Speculation in Standard Auctions with Resale

By Rod Garratt and Thomas Tröger*

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Abstract

We analyze the role resale creates for zero-value bidders, called speculators, in standard auctions with symmetric independent private values buyers. English/second-price auctions always have equilibria with active resale markets and positive profits for a speculator. In first-price/Dutch auctions, the unique equilibrium can involve an active resale market, but is never profitable for a speculator. In all standard auctions, allowing resale can increase the initial seller's revenue and lead to an inefficient allocation. First-price and second-price auctions are not revenue equivalent.

Keywords: second-price auction, first-price auction, English auction, speculation, resale, efficiency

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1 Introduction

Open English auctions and sealed-bid first-price auctions are among the most widely used sales mechanisms, historically and currently, with applications ranging from sales of used comic books on eBay to sales of government contracts worth billions of dollars. Beginning with Vickrey (1961), a

*Department of Economics, University of California at Santa Barbara, CA 93106-9210, U.S.A. We are grateful to Ted Bergstrom, Ken Binmore, James McAndrews, Eddie Dekel, David Easley, Gunnar Gunnarson, Matthew Jackson, Philippe Jehiel, Alexander Koch, Steve LeRoy, Benny Moldovanu, Georg Nöldeke, Jörg Oechssler, Thomas Palfrey, Hugo Sonnenschein, Ennio Stacchetti, John Wooders, Bill Zame, Charles Zheng, and three anonymous referees for helpful comments.

tremendous amount of research effort has focused on the strategic properties of these standard auctions, including the related second-price and Dutch auctions (see Klemperer's 1999 survey). The most fundamental results are that standard auctions allocate a good efficiently and yield identical revenues, provided the bidders are symmetric, have independent private values (SIPV), and there is no resale. In practice, however, active resale markets are common. Yet, relatively little is known about the strategic and economic properties of auctions with resale. Meanwhile, standard auction formats continue to be used without a full understanding of the implications of resale opportunities.

This paper shows that resale opportunities dramatically change the properties of standard auctions with SIPV buyers. In particular, we explore the role created by resale opportunities for bidders who have no consumption value for the goods in the market. Because such bidders buy solely in order to resell, we call them *speculators*. It is hard to argue that a model without speculators can be a complete description of a market with resale.

We consider a 2-period interaction. In period 1, the good is offered via a standard auction without a reserve price to SIPV buyers and a speculator (the results remain valid if multiple speculators can enter and, with some qualifications, if the auction is augmented by a reserve price). The winner of the auction either consumes the good or puts it up for resale in period 2. The period-2 seller chooses a resale mechanism that maximizes her expected period-2 payoff given her posterior beliefs about the market; in particular, the period-2 seller has full bargaining power in the resale market.[1] No new bidders arrive after period 1, and no information becomes public beyond what is revealed via bids. Because our focus is on the speculator, we restrict attention to perfect Bayesian equilibria that are symmetric across the SIPV buyers. We determine all such equilibria, modulo a monotonicity and a participation restriction.

We derive results for English/second-price auctions and first-price/Dutch auctions. Concerning the English/second-price auction, we show that—while bid-your-value remains an equilibrium outcome—the resale opportunity creates a continuum of new equilibria (in undominated strategies) with active resale markets. For every $q \in (0, 1]$, we construct an equilibrium such that the speculator wins the auction with probability q. All these equilibria are profitable for the speculator. Because the speculator keeps the good

[1]Wilson (1987, p. 36-37) stresses the practicality and robustness of detail-free standard auction mechanisms. However, factors that make standard auctions appropriate for the initial seller will not deter bidders, who are well-informed market traders, from using optimal resale mechanisms.

with positive probability, none of the equilibria yields an efficient allocation.

The key to understanding the English/second-price auction equilibria is that bid-your-value is not part of a dominant strategy when resale is possible. A buyer may want to reduce her chances to win in period 1 because she expects a lower price in period 2. In the equilibria we construct, low-value buyers pool at a bid of 0 in period 1 (which can be interpreted as abstaining from the auction). This allows the speculator to win the auction and get the good for free. The speculator's bid is higher than her resale price, in order to make it optimal for low-value buyers to wait for resale.

The first-price/Dutch auction results are quite different from the English/ second-price auction results. The equilibrium is always unique, and never profitable for the speculator. This is not to say that the speculator never becomes active. For any number of buyers in the market, there are examples of value distributions with increasing hazard rate such that the bidding competition among buyers is so weak that the speculator is attracted. In equilibrium buyers increase their bids until the speculator's payoff is driven down to zero, while the speculator randomizes her bid in a way that makes the buyers' increased bids optimal. Whenever the equilibrium is such that the speculator becomes active, there is delay and a probability of misallocation. Hence, the equilibrium is inefficient.

In some auction environments, the initial seller has the option to eliminate inefficiencies by taking actions that would prohibit resale. It is interesting, therefore, to know how prohibiting resale would effect initial-seller revenue. This enables the initial seller to weigh the desire for efficiency against the potential revenue loss from prohibiting resale. For English/second-price auctions we show that resale opportunities have an ambiguous effect on the initial seller's revenue: a little activity of the speculator ($q \approx 0$) increases the initial seller's expected revenue, but a lot of activity ($q \approx 1$) reduces it. Hence, depending on the prevailing equilibrium an initial seller who is restricted to an English or second-price auction without a (substantial) reserve price may or may not increase revenue by preventing resale. In the case of first-price/Dutch auctions, prohibiting resale never increases the initial seller's revenue, and reduces it in some cases.

Our revenue results imply that first-price and second-price auctions are not revenue-equivalent: the second-price auction always has an equilibrium that yields a lower revenue for the initial seller than the first-price auction equilibrium. Moreover, *none* of the inefficient equilibria in the second-price auction case yields the same allocation as the first-price auction equilibrium.

To the best of our knowledge, our model is the first that addresses how 0-value bidders can "invade" markets with private-value buyers. Bikhchandani and Huang (1989) and Bose and Deltas (1999) analyze auctions where 0-value bidders are allowed to participate, while final consumers are confined to the resale market. These models make sense for US Treasury Bill auctions or large real-estate auctions, where the 0-value bidders represent *intermediaries* or *middlemen* who go on to sell to the general public. In Bikhchandani and Huang, the 0-value bidders submit bids based on their private signals about the common resale value. Bikhchandani and Huang provide sufficient conditions such that a multiple-unit uniform-price auction yields a higher revenue than a discriminatory auction. Bose and Deltas show that admitting a single private-value buyer to an initial second-price auction will cause the 0-value bidders to reduce their bids or even abstain, thus lowering the initial seller's revenue.

