the midpoint z of two unit vectors x, y is sufficiently close to the "surface" of the unit ball then the points x and y must also be close to one another and, in addition, this must happen uniformly for all such points near the "surface". In a sense, it intends to capture the idea that the unit ball is "round" (or "rotund").

There are important examples of Banach spaces that are uniformly convex. For example, the norm induced by an inner product is uniformly convex, which implies that every Hilbert space is uniformly convex. In particular, the Euclidean norm on $\mathbb{R}^2$ is uniformly convex. On the other hand, it is not hard to verify that the max-norm on $\mathbb{R}^2$ is not uniformly convex. An important theorem of Clarkson states that the sequence spaces l_p and the Lebesgue spaces L_p for $p > 1$ are uniformly convex (Narici and Beckenstein, 1985, pp. 375-376).

Given the norm $||.||$ on E, the *distance from* $x \in E$ *to a subset* $S \subseteq E$ is defined according to: $d(x, S) = inf\{|| x - s|| : s \in S\}$. Then, a subset K of E is said to be *proximinal in* E if for each $x \in E$ there is $y_x \in K$ such that $|| x - y_x|| \leqslant || x - z||$ for all $z \in K$. This amounts to saying that for each $x \in E$ there is $y_x \in K$ such that $|| x - y_x|| = d(x, K)$, or, in words: that (for every possible $x \in E$) the distance from x to K is attained at some point $y_x \in K$. Then, one has:

LEMMA 1 *If E is a uniformly convex Banach space and A is a closed and convex subset of E then for each $x \in E$, there is a unique closest point x' to x in A so that, in particular, A is proximinal.*

Proof: See Jameson (1974), page 362. ■

Two vector spaces X, Y form a *pair*, denoted by (X, Y) if there is a bilinear function B defined on their product. They are a *dual pair* if the pairing separates points of each

[6] There are also other definitions of this concept, e.g. the norm $||.||$ is uniformly convex if and only if whenever $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ are sequences in $\overline{B(0,1)}$ with $||x_n + y_n|| \longrightarrow 2$ then $||x_n - y_n|| \longrightarrow 0$. For further detailed information about uniformly convex Banach spaces the reader is referred to Megginson (1998, Chapter 5).

space. For example, if E is a Banach space and E' its topological dual then (E, E') is a dual pair by the Hahn-Banach Theorem. Let (E, F) be a dual pair. Then the polar topology on E determined by the class of finite subsets of F is called the weak topology of E and is denoted by $w(E, F)$. The polar topology on E determined by the class of absolutely convex $w(F, E)$-compact subsets of F is called the Mackey topology on E and is denoted by $m(E, F)$. Polar topologies are discussed in Narici and Beckenstein (1985, Chapter 9).

2. Existence of Utility Functions

We begin by proving the following theorem on the existence of a continuous *metric* utility function on a uniformly convex Banach space. It is important to observe that the proof given below is elementary and direct and does *not* depend upon any deep set-theoretic principle such as e.g. the Axiom of Choice, the Continuum Hypothesis, Souslin's Hypothesis, or Martin's Axiom. [7]

THEOREM 1 *Let E be a uniformly convex Banach space and X a proper open and convex* [8] *subset of E. Assume that $\precsim$ is a binary relation on X that satisfies:*

(a) $\precsim$ is a lower semicontinuous total preorder on X;

(b) each upper section of $\precsim$ is convex and closed in E;

(c) for any $y \in X$ such that there is $\varepsilon > 0$ satisfying $X \cap B(y, \varepsilon) \subseteq L(y)$, it is also true that any $x \sim y$ can be associated with some $\delta > 0$ satisfying $X \cap B(x, \delta) \subseteq L(x) = L(y)$.

In addition, suppose that either one of the following two conditions holds:

(d) every norm-bounded and order-bounded increasing sequence in X has a convergent subsequence

[7] It is clear from recent developments in the literature that such set-theoretic principles are involved and may be used in proving the existence and non-existence of order-preserving functions (see, e.g. Beardon. et. al., 2002 and Roitman, 1990).

