Movement of equilibrium of Cournot duopoly and the visualization of bifurcations of its adjustment dynamics

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Abstract

This paper deals with the analytical and graphical representation of the bifurcations appearing from the adjustment dynamics of a 2-player Cournot duopoly, proposed by Puu (1997). We establish admissibility conditions on the initial state of the adjustment dynamics and visualize the dynamics in the space of orbits.

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1 Introduction

Cournot (1838) understood and clearly stated that the quantity competition in a non-cooperating duopoly leads to a quantity outcome that is stable and that any firm deviating from it, by self-interest alone, will be brought back to equilibrium in a sequence of adjustments. While not using the same exact terms, Cournot also stated in reference to the stability of the outcome that the duopoly equilibrium is an attractor while the cooperative monopoly equilibrium point is a repeller therefore, while most favorable to producers; it is unstable without self-enforcement or an enforcing mechanism. Friedman (1983) considers a myopic duopoly where each firm maximizes its profit in a given period given what its rival(s) produced in the previous period, obviously such a model can be criticized on many points but for us one salient failure, mentioned by the author, is the inability of firms to adjust their outputs to the other firms' output levels. In this paper, we consider an adjustment mechanism where a firm's output in period $t+1$ is the output in period t adjusted by a fraction of the output excess/shortage in the previous period (Puu, 1997).

This paper aims to analytically describe the critical bifurcation curves, while other authors did recognize the bifurcation effects using a trial and error approach (Agiza, 1998), here we derive the exact forms of bifurcation based on preset values of coordinates of Cournot equilibrium chosen as bifurcation parameters and visualize the dynamics by moving the equilibrium point along a given path. In the next section, we describe the adjustment dynamics and characterize its stability, in the third section we study the bifurcation emerging from the adjustment dynamics, in section four we represent bifurcation patterns of interest and put conditions on admissible initial points, and section five concludes the paper.

2 Model

Puu (1997, Ch.5) introduced an iterative process, which leads two oligopolies to the Cournot-Nash equilibrium via the following iterative process

$$\begin{cases} x_{t+1} = x_t + \lambda \left(\sqrt{\frac{y_t}{a}} - y_t - x_t \right) \\ y_{t+1} = y_t + \mu \left(\sqrt{\frac{x_t}{b}} - x_t - y_t \right) \end{cases} \tag{1}$$

where (x, y) are the supplies at time t of two competitors in a duopoly; a and b are their constant marginal costs, and λ and μ are the adjustment speeds such that $0 \leq \{\lambda, \mu\} \leq 1$. The firms produce an undifferentiated product sold at a market price $P = \frac{1}{x+y}$. The fixed point (x, y) of this iteration dynamics satisfies the system of algebraic equations

$$\begin{cases} x = x + \lambda \left(\sqrt{\frac{y}{a}} - y - x \right) \\ y = y + \mu \left(\sqrt{\frac{x}{b}} - x - y \right) \end{cases} \tag{2}$$

with a non-zero solution

$$x = \frac{b}{(a+b)^2}; y = \frac{a}{(a+b)^2} \tag{3}$$

which is the Cournot-Nash equilibrium.

The detailed analysis of the structure of the domain of attraction of this equilibrium and the bifurcations of its adjustment dynamics in (1) can be described analytically as follows (Sonis, 2000, pp. 340-341). The Jacobi approximation matrix for the dynamics in (1) is

$$J_{t,t+1} = \begin{bmatrix} 1-\lambda & \lambda\left(\frac{1}{2\sqrt{ay_t}}-1\right) \\ \mu\left(\frac{1}{2\sqrt{bx_t}}-1\right) & 1-\mu \end{bmatrix} \tag{4}$$

At the Cournot equilibrium, this matrix becomes

$$J = \begin{bmatrix} 1-\lambda & \frac{\lambda(b-a)}{2a} \\ \frac{\mu(a-b)}{2b} & 1-\mu \end{bmatrix} \tag{5}$$

therefore,

$$TrJ = 2-(\lambda+\mu)\,;\ \ \det J = (1-\lambda)(1-\mu)+\lambda\mu\frac{(a-b)^2}{4ab} \tag{6}$$