Standard private-value auctions with resale and no speculators have been examined by Haile (1999, 2000, 2003) and Gupta and Lebrun (1999). In these models, active resale markets emerge as a consequence of changes of the environment between periods. Haile (1999) allows for the arrival of new buyers in the resale market. In Haile (2000, 2003) and Gupta and Lebrun (1999), resale occurs because agents receive new information about their, initially uncertain, private values, or because values are made public after the initial auction. By construction, equilibria in these models lead to an efficient allocation. It is shown that the resale opportunity increases the initial seller's revenue if the resale seller has enough bargaining power in the resale market.

Zheng (2000, Section 5.2) considers what happens if resale is possible after an optimal auction à la Myerson (1981). He considers two possibly asymmetric buyers and shows that there is an equilibrium in undominated strategies where the buyers "collude": one buyer bids high in the auction, the other bids low; the high bidder wins at her reserve price and makes a resale offer to the low bidder. This equilibrium antedates the structure of one of the equilibria that we identify in the second-price auction case (see footnote 4). Like in our equilibria with $q \approx 1$, in Zheng's equilibrium the initial seller's revenue is smaller than in the situation where resale is not possible.

Zheng (2002) considers mechanism-design of the initial seller in (possibly asymmetric) private-value environments when she cannot prevent resale. With two buyers, and in some cases with three of more, the seller-optimal Myerson allocation can be implemented through repeated resale. In our case of SIPV buyers and a speculator, the optimal mechanism is straightforward: the initial seller simply excludes the speculator.

Jehiel and Moldovanu (1999), Ausubel and Cramton (1999), and Calzolari and Pavan (2002), also study optimal auctions with resale. Jehiel and Moldovanu consider environments with negative consumption externalities, but no private information. They examine whether the Coasian notion that the initial assignment of property rights has no effect on efficiency holds when resale is possible. Ausubel and Cramton assume that the resale market will eventually lead to an efficient allocation; it is then optimal for the initial seller to implement an efficient allocation right away. Calzolari and Pavan compute optimal auctions in environments with two buyers with high or low values. In contrast to most of the other literature, they assume that the distribution of bargaining power in the resale market is a function of the identity of the buyers. Calzolari and Pavan show that an important aspect of the auction design is the initial seller's bid announcement policy.

Because a bidder's expected payoff can be positive in the event that some other bidder wins the auction, auctions in environments with allocative externalities (see Jehiel and Moldovanu, 1996, 1999, 2000, 2001, and Jehiel, Moldovanu, and Stacchetti, 1996, 1999) share certain aspects of auctions with resale. For example, in Jehiel and Moldovanu (2000) buyers pool at a bid of 0. This feature also occurs in Haile (2000) and in our second-price auction equilibria.

In Section 2, we set up the model. Section 3 characterizes the situations where the standard no-resale equilibrium outcome remains valid when resale becomes possible. Section 4 is devoted to the equilibria with an active resale market in the English/second-price auction case. Section 5 deals with the first-price/Dutch auction case. The remarks in Section 6 pertain to the breakdown of revenue equivalence, alternative bid announcement policies, repeated resale, resale via standard auctions, and markets with a single buyer. The Appendix contains proofs.

2 Model

We consider $n \geq 2$ risk-neutral *buyers* who are interested in consuming a single indivisible private *good*. Buyer $i = 1, \ldots, n$ has the random value $\tilde{\theta}_i \in [0, 1]$ for the good. We also include a risk-neutral agent, called *speculator*, who has value $\theta_s = 0$ for the good. I.e., the speculator has no use value for the good. Our crucial innovation is the inclusion of at least one speculator; the results remain valid with free entry of speculators. Each of the $n + 1$ agents is called a *bidder*.

The buyers are ex ante identical and have independent private values (Vickrey, 1961); i.e., the random variables $\tilde{\theta}_1, \ldots, \tilde{\theta}_n$ representing buyers' values are stochastically independent and all have the same probability distribution F.

Assumption 1 *The distribution function F is continuous, $F(0) = 0$, $F(1) = 1$, and F has a positive and continuous density f on $[0, 1]$. Moreover, F has a weakly increasing hazard rate.*

We consider a 2-period interaction. Before period 1, buyer $i = 1, \ldots, n$ privately learns the realization of her value, $\tilde{\theta}_i = \theta_i$. In period 1, the good is offered via a sealed-bid second-price auction (shorthand: II) or first-price auction (I) without reserve price. The highest bidding agent becomes the new owner of the good. Our results do not depend on the tieing rule, but to simplify some proofs we assume the speculator loses all ties.

All agents have a discount factor of $\delta \in (0, 1]$ between periods 1 and 2.[2] In particular, in period 1 a buyer with value θ_i is willing to pay $\delta\theta_i$ for the right to consume the good in period 2. The agent who wins in period 1 either consumes the good in period 1 or offers the good for resale via an optimal mechanism in period 2; if she fails to resell the good she consumes it in period 2. This completes the interaction. Note that all market participants are present in both periods.

Actions taken in period 2 may depend on information that is revealed during period 1. Therefore, the bid announcement policy is important. Our assumptions on bid information revealed in the second-price auction make this auction format analogous to the standard English auction. Likewise, the bid announcement policy we assume for the first-price auction makes this format analogous to the Dutch auction.

Assumption 2 *After a second-price auction, the losers' bids become public; the winner's bid remains private.*

After a first-price auction, the winner's bid becomes public; the losers' bids remain private.

In an English auction the losers' dropping-out bids become public as the auction progresses, and the winner's dropping-out bid remains private because the auction stops at the second-highest bid. Assumption 2 ensures that after the second-price auction the same information is revealed as during an English auction. Our bias towards compatibility with the English auction is motivated by the prevalence of English auctions, and the scarcity

[2] Most other studies of auctions with resale do not allow discounting between periods. In our model, discounting only slightly complicates proofs, and increases the transparency of the presentation.

of second-price auctions.[3] We will conduct the analysis using the second-price auction framework. At the end of Section 4 we explain how the results can be applied directly to the English auction setting.

The first-price auction assumptions imply that the auction is strategically equivalent to a Dutch auction. Such an auction stops at the moment the highest bid is revealed, such that the losers' stopping bids remain private. This assumption is needed for tractability, but it is also common for real sellers to keep the losers' bids private in sealed-bid first-price auctions.

Since our focus is on the impact of the speculator, and the buyers are ex ante identical, we will focus on equilibria such that all buyers use the same bid function in period 1. In order to avoid complications in the definition of posterior beliefs and resale mechanisms, we assume right away that this bid function is strictly increasing in the winning range. Moreover, a buyer who does not expect to ever win does not participate (i.e., bids 0).

ASSUMPTION 3 *In equilibrium, all buyers use the same bid function b^* in period 1. For all $\theta, \theta' \in [0,1]$ with $\theta > \theta'$, we have $b^*(\theta) > b^*(\theta')$ if the bid $b^*(\theta')$ wins in equilibrium with positive probability, and have $b^*(\theta') = 0$ otherwise.*

As for further regularity properties of b^*, note that we will construct equilibria in the second-price auction case where b^* is not continuous. In the first-price auction case, there will be a unique equilibrium, where b^* is always continuous, but not necessarily differentiable.