[8] Observe that convexity of X is implied by the rest of our assumptions since $X = \bigcup_{x \in X} U(x)$ and the upper sections form a chain of sets under inclusion. We have stated it explicitly because convexity arguments are used extensively along the proof and also because it is a natural requirement that already appeared in the Arrow-Hahn theorem, to which we shall refer afterwards in detail.

(d′) for any $\varepsilon > 0$, every closed ball B and every increasing sequence $\{x_n\}_{x\in\mathbb{N}}$ in X that converges to $x \in X$, there is n_0 such that $[x_n]_\sim \cap B \subseteq B([x]_\sim, \varepsilon)$ whenever $n \geqslant n_0$, where $B(A,\varepsilon)$ denotes $\bigcup_{a\in A} B(a,\varepsilon)$

Then for any $x_0 \in E\setminus X$, the expression $u(x) = d(x_0, U(x))$ defines a continuous utility function for $\preccurlyeq$ on X.

Proof: Since $U(x)$ is closed in E and convex for each $x \in X$, Lemma 1 implies that u is well defined. We let $M(x) = \{y \in U(x) : d(x_0, y) = d(x_0, U(x))\}$. It follows from Lemma 1 that we can write $M(x) = \{x'\}$. We claim that $x' \sim x$. Indeed, $x \preccurlyeq x'$ by definition of $M(x)$. Suppose that $x \prec x'$. Then there is $x' \in V$, an open subset of E, such that $V \subseteq X \setminus L(x)$ -this latter subset is open in E because $\preccurlyeq$ is lower semicontinuous and X is open in E- and $x \prec v$ when $v \in V$. For $\alpha \in (0,1)$ sufficiently small, $z = (1-\alpha)x' + \alpha x_0 \in V$. But then since $z \in V \subseteq U(x)$, one has $u(x) \leqslant ||z - x_0|| = (1-\alpha)u(x)$. This contradiction proves the claim. Observe that $y \preccurlyeq x$ yields $u(x) \geqslant u(y)$, since $U(x) \subseteq U(y)$. If we show that $y \prec x$ and $u(x) = u(y)$ are not compatible, this will then entail that u is a utility function on X. But assuming $u(x) = ||x' - x_0|| = ||y' - y_0|| = u(y)$, with $x' \in U(x) \subseteq U(y)$, means $x' = y'$ because the distance from x_0 to $U(y)$ is attained at a single point. However, this coincidence is impossible, since $y' \sim y \prec x \sim x'$. This argument proves that the function u is a utility function on X.

We prove now that the function u is upper semicontinuous. To that end, we need to verify that for all $t \in \mathbb{R}$ the set $A_t = \{x \in X : u(x) \geqslant t\}$ is closed in X. Consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq A_t$ with limit $x \in X$. By reductio ad absurdum, assume that $u(x) = \|x' - x_0\| < t$, where $x' \in M(x)$. In this event, there is U open such that $x' \in U$, and $z \in U \cap X \Rightarrow \|z - x_0\| < t$. Observe that $x' \in U \cap X$, which is open in E, and also recall that $x \sim x'$.

Using assumption (c), we may assume that x' is not a point of local satiation, since $B(x', \varepsilon) \subseteq L(x')$ with $\varepsilon > 0$ would mean $B(x, \delta) \subseteq L(x)$ for some $\delta > 0$, therefore yielding $x_n \preccurlyeq x$ eventually, which is a contradiction.

We may conclude that there exists $y \in U \cap X$ with $x \sim x' \prec y$, and, therefore, $x \prec y$. By the definition of u, $u(y) \leqslant \|y - x_0\| < t$. However, the upper continuity of $\preccurlyeq$ implies that there is an index n_0 such that $x_n \preccurlyeq y$ whenever $n \geqslant n_0$, and therefore $u(y) \geqslant u(x_n) \geqslant t$ whenever $n \geqslant n_0$. This final contradiction concludes the proof.