It is well known (Hsu, 1977) that the domain of attraction of every 2-dimensional discrete dynamics has a form

$$-1 \pm TrJ < \det J < 1 \tag{7}$$

It is easy to see that for the adjustment dynamics in (2) we always have

$$-1 \pm TrJ < \det J \tag{8}$$

Hence the domain of attraction of the Cournot equilibrium is defined by the inequality $\det J < 1$, which gives

$$(1-\lambda)(1-\mu)+\lambda\mu\frac{(a-b)^2}{4ab} < 1 \tag{9}$$

or

$$\frac{(a+b)^2}{4ab} < \frac{1}{\lambda}+\frac{1}{\mu} \tag{10}$$

Let us introduce the ratio of the marginal costs;

$$\frac{a}{b} = k\ \left(=\frac{x}{y}\right) \tag{11}$$

and let

$$\frac{1}{\lambda}+\frac{1}{\mu}=\Lambda \tag{12}$$

Obviously we have $k>0$ and $\Lambda \geq 2$. Then from (10)-(12), the domain of attraction of the Cournot equilibrium is

$$\frac{(1+k)^2}{4k}<\Lambda \tag{13}$$

or

$$k^2-2k\,(2\Lambda-1)+1<0 \tag{14}$$

This implies that the following inequality represents the domain of attraction for the Cournot equilibrium

$$k_1<k<k_2 \tag{15}$$

where $k_1, k_2>0$ are the (positive) roots of (14) when transformed into an equality

$$k_{1,2}=(2\Lambda-1)\mp 2\sqrt{\Lambda\,(\Lambda-1)} \tag{16}$$

Obviously, these roots are reciprocal, since we can use the reciprocal ratio $k=\frac{b}{a}$. The character of bifurcations in these roots is the same and is defined by the value of α (Sonis, 2000, p. 341)

$$\alpha=TrJ=2-(\lambda+\mu)>0 \tag{17}$$

If the quantity $\Omega=\frac{1}{2\pi}\arccos\frac{\alpha}{2}$ is a rational number $\frac{p}{q}$ then the adjustment dynamics will be q-periodic; if Ω is irrational, then the adjustment dynamics will be quasi-periodic.

Few examples are of special interest; if the adjustments (λ,μ) are unitary, i.e. $\alpha=0$ then we have a 4-period cycle starting the Feigenbaun double-periodic way to chaos; if $\lambda+\mu=1$, then $\alpha=1$ then $\alpha=1$ and we have 6-period cycle bifurcations. Moreover, from (17) $\alpha>0$, the duopoly adjustment cannot have 2-periodic bifurcation (with $\alpha=-2$) and 3-period bifurcation (with $\alpha=-1$), but it can have the 5-period cycle corresponding to $\alpha=0.61803$, i.e., $\lambda+\mu=1.38157$ (see figure 1 for the 5-periodic cycle), etc.

3 Movement of Cournot equilibrium and corresponding bifurcations of adjustment dynamics in the space of orbits

In formulae (3), if the marginal costs a and b are changing, i.e. the point (a,b) is moving in the space of marginal costs, then the Cournot equilibrium (x,y) also changes its position in the space of orbits (x_t,y_t). For example, if the point (a,b) is moving on the straight line $b=sa+r$ in the space of marginal costs then the Cournot equilibrium (x,y) is moving on the curve $y=rx+s(x+y)^2$ in the space of orbits.

It is possible to see, that equation (3) implies that

$$a = \frac{y}{(x+y)^2}; \; b = \frac{x}{(x+y)^2} \tag{18}$$

this means that we can consider the coordinates of the Cournot equilibrium as playing the role of bifurcation parameters. In such a way we are replacing the set of bifurcation parameters $(a, b; \lambda, \mu)$ by a new set $(x, y; \lambda, \mu)$. Fixing the speeds of change (λ, μ) we can visualize in the same space of orbits the domain of attraction of Cournot equilibrium, its movement, and the bifurcations of actual adjustment dynamics. This is convenient because the boundaries of the domain of attraction in the space of orbits depend on the parameters (λ, μ) alone.