The perfect Bayesian equilibrium conditions are (i) that posterior beliefs are determined by Bayes rule whenever possible, (ii) the resale mechanism is optimal in the sense of maximizing the expected resale payoff of the resale seller (including her payoff when she fails to resell and consumes herself) given her posterior beliefs, (iii) the period-1 winner decides optimally whether to consume the good in period 1 or to offer it for resale in period 2, and (iv) the period-1 bidding behavior is optimal. Let us spell out these conditions.

Posterior beliefs after period 1

Consider a bidder who observes that buyer i's value is in an interval $I \subseteq [0,1]$. Let $a = \inf I$ and $b = \sup I$. According to Bayes rule, the resulting

[3] See Rothkopf, Teisberg and Kahn (1990) for a discussion on why second-price auctions are rare.

posterior distribution function $\hat{F}_I$ for buyer i's value is given by

$$\hat{F}_I(\theta_i) = \begin{cases} \frac{F(\theta_i)-F(a)}{F(b)-F(a)} & \text{if } \theta_i \in [a,b), \\ 1 & \text{if } \theta_i \geq b, \\ 0 & \text{if } \theta_i < a. \end{cases}$$

Note that $\hat{F}_I$ is a point distribution if $b = a$. If $b > a$, the distribution $\hat{F}_I$ has (on its support) the same regularity properties as F: a positive and continuous density and an increasing hazard rate. By Assumption 3, the case $b > a$ becomes relevant only if an off-equilibrium bid or bid 0 is observed.

We define posterior beliefs about period-1 losers, for any possible bid profile $b_1, \ldots, b_n, b_s \geq 0$ (posterior beliefs about the period-1 winner will not be relevant). Consider a buyer i who bids b_i and loses the second-price auction. Then, the other bidders' posterior belief $\Pi_i^{\text{II}}(\cdot \mid b_i)$ about buyer i is given by

$$\Pi_i^{\text{II}}(\cdot \mid b_i) = \hat{F}_{b^{*-1}(b_i)} \qquad \text{if } b_i \in b^*([0,1]), \tag{1}$$

and is undetermined otherwise.

Now consider a buyer i who loses the first-price auction. Let j denote the label of the winner, and $b_j > 0$ the winner's bid. Then, the posterior belief about i's value of bidders other than i is given by

$$\Pi_i^{\text{I}}(\cdot \mid j, b_j) = \hat{F}_{b^{*-1}([0,b_j])} \qquad \text{if } [0,b_j] \cap b^*([0,1]) \neq \emptyset, \tag{2}$$

and is undetermined otherwise. Note that, by Assumption 3, the posterior belief does not depend on the tieing rule. If $b_j = 0$ the posterior belief can depend on the tieing rule, but it is not necessary to spell out this case.

According to her posterior beliefs, the resale seller faces a group of bidders with independent private values in period 2. Moreover, the posterior beliefs are either point distributions or have an increasing hazard rate (for simplicity, let us assume this for the undetermined off-path beliefs as well).

Optimal Resale Mechanism

The posterior beliefs determine which resale mechanism is optimal (in the sense of maximizing expected period-2 payoff) for the period-1 winner, should she decide to offer the good for resale. Because the posterior beliefs induce independent private values among the bidders in the resale mechanism, an optimal mechanism exists and can be computed using Myerson's (1981) methods. Myerson does not take account of point distributions, but this is easily incorporated: the virtual valuation equals the value where the distribution is concentrated. Moreover, Maskin-Tirole (1990) show that Myerson's mechanism remains optimal when the seller has an independent private value, as may be the case for our resale seller. Let $\mathcal{M}(\Pi, \theta_i)$ denote any Myerson mechanism that is optimal for a resale seller with value θ_i and posterior beliefs Π.

For our analysis, the following posterior beliefs are of particular importance. The resale seller has value 0 and there are n bidders with independent private values each distributed according to $\hat{F}_{[0,b]}$, where $b \in (0,1]$. In such an environment, it is well-known that the optimal mechanism $\mathcal{M}(b) \stackrel{\text{def}}{=} \mathcal{M}(\hat{F}_{[0,b]}, \ldots, \hat{F}_{[0,b]}, 0)$ induces the same expected payments and allocation probabilities as a second-price auction with optimal reserve price, denoted $r^*(b)$. Note that, because the virtual valuation function for F_b is strictly increasing, the reserve price is uniquely determined, and the reserve price function r^* is strictly increasing and continuous. Denote by $P_b(\theta_i)$ the expected payment of a bidder with value $\theta_i \in [0,1]$ in the mechanism $\mathcal{M}(b)$, and by $Q_b(\theta_i)$ the probability that bidder i obtains the good. Important properties of these functions are summarized in Lemma 2 in the Appendix. The seller's expected revenue in the mechanism $\mathcal{M}(b)$ will be denoted $M(b)$. Of course, M is strictly increasing. Other important properties of M are summarized in Lemma 3 in the Appendix. The following inequality, proved in the Appendix, will play a crucial role.

LEMMA 1 *There exists a strictly positive function η on $(0,1]$ such that*

$$\forall \epsilon > 0, \ \theta \in [\epsilon, 1] : \ P_\theta(\theta) \geq M(\theta) + \eta(\epsilon). \tag{3}$$

I.e., the payment of the highest type of bidder who participates in the resale market is higher than the resale seller's expected revenue.

A second important class of posterior beliefs are beliefs with the property that the maximum value in the market is known. The optimal mechanism then yields the same expected payments and allocation probabilities as a take-it-or-leave-it offer to a bidder with the highest value, unless the seller's value is higher, in which case the seller keeps the good.

Other posterior beliefs can also occur, but it is not necessary to compute the resulting resale mechanisms. All that is needed for our results is that the resale seller's expected revenue cannot exceed the total expected surplus available in the resale market.