We prove next that the function u is lower semicontinuous. Select an arbitrary $t \in \mathbb{R}$. Let us see that the set $B_t = \{x \in X : u(x) \leqslant t\}$ is closed in X.

Take a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq B_t$ with limit $x \in X$. By reductio ad absurdum, assume that $u(x) > t$. Thus, $x_n \prec x$ for all $n \in \mathbb{N}$ and the sequence $\{x_n\}_{n\in\mathbb{N}}$ is order-bounded above.

Fix $x'_n \in M(x_n)$ for each $n > 0$. We have that $\{x'_n\}_{n>0}$ is norm-bounded because $\|x'_n - x_0\| = u(x_n) \leqslant t$ for each $n > 0$.

We may also construct an increasing subsequence $\{x'_{n_k}\}_{k\in\mathbb{N}}$ of $\{x'_n\}_{n\in\mathbb{N}}$ - which amounts to obtaining an increasing subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ too. Define $x'_{n_1} = x'_1$. As $Z_1 = \{y \in X : x'_1 \prec y\}$ is open in X, which is open itself, and because $x \in Z_1$ and $\{x_n\}_{n\in\mathbb{N}}$ converges to x, there must be $x_{n_2} \in Z_1$. Therefore $x'_{n_1} \prec x'_{n_2}$. Continuing in this way, we construct the desired subsequence(s). But we proceed to check that this produces a contradiction, depending on which of either (d) or (d') is satisfied.

In case (d) holds, because $\{x'_n\}_{n>0}$ is norm- and order-bounded there must be another (increasing sub-)subsequence $\{x'_{m_k}\}_{k\in\mathbb{N}}$ with limit x', and thus $x' \in X$. Indeed, by construction x' lies in the closure (relative to E) of $V = \{z \in X : x'_{m_1} \prec z \prec x\}$, and $cl_E(V) = cl_E((X \setminus L(x'_{m_0})) \cap (X \setminus U(x))) \subseteq cl_E(X \setminus L(x'_{m_0})) \subseteq cl_E(U(x'_{m_0})) \subseteq X$. Also, we have $x' \sim x$ by continuity of $\preccurlyeq$. We conclude, since $u(x) = u(x') \leqslant \|x' - x_0\| \leqslant t$.

In case (d') holds, take $\varepsilon = \frac{u(x)-t}{2} > 0$, $B = B(x_0, t+\varepsilon)$ and then an index k_0 for which $[x_{n_{\overline{k}}}]_\sim \cap B \subseteq B([x]_\sim, \varepsilon)$, whenever $\overline{k} > k_0$. Fix any $k > k_0$. We get the contradiction $B(x_0, t+\varepsilon) \cap [x'_{n_k}]_\sim = \varnothing$ though $d(x_0, x'_{n_k}) \leqslant t$: notice that $a \sim x'_{n_k}$ with $d(a, x_0) < t + \varepsilon$ would yield the existence of $y \sim x$ such that $d(y, a) < \varepsilon$, which in turn produces $d(a, x_0) \geqslant d(y, x_0) - d(a, y) > u(x) - \varepsilon = t + \varepsilon$.

This proves that the function u is continuous and completes the proof of the theorem. ∎

REMARK 1 *Condition (d′) holds under the following requirement:*

For any $\varepsilon > 0$ and every increasing sequence $\{x_n\}_{x \in \mathbb{N}}$ in X that converges to $x \in X$, there is n_0 such that $[x_n]_\sim \subseteq B([x]_\sim, \varepsilon)$ whenever $n \geqslant n_0$.