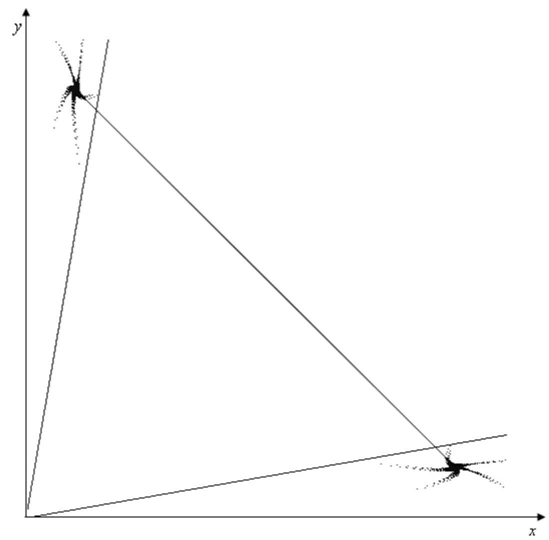

Figure 1: Domain of attraction of the Cournot equilibrium in the space of orbits and 5-period dynamics ($\lambda = \mu = 0.690985$, time periods: 5000, fixed points: 2000)

Indeed, after the substitution of (18) into (15), the analytical description of the domain of attraction obtains the form:

$$(2\Lambda - 1) - 2\sqrt{\Lambda(\Lambda - 1)} = \frac{1}{(2\Lambda - 1) + 2\sqrt{\Lambda(\Lambda - 1)}} < \frac{y}{x} < (2\Lambda - 1) + 2\sqrt{\Lambda(\Lambda - 1)} \tag{19}$$

Geometrically, this inequality defines the angle in the space of orbits with the vertex in the origin of coordinates and the sides defined by the straight lines

$$\begin{cases} \frac{y}{x} = (2\Lambda - 1) + 2\sqrt{\Lambda(\Lambda - 1)} \\ \frac{y}{x} = (2\Lambda - 1) - 2\sqrt{\Lambda(\Lambda - 1)} \end{cases} \tag{20}$$

Since $\Lambda \geq 2$ then

$$\begin{cases} \frac{y}{x} = (2\Lambda - 1) + 2\sqrt{\Lambda(\Lambda - 1)} \\ \frac{y}{x} = (2\Lambda - 1) - 2\sqrt{\Lambda(\Lambda - 1)} \end{cases} \tag{21}$$

This means that the domain of stability of Cournot equilibrium in the space of orbits always includes the angle (see figure 1)

$$3 - 2\sqrt{2} \leq \frac{y}{x} \leq 3 + 2\sqrt{2} \tag{22}$$

If the Cournot equilibrium lies on the boundaries (20) of this domain, the adjustment dynamics undergoing the bifurcation defined by the values of $\Lambda = \lambda + \mu$ (see eq. (17)); if the quantity $\Omega = \frac{1}{2\pi} \arccos \frac{2-\Lambda}{2}$ is a rational number $\frac{p}{q}$ then the adjustment dynamics will be q-periodic; if Ω is irrational, then the adjustment dynamics will be quasi-periodic.

4 Visualization of bifurcations of adjustment dynamics

The transfer from the space of marginal costs to the space of orbits, together with the immovability of the boundaries of the domain of attraction in the space of orbits when the Cournot equilibrium is changed with constant speeds of change provides the possibility to visualize all admissible qualitative features of the behavior of the adjustment dynamics, for the equilibrium near and on the boundaries of the attraction domain. The movements of the equilibrium in the space of orbits on the segments of straight lines and the crossing of the boundaries of the attraction domain reveal the plethora of possible ways from stability, periodicity, Arnold horns and quasi-periodicity to chaos.

4.1 Admissible initial states of the adjustment dynamics

The Cournot-Nash equilibrium is found by solving a system of reaction functions determining the output of one firm as a function of the output of the other firm, such reaction functions must lead to non-negative levels of output, this translates into having

$$\begin{cases} x = x + \lambda\left(\sqrt{\frac{y}{a}} - y - x\right) \\ y = y + \mu\left(\sqrt{\frac{x}{b}} - x - y\right) \end{cases} \tag{23}$$

and provides the condition on the initial point $(x_0, y_0) \in [0, \frac{1}{b}] \times [0, \frac{1}{a}]$.

We call the initial state (x_0, y_0) admissible if the adjustment dynamics at each step t produces the non-negative states (x_t, y_t). The conditions of non-negativity can be presented in the following form:

Lemma 1. *Let for some fixed t the following conditions hold*

$$\begin{cases} 0 \le x_t \le \frac{1}{b} \\ 0 \le y_t \le \frac{1}{a} \end{cases} \tag{24}$$

then

$$\begin{cases} 0 \le x_{t+1} \le \frac{1}{a} + \mu\left(\frac{1}{4a} - \frac{1}{b}\right) \\ 0 \le y_{t+1} \le \frac{1}{b} + \lambda\left(\frac{1}{4b} - \frac{1}{a}\right) \end{cases} \tag{25}$$