In the second-price auction case, let $\mathcal{M}^{\text{II}}(i, b_{-i}, \theta_i)$ denote the period-2 mechanism offered by the period-1 winner $i = 1, \ldots, n, s$ when the vector of losers' bids equals b_{-i} and i's value is θ_i. In equilibrium, the resale seller's mechanism is optimal given her posterior beliefs,

$$\forall i, b_{-i}, \theta_i : \ \mathcal{M}^{\text{II}}(i, b_{-i}, \theta_i) = \mathcal{M}((\Pi_j^{\text{II}}(\cdot \mid b_j))_{j \neq i,s}, \theta_i). \tag{4}$$

Similarly, in the first-price auction case let $\mathcal{M}^{\text{I}}(i, b_i, \theta_i)$ denote the period-2 mechanism offered by the period-1 winner $i = 1, \dots, n, s$ when b_i denotes her bid and i's value is θ_i. The optimality condition for the resale seller's mechanism is as follows.

$$\forall i, b_i, \theta_i : \ \mathcal{M}^{\text{I}}(i, b_i, \theta_i) = \mathcal{M}((\Pi_j^{\text{I}}(\cdot \mid i, b_i))_{j \neq i,s}, \theta_i). \tag{5}$$

Optimal Period-1 Behavior in the Second-Price Auction

We need some notation for the outcome of the period-2 mechanism, because that determines the bidding incentives in period 1. For all bidders $i, j = 1, \dots, n, s$ with $j \neq i$, all $b_{-i} \in [0, \infty)^n$, and all $\theta_i, \theta_j \in [0, 1]$, let $P_j^{\text{II}}(i, b_{-i}, \theta_i, \theta_j)$ denote the expected transfer from bidder j of type θ_j to the resale seller i in the mechanism $\mathcal{M}^{\text{II}}(i, b_{-i}, \theta_i)$. Let $Q_j^{\text{II}}(i, b_{-i}, \theta_i, \theta_j)$ denote the respective probability that bidder j obtains the good. Similarly, let $P^{\text{II}}(i, b_{-i}, \theta_i)$ denote the expected transfer *to* the resale seller i, and $Q^{\text{II}}(i, b_{-i}, \theta_i)$ the probability that the resale seller keeps the good. Note that

$$P^{\text{II}}(i, b_{-i}, \theta_i) = \sum_{j \neq i} E[P_j^{\text{II}}(i, b_{-i}, \theta_i, \tilde{\theta}_j) \mid b^*(\tilde{\theta}_j) = b_j],$$

and

$$Q^{\text{II}}(i, b_{-i}, \theta_i) = 1 - \sum_{j \neq i} E[Q_j^{\text{II}}(i, b_{-i}, \theta_i, \tilde{\theta}_j) \mid b^*(\tilde{\theta}_j) = b_j],$$

where the expectations are computed using the posterior beliefs $\Pi_j^{\text{II}}(\cdot \mid b_j)$.

We are now in a position to spell out the bidders' payoff functions. Let b_s^* denote the speculator's equilibrium bid. Define $\tilde{b}_i = b^*(\tilde{\theta}_i)$ for all $i \neq s$, and $\tilde{b}_s = b_s^*$. For all i, let $\tilde{b}_{-i} = (\tilde{b}_j)_{j \neq i}$ and let $\tilde{b}_{-i}^{(1)} = \max_{j \neq i} \tilde{b}_i$. For all i, j, let $\tilde{b}_{-i-j} = (\tilde{b}_k)_{k \notin \{i,j\}}$. Buyer i's expected payoff when she bids b_i and has the value θ_i equals

$$\begin{aligned} &u_i(b_i, \theta_i) = E[\\ &\left(-\tilde{b}_{-i}^{(1)} + \max\{\theta_i, \delta\left(\theta_i Q^{\text{II}}(i, \tilde{b}_{-i}, \theta_i) + P^{\text{II}}(i, \tilde{b}_{-i}, \theta_i)\right)\}\right) \mathbf{1}_{w(b_i, \tilde{b}_{-i}) = i} \\ &+ \sum_{j \neq i} \delta\left(\theta_i Q_i^{\text{II}}(j, (b_i, \tilde{b}_{-j-i}), \tilde{\theta}_j, \theta_i) - P_i^{\text{II}}(j, (b_i, \tilde{b}_{-j-i}), \tilde{\theta}_j, \theta_i)\right) \mathbf{1}_{w(b_i, \tilde{b}_{-i}) = j} \ ], \end{aligned}$$

where w denotes the period-1 winner as a function of the bid profile. The max-term reflects the condition that after winning in period 1, i decides optimally whether to consume the good or offer it for resale.

For the speculator, the expected payoff function is given by

$$u_s(b_s) \quad = \quad E[\left(-\tilde{b}_{-s}^{(1)} + \delta P^{\text{II}}(s, \tilde{b}_{-s}, \theta_s)\right) \mathbf{1}_{w(b_s, \tilde{b}_{-s})=s} \ ].$$

The optimal bidding conditions are

$$\forall i \neq s, \theta_i : \ b^*(\theta_i) \in \arg\max_{b_i \geq 0} u_i(b_i, \theta_i), \tag{6}$$

$$b_s^* \in \arg\max_{b_s \geq 0} u_s(b_s). \tag{7}$$

Optimal Period-1 Behavior in the First-Price Auction

Analogously to the second-price auction case, for all $i, j = 1, \ldots, n, s$ with $j \neq i$, all $b_i \in [0, \infty)$, and all $\theta_i, \theta_j \in [0, 1]$, let $P_j^{\text{I}}(i, b_i, \theta_i, \theta_j)$ denote the expected transfer from bidder j of type θ_j to the resale seller i in the mechanism $\mathcal{M}^{\text{I}}(i, b_i, \theta_i)$. Let $Q_j^{\text{I}}(i, b_i, \theta_i, \theta_j)$ denote the respective probability that bidder j obtains the good. Similarly, let $P^{\text{I}}(i, b_i, \theta_i)$ denote the expected transfer to the resale seller i, and $Q^{\text{I}}(i, b_i, \theta_i)$ the probability that the resale seller keeps the good.

We allow the speculator to randomize her bid because otherwise no equilibrium may exist. Let $\tilde{b}_s$ denote an independent random variable for the speculator's bid.

Buyer i's expected payoff when she bids b_i and has the value θ_i equals

$$\begin{aligned} & u_i(b_i, \theta_i) = E[\\ & (-b_i + \max\{\theta_i, \delta\,(\theta_i Q^{\text{I}}(i, b_i, \theta_i) + P^{\text{I}}(i, b_i, \theta_i))\})\, \mathbf{1}_{w(b_i, \tilde{b}_{-i})=i} \\ & + \sum_{j \neq i} \delta \left(\theta_i Q_i^{\text{I}}(j, \tilde{b}_j, \tilde{\theta}_j, \theta_i) - P_i^{\text{I}}(j, \tilde{b}_j, \tilde{\theta}_j, \theta_i)\right) \mathbf{1}_{w(b_i, \tilde{b}_{-i})=j} \ ]. \end{aligned}$$

For the speculator, the expected payoff function is given by

$$u_s(b_s) \quad = \quad E[(-b_s + \delta P^{\text{I}}(s, b_s, \theta_s))\, \mathbf{1}_{w(b_s, \tilde{b}_{-s})=s} \ ].$$

The optimal bidding conditions are

$$\forall i \neq s, \theta_i : \ b^*(\theta_i) \in \arg\max_{b_i \geq 0} u_i(b_i, \theta_i), \tag{8}$$

$$\Pr[\tilde{b}_s \in \arg\max_{b_s \geq 0} u_s(b_s)] = 1. \tag{9}$$

Perfect Bayesian Equilibrium

A vector (b^*, b_s^*) is a *(perfect Bayesian) equilibrium outcome for the second-price auction with resale* if there exists $\Pi_\cdot^{\text{II}}$ and $\mathcal{M}^{\text{II}}$ such that (1), (4), (6), and (7) are satisfied.