REMARK 2 *It is instructive to compare the assumptions in Theorem 1 with the Arrow-Hahn assumptions. The finite-dimensional Arrow-Hahn theorem requires all of the assumptions with the following exceptions. First, the assumption that each upper section is convex is not needed because in $\mathbb{R}^n$ each closed set is proximinal. However, the convexity of upper sections is essential in an infinite-dimensional context. Second, it is interesting to observe that the assumption stated in (d′) has one advantage: it is automatically fulfilled by any continuous preference in the setting of a finite-dimensional normed spaces, where closed balls are compact (cf. Jameson, 1974, Section 20) and the same property holds for (d). Third, we emphasize that we are not assuming that the indifference classes are "thin"; the only restriction about their forms is given by condition* (c). *Roughly speaking, condition* (c) *says that if an indifference class is not "thin" at some point, then it is not "thin" at any other point in it. This condition replaces the requirement that the binary relation is locally non-satiated, which is needed in the Arrow-Hahn theorem. We recall that it is also possible to weaken the local non-satiation condition in the Arrow-Hahn theorem by the assumption that any point of local satiation is also a point of global satiation (see, Alcantud, 2002 and Mehta, 1991a, p. 977). Observe that with this assumption our condition* (c) *is an easy consequence. Indeed, either there is no point of local satiation (and thus our requirement is met vacuously) or, if there is such a point -i.e. if there is $\varepsilon > 0$ with $X \cap B(y, \varepsilon) \subseteq L(y)$-, then for any $x \sim y$ we would have $L(x) = L(y) = X$, so it would always be true that any*

$\delta > 0$ satisfies $X \cap B(x, \varepsilon) \subseteq L(x)$. On the other hand in the Arrow-Hahn theorem, where $X \subseteq \mathbb{R}^n$ is closed and convex and therefore (path-)connected, our assumptions preclude the existence of points of local satiation other than global satiation points, since the lower section associated with any such point would be open and closed in X and this disconnects X unless that point is also a point of global satiation. Fourth, in the Arrow-Hahn theorem and the infinite-dimensional generalizations of this theorem each upper section is only required to be closed in X. Here, a stronger assumption is needed, namely, that each upper section is closed in the whole space E. These considerations show that the approach we used in Theorem 1 does not provide an extension of the Arrow-Hahn theorem, but rather a closely related variation of it.

REMARK 3 *We do* not *assume that the Banach space satisfies the topological separability requirement. This is quite a stringent assumption in the infinite-dimensional context. For more about this assumption, in particular in the context of infinite-dimensional economic models, see Mehta-Monteiro (1996) and Mehta (1998).*

In the first part of this paper we have given a simple and appealing argument to prove the existence of continuous utility functions for special kinds of Banach spaces using the metric approach. However, the assumption that the space is uniformly convex is restrictive. Therefore, we now prove some more general results. [9]

In Theorem 1, the Banach space E is endowed with the norm topology which is metrizable so that sequential convergence is adequate. However, for an infinite-dimensional Banach space E with other topologies convergence cannot, in general, be described by sequences. We need to replace sequences by nets or filterbases as we see in the following remark.

[9] The price that we pay for this generality is that now the argument is no longer elementary and direct because of the fact that essential use is made of the Banach-Alaoglu Theorem and this theorem is proved by using Tychonoff's theorem which is equivalent to the Axiom of Choice. We do not know if the Axiom of Choice can be avoided in this context. However, it should be observed that for separable Banach spaces there are proofs in intuitionistic mathematics of both the Hahn-Banach Theorem and the Banach-Alaoglu Theorem which do *not* use the Axiom of Choice or Zorn's Lemma (see, e.g. Bishop and Bridges, 1985).

LEMMA 2 *Let E be an infinite-dimensional Banach space. Then neither the weak topology on E nor the weak* topology on E' is first countable; in particular neither of these topologies is metrizable (Fabin et. al., 2001, p. 95).*

THEOREM 2 *Let E be a Banach space with a pre-dual F and let X be a proper $w(E,F)$-open and convex subset of E. Suppose that $\preccurlyeq$ is a binary relation on X that satisfies the following conditions:*

(a) $\preccurlyeq$ is a total preorder;

(b) the upper section $U(x)$ of each $x \in X$ is $w(E,F)$-closed in E;