Proof. Conditions (21) imply that

$$\frac{\sqrt{x_t}}{\sqrt{a}} - x_t \ge 0; \;\; \frac{\sqrt{y_t}}{\sqrt{b}} - y_t \ge 0 \tag{26}$$

Therefore,

$$\begin{cases} x_{t+1} = \lambda\left(\frac{\sqrt{y_t}}{\sqrt{b}} - y_t\right) + x_t\left(1-\lambda\right) \ge 0 \\ y_{t+1} = \mu\left(\frac{\sqrt{x_t}}{\sqrt{a}} - x_t\right) + y_t\left(1-\mu\right) \ge 0 \end{cases} \tag{27}$$

Further

$$\begin{aligned} x_{t+1} &= \tfrac{\lambda}{4a} - \lambda\left(\sqrt{y_t} - \tfrac{1}{2\sqrt{a}}\right)^2 + x_t\left(1-\lambda\right) \\ &\le \tfrac{\lambda}{4a} + \tfrac{1-\lambda}{b} = \tfrac{1}{b} + \lambda\left(\tfrac{1}{4a} - \tfrac{1}{b}\right) \end{aligned} \tag{28}$$

and

$$\begin{aligned} y_{t+1} &= \tfrac{\mu}{4b} - \mu\left(\sqrt{x_t} - \tfrac{1}{2\sqrt{b}}\right)^2 + y_t\left(1-\mu\right) \\ &\le \tfrac{\mu}{4b} + \tfrac{1-\mu}{a} = \tfrac{1}{a} + \mu\left(\tfrac{1}{4b} - \tfrac{1}{a}\right) \end{aligned} \tag{29}$$

□

Lemma 2. *If*

$$\frac{1}{4} \le \frac{b}{a} \le 4 \tag{30}$$

then each initial state (x_0, y_0) such that:

$$0 \le x_0 \le \frac{1}{b}; \;\; 0 \le y_0 \le \frac{1}{a} \tag{31}$$

is admissible and the adjustment dynamics (1) converges to the attractor point $x = \frac{b}{(a+b)^2}$ and $y = \frac{a}{(a+b)^2}$.

Proof. Conditions (30) imply that $\frac{1}{4a} - \frac{1}{b} \leq 0$ and $\frac{1}{4b} - \frac{1}{a} \leq 0$, therefore, from (28) and (29) we have $x_{t+1} \leq \frac{1}{b} + \lambda \left(\frac{1}{4a} - \frac{1}{b} \right) \leq \frac{1}{b}$ and $y_{t+1} \leq \frac{1}{a} + \mu \left(\frac{1}{4b} - \frac{1}{a} \right) \leq \frac{1}{a}$. For each t, the state (x_t, y_t) is non-negative, i.e. each initial state satisfying (31) is admissible. Condition (30) geometrically represents the angle $\frac{1}{4} \leq \frac{x}{y} \leq 4$ which lies inside the domain of attraction defined by (20) of the adjustment dynamics and the fixed point $x = \frac{b}{(a+b)^2}$ and $y = \frac{a}{(a+b)^2}$ is the attractor point for this dynamics. □

Remark. If condition (25) does not hold, then for visualization of the adjustment dynamics one should chose the admissible initial state of the form, for example, $x_0 = x + \varepsilon$ and $y_0 = y + \varepsilon$ with a suitable small $\varepsilon > 0$.

4.2 Visualization of adjustment dynamics

The numerical procedure of the description of such phenomena includes the construction of spatial bifurcation diagrams in which the bifurcation parameter is the equilibrium itself. The construction of two-dimensional bifurcation diagram, i.e. the visualization of the movements of equilibria in the space of orbits on the segments of straight lines can be accomplished in the following way. Let us fix the speeds of change (λ, μ) and the same initial state $(\hat{x}, \hat{y})$ for adjustment dynamics (1) at all bifurcation steps. Further let us chose the number of bifurcation steps S and the segment of the straight line of movement of equilibrium $[(x_0, y_0), (x_S, y_S)]$. It is possible to parameterize the segment of the straight line between (x_0, y_0) and (x_S, y_S) as

$$\left\{ \begin{array}{l} x\left(j\right) = x_0 \left(1 - \frac{j}{S}\right) + x_S \frac{j}{S} \\ y\left(j\right) = y_0 \left(1 - \frac{j}{S}\right) + y_S \frac{j}{S} \end{array} \right. ;\ j = 0,\ 1, ..., S \tag{32}$$

where j is a bifurcation parameter and S is a number of bifurcation steps.