A vector (b^*, H) is a *(perfect Bayesian) equilibrium outcome for the first-price auction with resale* if there exists $\Pi_\cdot^{\text{I}}$ and $\mathcal{M}^{\text{I}}$ such that (2), (5), (8), and (9) are satisfied, where H denotes the distribution function for $\tilde{b}_s$.

Speculation-Proof Auctions

An important question to ask is whether the anticipation of a resale opportunity will change bidding in an auction. Let b^{I} denote the standard equilibrium bid function for the first-price auction without resale, and let b^{II} denote the weakly dominant strategy for every buyer in the second-price auction without resale. Let us call an auction $a \in \{\text{I}, \text{II}\}$ *weakly speculation-proof* (in the environment (n, F, δ)) if auction a with resale has a perfect Bayesian equilibrium where the buyers use the bid function b^a and the speculator bids 0. Let us call an auction *strongly speculation-proof* if this is the case for *all* perfect Bayesian equilibria. Note that if in auction $a \in \{\text{I}, \text{II}\}$ the buyers use b^a and the speculator bids 0, all bidders believe that the auction assigns the good to the buyer with the highest value. Hence, there will be no activity in the resale market.

A standard auction is weakly speculation-proof if and only if two things hold. First, no buyer has an incentive to increase her bid and subsequently offer the good for resale. Second, the speculator cannot make a profit by submitting a positive bid and then offer the good for resale. Haile (1999, Theorem 1) has shown that the first condition is satisfied for all standard auctions. What is new to our analysis is the second condition.

3 Are Standard Auctions Speculation-Proof?

Consider the second-price auction. Suppose buyers use the bid function b^{II}. If the speculator makes a positive bid and wins, her optimal resale mechanism is a take-it-or-leave-it offer at a price equal to her own payment in the auction; hence, it is optimal to bid 0. In other words:

PROPOSITION 1 *The second-price auction is weakly speculation-proof in all environments* (n, F, δ).

Proposition 3 shows that second-price auctions are not strongly speculation-proof.

The situation is different in the case of first-price auctions. In Proposition 5 we will show that there always is a unique equilibrium. Therefore, the first-price auction is strongly speculation-proof if and only if it is weakly speculation-proof. Proposition 2 below shows two things. On the one hand, for any number of buyers n, it can happen that the auction is *not* speculation-proof. On the other hand, if the number of buyers is large, the auction is speculation-proof even for discount factors very close to 1.

PROPOSITION 2 *For every buyer number $n \geq 2$, there exists a number $\delta_n < 1$ with the following property.*

> *If and only if $\delta > \delta_n$, there exists a distribution F such that the first-price auction is not (weakly or strongly) speculation-proof in the environment (n, F, δ).*

Moreover, $\delta_n \to 1$ as $n \to \infty$.

The result that the first-price auction can fail to be weakly speculation-proof is particularly striking. It strongly suggests that any analysis of first-price auctions with resale that ignores speculators is incomplete.

For an example of a first-price auction that is not weakly speculation-proof, consider a distribution F such that, except at points close to 1, the density is that of an exponential distribution; i.e., $F(\theta) = 1 - e^{-z\theta}$, for some $z \in (0,1)$. When the point 1 is approached, the density suddenly peaks. Such a distribution satisfies Assumption 1: it has a constant hazard rate up to values close to 1, but then the hazard rate increases sharply. Suppose that all buyers use the standard no-resale bid function b^{I}. The speculator can make a profit by bidding $b^{\mathrm{I}}(1)$ in the auction, followed by a take-it-or-leave-it-offer just below 1 in period 2. By definition of b^{I}, the bid $b^{\mathrm{I}}(1)$ equals the expectation of the distribution F^{n-1}, which approximates

$$e(z,n) = (n-1)\frac{1}{z}\int_0^z \tau e^{-\tau}(1-e^{-\tau})^{n-2}\mathrm{d}\tau + 1 - (1-e^{-z})^{n-1}. \tag{10}$$

The resale revenue approximates the probability that at least one buyer has a value close to 1,

$$r(z,n) = 1 - (1-e^{-z})^n. \tag{11}$$

Using the Taylor expansion for e^{-z}, one finds that for $z \approx 0$,

$$r(z,n) = 1 - z^n + \text{higher order terms}$$

and

$$e(z,n) = 1 - \frac{1}{n} z^{n-1} + \text{higher order terms.}$$

Therefore, $r(z,n) > e(z,n)$ for small z. Hence, when the discount factor is close to 1, the auction is not speculation-proof.

The remaining steps of the proof of Proposition 2 are in the Appendix. The main idea of the proof is to show that the type of distribution used in the example above gives the speculator the best chances to make a profit; consequently, we can define

$$\delta_n = \min_{z \in [0,1]} \frac{e(z,n)}{r(z,n)}. \tag{12}$$

Note that in cases where the first-price auction is not speculation-proof, it is not obvious what constitutes an equilibrium or indeed whether an equilibrium exists at all. Proposition 5 solves this problem.

4 Equilibria for the Second-Price Auction with Resale

The main results in this section are Proposition 3 which identifies the equilibrium outcomes, and Proposition 4 which discusses the impact of resale on initial seller revenue. We also explain why our equilibria remain valid if multiple speculators can participate, and discuss the consequences of having a reserve price in the initial auction. The section concludes with an explanation of how to adapt the equilibria to the case of English auctions.

The equilibrium outcomes are described in Proposition 3 below. Buyers with values below a certain cutoff θ^* pool at a bid of 0 in period 1, while buyers with values above the cutoff bid their values. The speculator's bid wins if and only if all buyers have values below the cutoff. The exact size of the speculator's bid is determined by the optimality condition for the buyer's bid function.[4] [5]

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[4] The equilibrium with $\theta^* = 1$ and $b_s^* = 1$ is similar to the one constructed by Zheng (2000, Section 5.2) in a slightly different context with two bidders.

[5] None of the equilibria we construct in the second-price auction case can be discarded using standard equilibrium refinements: the equilibria are in undominated strategies and the intuitive criterion is satisfied; see Garratt and Tröger, 2003.