(c) for any $y \in X$ such that there is $\varepsilon > 0$ satisfying $X \cap B(y, \varepsilon) \subseteq L(y)$, it is also true that any $x \sim y$ can be associated with some $\delta > 0$ satisfying $X \cap B(x, \delta) \subseteq L(x) = L(y)$

Suppose that, in addition each lower section $L(x)$ is $w(E,F)$-closed in X and that either (d) or (d′) holds:

(d) every norm-bounded and order-bounded increasing net in X has a convergent subnet;

(d′) for any $\varepsilon > 0$, every closed ball B and every increasing net $\{x_s\}_{s\in\mathcal{D}}$ in X that converges to $x \in X$, there is s_0 such that $[x_s]_\sim \cap B \subseteq B([x_s]\sim, \varepsilon)$ whenever $s \geqslant s_0$, where $B(A, \varepsilon)$ denotes $\bigcup_{a\in A} B(a, \varepsilon)$

Then for any $x_0 \in E \setminus X$, the expression $u(x) = d(x_0, U(x))$ defines a $w(E,F)$-continuous utility function for $\preccurlyeq$ on X.

Proof: We observe first that each upper section $U(x)$ is proximinal in E. This is because $U(x)$ is $w(E,F)$-closed in E by condition (b) and each $w(E,F)$-closed subset of E is proximinal in E by the Banach-Alaoglu Theorem (Holmes, 1975, p. 116). This proves that the function u is well-defined on each $U(x_0)$.

We claim that for all $x \in U(x_0)$, $x^\star \in M(x)$ implies that $x \sim x^\star$. Suppose, to the contrary, that $x \prec x^\star$. Clearly, we may assume that the line segment $[x_0, x^\star]$ is non-degenerate. Let $z_\lambda = (1-\lambda)x_0 + \lambda x^\star$ and consider the net $\{z_\lambda : \lambda \in [0,1]\}$. The net $\{z_\lambda\}$ converges to $x^\star$ in the norm topology. Therefore, it converges to $x^\star$ in the w(E,F) topology. Since $\preccurlyeq$ is w(E,F)-lower semicontinuous each strict upper section is w(E,F)-open and it follows that there exists λ_0 such that $x \prec z_{\lambda_0}$ contradicting the definition of the function $u(x)$. Hence, the claim is proved.

The rest of the argument is similar to the proof of Theorem 1 except that, one works with the w(E,F) topology instead of the norm topology and in view of the above lemma, we replace sequences by nets or filterbases. ■

REMARK 4 *In contrast with Theorem 1, we observe that in the above theorem we have* not *assumed that the preorder has convex upper sections. This condition is not needed because each $w(E, F)$-closed subset of E is proximinal in E essentially because of the properties of the weak* topology. The situation is the same as in the Arrow-Hahn theorem because in $\mathbb{R}^n$ the Euclidean topology is equal to the weak* topology (Fabián, et. al., 2001, p. 66).*

REMARK 5 *It is worthwhile to point out that Theorem 2 applies to the space $L^\infty(\mu)$, where μ is a σ-finite measure, and to the space $M(K)$ of finite signed Baire measures on a compact Hausdorff space K with the variation norm because these spaces are conjugate Banach spaces by the Riesz Representation Theorem (Royden, 1988, p. 246 and p. 311). These spaces have been used in economic applications (see, e.g. Bewley, 1972 and Mas-Colell, 1975). Suppose now that E is a reflexive space. Then by the Banach-Bourbaki Theorem (Narici and Beckensterin, 1985, p. 336) we may conclude that the unit ball in E is weakly compact. Hence, under the conditions of the above theorem we also get a continuous utility representation. Finally, observe that this enables us to generalize Theorem 1 because every uniformly convex Banach space is a reflexive Banach space (see Megginson, p. 452).*

In Theorem 2 we have proved the existence of a continuous constructive utility function on a dual Banach space with a preference relation that is continuous with respect to the *weak** topology. It is desirable to have such a utility representation for preferences that are continuous with respect to the Mackey topology in view of the importance of this topology in infinite-dimensional equilibrium theory (see, e.g. Bewley, 1972, Brown and Lewis, 1981 and Toussaint, 1984). We now prove such a theorem for a Mackey continuous utility function for a convex preference relation.