Formulae (17) helps to construct the marginal costs $(a(j), b(j))$ for each bifurcation step. Choosing these marginal costs we can calculate with the help of (1) the orbit of corresponding adjustment dynamics. The two-dimensional bifurcation diagram is the presentation of the "tails" $(t = P, P+1, ...T)$ of orbits of the adjustment dynamics (x_t, y_t) on all bifurcation steps. The usual one-dimensional bifurcation diagram can be obtained from (32) by presenting for each $(t = P, P+1, ...T)$ the diagram of change of coordinates x_t and y_t separately against the bifurcation parameter j. The important conclusion is that these steps provide the visualization of adjustment dynamics with a *preset* of qualitative properties.

Let us start the visualization of the adjustment dynamics with the case of transfer from the attraction to 4-periodic cycle (see figure 2). The 4-periodic cycle corresponds to the maximal speeds of adjustment $\lambda = \mu = 1$. If the Cournot equilibrium is crossing over the critical straight lines $\frac{y}{x} = 3 \pm 2\sqrt{2}$ the adjustment dynamics undergoes the bifurcation from the attraction to Cournot equilibrium to the attractive 4-periodic cycle.

The 5-perido bifurcation diagrams is provided in figure 1 and the following bifurcation diagram represents the adjustment dynamics, corresponding to the 9-period dynamics (figure 3).

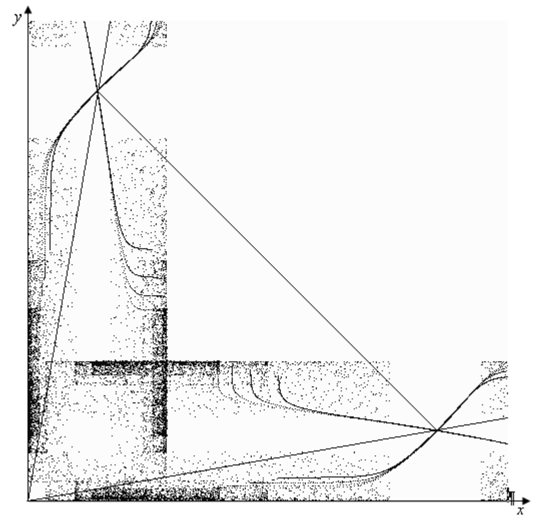

Figure 2: Two-dimensional bifurcation diagram describing the 4-periodic cycle

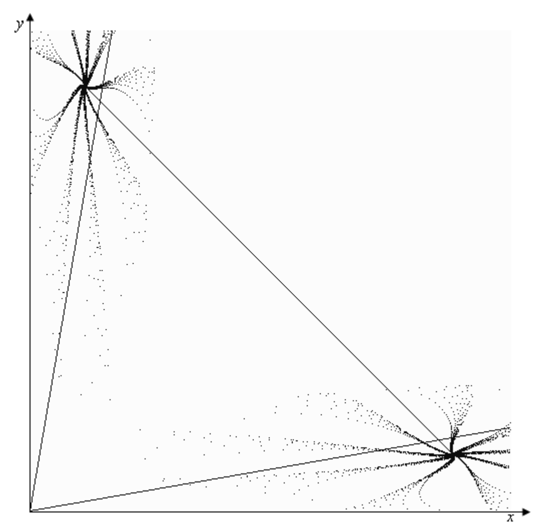

Figure 3: Two-dimensional bifurcation diagram describing the 9-periodic cycle

5 Conclusion

In this paper the analytical and geometrical constructions of the domain of attraction of 2-player Puu adjustment dynamics was achieved by choosing the coordinates of Cournot equilibrium as bifurcation parameters. Such a choice of bifurcation parameters allows to describe analytically the set of all possible bifurcation phenomena and to visualize the preset bifurcation events.

In Sonis (2000), analogical method was elaborated for a 3-player Cournot game in their space of orbits while Agiza (1998) studies the stability of 3-player and 4-player Cournot games and characterizes their bifurcation in the space of orbits. Variations introduced to the study of the adjustment dynamic of the Cournot model involve heterogeneous expectations in Agiza, Hegazi and Elsadany (2001) where the authors showed that varying the adjustment speed of the bounded rational player leads to unstable behavior, or capacity constraint (Puu and Norin, 2003).

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