Proposition 3 *If* (b^*, b_s^*) *is a perfect Bayesian equilibrium outcome of the second-price auction with resale then there exists a* $\theta^* \in [0,1]$ *such that*

$$b^*(\theta_i) = \begin{cases} 0 & \text{if } \theta_i \in [0, \theta^*), \\ \theta_i & \text{if } \theta_i \in (\theta^*, 1), \end{cases} \tag{13}$$

$$\begin{array}{l} \text{if } \theta^* < 1 \text{ then } b_s^* = \theta^* - \delta\left(\theta^* - P_{\theta^*}(\theta^*)\right), \\ \text{if } \theta^* = 1 \text{ then } b_s^* \geq 1 - \delta(1 - P_1(1)). \end{array} \tag{14}$$

Conversely, for every $\theta^* \in [0,1]$ *there exists an equilibrium outcome* (b^*, b_s^*) *such that (13) and (14) hold. Moreover,*

$$\forall \theta^* \in [0,1]: \; b_s^* \geq P_{\theta^*}(\theta^*). \tag{15}$$

Any speculator payoff in the interval $[0, \delta M(1)]$ *is supported by an equilibrium.*

A striking aspect of this result is the existence of equilibria that are profitable for the speculator. The speculator makes profits by buying low and selling high, even though all bidders are present at all times. Speculation works because the speculator does not have to pay her own bid. She submits an aggressive bid that makes low-value buyers willing to wait for a resale offer, and pays the second-highest bid, 0. This way the speculator buys at price 0 and sells, in expectation, at a positive price.

The speculator's profit depends on the prevailing equilibrium. In a θ^*-equilibrium the speculator wins in period 1 with probability $F^n(\theta^*)$. Thus, her payoff equals $\delta F^n(\theta^*) M(\theta^*)$. The speculator's largest profit, $\delta M(1)$, occurs when she completely captures the market ($\theta^* = 1$). She makes, of course, no profit in the bid-your-value equilibrium outcome ($\theta^* = 0$; cf. Proposition 1). The participation requirement in Assumption 3 precludes the existence of additional equilibria where buyers with values below θ^* would make a positive bid and the speculator's profit would be reduced accordingly.

On first glance, the speculator seems the least likely candidate to become a successful trader because she is the weakest bidder in terms of her consumption value. However, the fact that her zero value is common knowledge among buyers actually *simplifies* the construction of equilibria: everybody knows that if the speculator becomes the resale seller, she will offer the good cheaper than anybody else would, and there is no uncertainty about the resale mechanism.[6]

[6]We do, however, not claim that the possibility of an active resale market always

Because buyers' bids in the range $(0, \theta^*]$ occur with probability 0, the equilibria rely on appropriately defined off-path posterior beliefs. In particular, buyers with values below θ^* must prefer to bid 0 rather than make a bid in the losing range $(0, b_s^*)$. One way to guarantees this is to define beliefs about a buyer i who bids b_i and loses as follows,

$$\Pi_i^{\mathrm{II}}(\theta_i \mid b_i) \stackrel{\mathrm{def}}{=} \mathbf{1}_{\theta_i \geq \theta^*} \quad \text{if } b_i \in (0, \theta^*]. \tag{16}$$

Given such beliefs, a deviation to a bid in $(0, b_s^*)$ would raise the resale price to θ^* because the speculator believes that θ^* is the highest value in the market. Generally, the equilibria can be supported by any off-path posterior beliefs that do not reduce the resale price. For example, the posterior beliefs could be identical to the beliefs about a buyer who bids 0, leaving the resale price unchanged.[7]

Let us sketch the main steps of the proof of Proposition 3. We begin with the necessary equilibrium conditions (13) and (14). On any equilibrium path, a buyer who wins consumes the good rather than offering it for resale. This is because by Assumption 3, a buyer who wins has the highest value in the market. But when a buyer consumes the good upon winning, she finds it optimal to bid her value, by the same argument as in a second-price auction without resale. On the other hand, buyer types who are overbid by the speculator never win the auction, and consequently bid 0 by Assumption 3. We obtain (13) when we define θ^* as the largest type who does not overbid the speculator. The speculator's bid (14) is chosen such that buyers with value above θ^* prefer to bid their value rather than 0, and vice versa for buyers with value below θ^*. If $\theta^* < 1$, a buyer with value θ^* is just indifferent between these possibilities; if $\theta^* = 1$, all buyer types may strictly prefer to bid 0.

A buyer with value θ^* can be willing to wait for resale only if the expected resale payment is not higher than the price at which she could buy the good today. Accordingly, (15) follows from (14).

To complete the proof of Proposition 3, we have to show that an equilibrium satisfying (13) and (14) exists. Consider the case $\theta^* < 1$; the case $\theta^* = 1$ is similar. Optimality of the speculator's bid is straightforward. The bid b_s^* lies in the range $(0, \theta^*]$. All bids in this range are equally good for the

requires activity of a speculator; asymmetric bidding among symmetric buyers can create a continuum of equilibria with active resale markets as well, at least in markets with 2 buyers and no discounting. Garratt and Tröger (2003, p.26) show this in the case where buyers' values are distributed uniformly, but equilibria can also be constructed for arbitrary value distributions (personal communication by Charles Zheng, Summer 2003).

[7] We thank Bill Zame for suggesting these beliefs.

speculator because they lead to the same chances of winning the auction. Bidding more than θ^* is not as good because in the event that the speculator needs such a high bid in order to win, she wins at a price equal to the highest value among all buyers. Optimality of the buyers' bid function is more involved. A low-value buyer can deviate by overbidding the speculator and attempting resale herself; if she fails to resell the good, she still obtains her value from consuming it. An explicit computation shows that such a deviation is not profitable. We also show that a buyer with value above θ^* has no incentive to bid 0 and wait for a resale offer. By (14), type θ^* is indifferent between doing that and bidding her value. Higher types have an additional incentive to bid their value because the highest opponent value might be in between θ^* and their own value.

All equilibria remain valid with free entry of multiple speculators. If there is nothing to prevent the presence of a speculator in a market with resale, then multiple speculators might be present as well. Consider a second speculator. She can make a profit only if she overbids the first speculator's bid b_s^*. Bidding higher than θ^* can only harm her: if she needs such a high bid in order to win, the highest value in the market does not exceed her payment in the auction. So her best hope is a bid in the interval (b_s^*, θ^*). Any such bid yields the resale revenue $\delta M(\theta^*)$, which by Lemma 1 is lower than the expected period-2 payment of type θ^*. By (15), however, that payment is at most as high as b_s^*, showing that the second speculator would make a loss.

When the initial auction is augmented by a reserve price $r > 0$, the equilibrium outcomes where the speculator's resale revenue $\delta M(\theta^*) \geq r$ remain valid. This is because Lemma 1 together with (15) implies $b_s^* \geq r$ for such equilibrium outcomes. The bid-your-value equilibrium outcome ($\theta^* = 0$) also remains valid. All other equilibria break down. Proposition 3 is robust to small reserve prices in the sense that that for all $\theta^* \in [0, 1]$, the θ^*-equilibrium remains valid if r is sufficiently small.