THEOREM 3 *Let E be a Banach space with pre-dual F and X a proper $w(E,F)$-open and convex subset of E. Suppose that $\preccurlyeq$ is a binary relation on E that satisfies the following conditions:*

(a) $\preccurlyeq$ is a total preorder;

(b) the upper section $U(x)$ of each $x \in X$ is convex and m(E,F)-closed in E;

(c) $\preccurlyeq$ is $m(E,F)$ is lower semicontinuous on X;

(d) for any $y \in X$ such that there is $\varepsilon > 0$ satisfying $X \cap B(y,\varepsilon) \subseteq L(y)$, it is also true that any $x \sim y$ can be associated with some $\delta > 0$ satisfying $X \cap B(x,\delta) \subseteq L(x) = L(y)$.

Suppose that, in addition either (e) or (e′) holds:

(e) every norm-bounded and order-bounded increasing net in X has a convergent subnet;

(e′) for any $\varepsilon > 0$, every closed ball B and every increasing net $\{x_s\}_{s\in\mathcal{D}}$ in X that converges to $x \in X$, there is s_0 such that $[x_s]_\sim \cap B \subseteq B([x_s]\sim,\varepsilon)$ whenever $s \geqslant s_0$, where $B(A,\varepsilon)$ denotes $\bigcup_{a\in A} B(a,\varepsilon)$

Then for any $x_0 \in E \setminus X$, the expression $u(x) = d(x_0, U(x))$ defines an $m(E,F)$-continuous utility function on X that is $w(E,F)$-upper semicontinuous.

Proof: In view of the permanence in duality of closed convex sets (Narici and Beckenstein, 1985, 207), any convex set has the same closure with respect to any topology of a dual pair. Therefore, the Mackey-Arens Theorem (Narici and Beckenstein, 1985, p. 205) implies that $\preccurlyeq$ is $w(E,F)$-upper semicontinuous because the *weak** topology $w(E,F)$ is the weakest and the $m(E,F)$ topology the strongest topology of the dual pair (E,F). Therefore, for each $x \in X$ the upper section $U(x)$ is $w(E,F)$-closed and so the function u may be defined as in Theorem 2.

The rest of the proof is concluded, *mutatis mutandis*, as in Theorem 2. ■

4. Conclusions and Future Research

In this paper we have proved the existence of continuous utility functions on subsets of Banach spaces by using a constructive procedure based on the concept of Euclidean distance similar to the one employed by Wold and Arrow-Hahn. An approach similar to that of this paper is employed in Mehta (1995) where it is proved that utility functions of the Arrow-Hahn type and the so-called "money-metric" utility functions have a common basis and are not really different problems as might seem from the literature. In that paper a theorem is proved which subsumes as special cases results of the Arrow-Hahn type and other results in the literature dealing with "money-metric" utility functions; this is accomplished by again generalizing the Arrow-Hahn method. In the paper of Alcantud-Manrique (2001) results on the existence of "money-metric" utility functions are proved. It would be interesting to try to unify these two papers and to obtain common infinite-dimensional generalizations and extensions of the results and methods of Alcantud-Manrique (2001) and Mehta (1995) since both these papers work in the context of finite-dimensional spaces.

In an interesting paper Beardon (1997) used the Euclidean distance approach to prove the existence of a (continuous) utility function on a metric space which need not have any linear structure. It is an open question whether the ideas and results of the present paper can be related to Beardon's methods.

Throughout this paper we have assumed that the preorder is total. It would be interesting to try to weaken this condition since many preference relations that arise are not total and some are not even preorders. [10]

Finally, we refer the reader to the recent Ph.D thesis of Campión (2004) for further utility representation theorems on Banach spaces.

[10] For some partial results in this direction in the context of finite-dimensional Euclidean spaces the reader is referred to Bridges (1988).

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