Note that all equilibria established in Proposition 3, except the one with $\theta^* = 0$, are inefficient. Because the speculator wins the auction with positive probability, the final allocation is delayed with positive probability. Even ignoring inefficiencies due to delay, the resale market will induce an inefficient allocation in the sense that the speculator keeps the good with positive probability (due to a positive reserve price). Therefore, in the context of resale the second-price auction loses what has been propagated as one of its main advantages in environments with private values: to implement an efficient allocation as the unique equilibrium outcome in undominated strategies.

Let us compare the agents' payoffs in the inefficient equilibria with their payoffs in the efficient equilibrium where all bidders bid their values. The speculator is obviously better off in the inefficient equilibria compared to the efficient one, and it can be shown that the buyers are worse off regardless of their values. What about the initial seller? It is interesting to compare her expected revenue in the inefficient equilibria to her expected revenue in the efficient equilibrium. Proposition 4 below shows that a little activity of the speculator increases the initial seller's expected revenue, but a lot of activity reduces it.

PROPOSITION 4 *For all* $\theta^* > 0$ *that are sufficiently close to 0 (resp., 1), the initial seller's expected revenue in a* θ^**-equilibrium is larger (resp, smaller) than the expected revenue that results when all bidders bid their values.*

The proof begins by distinguishing two distinct events that cause the revenue in a θ^*-equilibrium to differ from the revenue that arises in the bid-your-value equilibrium. Event (i) is that θ^* lies between the highest and the second-highest value, in which case revenue changes from the second-highest value to b_s^*. Event (ii) is that θ^* is larger that the highest value, in which case revenue falls from the second-highest value to 0. If θ^* is close to 1 the probability of (i) becomes small and the probability of (ii) does not, implying that expected revenue is reduced. If θ^* tends to 0 both events' probabilities tend to 0, but event (i) allows one buyer's value to stay above θ^* and thus becomes infinitely more likely than event (ii). The expected payoff loss from event (ii) is of the order θ^*. It is thus sufficient to show that event (i) results in an expected payoff gain of the order θ^*. Using (15), one obtains a lower bound for the expected payoff difference. Using properties of order statistics, one can verify that the payoff difference is positive of the order θ^* if the value distribution F is uniform. However, any continuously differentiable distribution is approximately uniform on any small interval, which completes the proof.

Proposition 4 remains qualitatively valid when the initial auction is augmented by a small reserve price. The result establishes strict revenue inequalities. The inequalities remain intact because the revenue from a θ^*-equilibrium is continuous in the reserve price.

By setting a large reserve price, the initial seller can shut out the speculator. However, this might require a reserve price larger than the price that would be revenue maximizing in the absence of resale. Using this, it can be shown that the existence of a resale opportunity can harm a revenue-maximizing seller who can set any reserve price (see Garatt and Tröger, 2003).

An efficiency-minded (i.e., expected surplus maximizing) seller might want to allow activity of a speculator rather than set a large reserve price. If following a small or no reserve price a θ^*-equilibrium with small $\theta^* > 0$ is played, and following intermediate reserve prices a θ^*-equilibrium with large θ^* is played, it can be less inefficient to not set a reserve price rather than set a large one that would shut out the speculator but keeps the good in the hands of the initial seller with high probability.

English Auction with optimal resale

The equilibrium outcomes that we have constructed remain valid if the second-price auction in period 1 is replaced by an English auction, as modelled by Milgrom and Weber (1982). At the end of period 1, exactly the same information is revealed after an English auction as after a second-price auction, provided the same bids are made in both auctions. Hence, the only thing we need to argue is that bidding incentives in period 1 do not change. The difference between the second-price auction and the English auction is that the losing bids become public during instead of after the auction, so that bidders can revise their beliefs each time a buyer drops out.

We define bidding in the English auction as follows. Every low-value buyer ($\theta_i < \theta^*$) stops bidding at 0, and every high-value buyer ($\theta_i > \theta^*$) is willing to bid up to her value, independently of who stays in and how long. The speculator is willing to bid up to b_s^*, also independently of who stays in and how long.

To see that these bidding strategies are optimal, consider first the speculator's bidding incentives, given the buyers' strategies. At the bid 0, it is better to stay in than to drop out because the latter means she foregoes her chances of winning and making a resale profit. If some buyer stays in at 0 as well, Bayesian updating requires the speculator to believe that the value of this buyer is distributed on the support $[\theta^*, 1]$. She expects the buyer to stay in up to her value, which is beyond b_s^*. Hence, it is optimal for the speculator to drop out at b_s^*. If one buyer deviates by stopping at a bid in $(0, \theta^*]$, the speculator switches to the belief that the value of this buyer equals θ^* and will thus offer the same take-it-or-leave-it resale mechanism as in the second-price auction case.

Now consider a buyer's bidding incentives, given the other bidders' strategies. A low-value buyer has no incentive to stay in beyond bid 0, for the same reasons as in the second-price auction case. A high-value buyer will stay in up to her value because she expects the same from all other buyers once they stay in at positive bids. When a high-value buyer observes that a competing buyer drops out at a bid in $(0, \theta^*]$, she switches to the belief that this buyer has value θ^*, and it remains optimal for the high-value buyer to bid up to her value.

5 The Equilibrium in the First-Price Auction with Resale

The main result in this section is Proposition 5 which establishes existence and uniqueness of equilibrium. Corollary 1 concerns the impact of a resale opportunity on the initial seller's revenue. We also discuss the impact of augmenting the initial auction by a reserve price, and explain why the equilibrium remains valid if multiple speculators can participate.

PROPOSITION 5 *The first-price auction with resale has a unique perfect Bayesian equilibrium outcome* (b^*, H). *We have*

$$\forall \theta \in [0,1]: \; b^*(\theta) \geq \max\{b^I(\theta), \delta M(\theta)\}, \tag{17}$$

where b^I *denotes the standard no-resale first-price auction equilibrium bid function.*

The following three statements are equivalent:

$$\exists \theta \in [0,1]: \; \delta M(\theta) > b^I(\theta), \tag{18}$$

$$H(0) < 1, \tag{19}$$

$$\exists \theta \in [0,1]: \; b^*(\theta) > b^I(\theta). \tag{20}$$

This result has three immediate implications. First, from (17) one sees that the speculator's equilibrium payoff always equals 0—speculation is not profitable. This contrasts the second-price auction where a speculator can make profits. Second, the auction fails to be speculation-proof if and only if (20) holds. Therefore,

COROLLARY 1 *Whenever the first-price auction is not (weakly or strongly) speculation-proof, the initial seller's equilibrium revenue is higher than in the first-price auction without resale.*

Third, whenever the auction is not speculation-proof, the final allocation will not be fully efficient. The speculator keeps the good with positive probability. Without a resale opportunity, the speculator would bid 0 and the allocation would be efficient. I.e., in the context of resale the first-price auction can lose its ability to implement an efficient allocation when buyers have SIPV.

The proof of Proposition 5 is split up into a series of lemmas. First, a unique equilibrium candidate is established. Then we prove existence. Lemma 4 shows that the buyer will not bid above her value. By Assumption 3, such a bid would win with positive probability. In that event, the buyer would obtain a negative payoff. In the event the bid does not win, lowering her bid does not change the buyer's payoff because losing bids are not observable to the winner.

Lemma 5 shows that, with positive probability, the speculator makes arbitrarily small bids. Suppose not. Then, by Lemma 4, buyers with values below the support of the speculator's bid distribution never win the auction. By Lemma 1, the highest among these buyer types pays a price in period 2 that is higher than the speculator's revenue when making her lowest equilibrium bid. This revenue, in turn, is at least as high as the speculator's payment in the auction (otherwise the speculator would make expected losses). Therefore, the buyer can get the good cheaper with positive probability when she slightly overbids the speculator's lowest equilibrium bid.

Lemma 6 shows that the speculator's equilibrium payoff equals 0. Because she makes arbitrarily small bids by Lemma 5, she could obtain a positive payoff only if a positive mass of buyer types bid exactly 0. An argument similar to that in the proof of Lemma 5 shows that some of these buyer types would prefer to deviate to a small positive bid.

Lemma 7 shows that the speculator's bid distribution has no atoms. If there were an atom, some buyer types would bid just below it because otherwise the speculator would prefer to bid lower. However, using Lemma 1 again, buyer types who bid just below the atom have to pay so much in period 2 that they prefer to overbid the atom.

Lemma 8 states that the buyers' equilibrium bid function is continuous. The proof is standard. Lemma 9 states that every buyer with a positive value will bid below her value; this is because even arbitrarily small bids have a positive chance of winning by Lemma 5.

Lemma 10 uses the buyers' incentive compatibility constraints in period 1 to show that the buyers' bid function b^* satisfies a certain differential equation. The derivative of b^* depends on two factors. Firstly, it depends on the bidding competition among the buyers themselves. This competition yields a lower bound for the derivative of b^*. This lower bound is identical to the right-hand side of the differential equation for the standard no-resale equilibrium bid function b^{I}. Secondly, the derivative of b^* depends on the added competition from the speculator's bidding. At every point where competition among the buyers has dropped to a point that would allow the speculator to make a positive profit without added competition, the derivative of b^* increases so that the speculator makes zero profit. Lemma 10 then goes on and determines a differential equation for H, the distribution function for the speculator's bid, such that b^* becomes in fact optimal for the buyers. At every point where the competition among buyers is not strong enough to keep the speculator's profit equal to 0, the speculator must add in just so much probability weight that her profit becomes equal to 0.

We then turn to the equilibrium existence proof. The differential equation for b^* constructed in Lemma 10 is not continuous: at each point where the speculator's starts adding in probability weight, the right-derivative of b^* may jump upwards. In Lemmas 11 and 12, we apply techniques from the theory of differential inclusions[8] to show that the differential equation for b^* has a solution.

To complete the existence proof, we show that each buyer type's equilibrium bid is globally optimal. The differential equations constructed in Lemma 10 only guarantee that first-order conditions are satisfied. Also, we show that no buyer has an incentive to bid higher in an attempt to offer the good for resale herself. Optimality of the speculator's bid follows by construction. The equivalence of the three statements (18) to (20) is easily seen.

The equilibrium established in Proposition 5 remains valid with free entry of multiple speculators because a second speculator obtains the same zero payoff as the first from any bid. There also exist equilibria with multiple active speculators. Buyers will use the same bid function as in the equilibrium constructed above, and the distribution function for the *maximum* bid among speculators will be equal to the function H constructed for the equilibrium with one speculator.

If the initial auction is augmented by a small reserve price $r > 0$, the equilibrium must still involve an active speculator in some cases. The no-resale equilibrium bid of any type $\theta \in (r, 1]$ is continuous in r. Therefore, Proposition 2 shows that when r is small there exist cases where in equilibrium the speculator must be active. Any sufficiently large reserve price will shut out the speculator, which extends the result that speculators cannot make profits in first-price auctions with resale. An efficiency-minded seller has, in cases where the first-price auction is not speculation-proof, a choice

[8] We thank Jörg Oechssler for help with this part of the proof.

between two inefficiencies: either she uses a reserve price that leaves the good in her hands with some probability, or she uses no reserve price, which makes the speculator keep the good with some probability.

6 Remarks

Breakdown of Revenue Equivalence

A resale opportunity, by attracting a speculator, destroys the revenue equivalence of standard auctions. By Proposition 4, Corollary 1, and revenue-equivalence in the absence of resale, the second-price auction with resale always has an equilibrium that yields a *lower* revenue for the initial seller than the unique equilibrium in the first-price auction with resale. Moreover, whenever the first-price auction is speculation-proof, the second-price auction with resale has an equilibrium that yields a *higher* revenue for the initial seller than the first-price auction.

In cases where the first-price auction is *not* speculation-proof, none of the equilibria of the second-price auction with resale induces the same allocation as the equilibrium in the first-price auction with resale. This follows from Lemma 5. Because the speculator makes arbitrarily small bids with positive probability, buyers with arbitrarily small positive types have a chance to obtain the good. In every θ^*-equilibrium with $\theta^* > 0$, however, types below the speculator's resale reserve price have no chance to get the good.

Bid Announcement Policy

The structure of our second-price auction equilibria suggests that they remain valid under any bid announcement policy of the initial seller. Even if the initial seller keeps all bids secret, the winner learns the second-highest bid from her payment, and this information is sufficient to make the inferences that support the speculator's resale mechanism.[9]

In the first-price auction case, the equilibrium outcome can change if losers' bids are made public. One can verify that when losers' bids remain

[9] Establishing general equilibrium conditions for the second-price auction with secret bids would, however, be tedious. Among other complications, two bidders can have different posterior beliefs about a third. To see this, suppose bidder j wins at price p^1. If bidder $i \neq j$ has made a bid $b_i < p^1$ then she learns that there exists at least one other bidder who has made the bid p^1. If $b_i = p^1$ then she only learns that no other bidder has made a bid above p^1; she cannot learn anything about whether somebody has bid precisely p^1. Because b_i is not publicly observable, two agents may thus have different beliefs about a third agent.

private, the auction is speculation-proof in an environment with two buyers with uniformly distributed values and small discounting. In the same environment with losers' bids made public, the auction is not speculation-proof. If it were, the speculator could win the auction for sure at a price of $1/2$, and from the observed losers' bids she would infer the maximum value among buyers, which has the expectation $2/3 > 1/2$.

Repeated Resale

Our equilibria remain valid in a model with more than 2 periods; i.e., where repeated resale is permitted. Our equilibrium constructions rely on only two classes of posterior beliefs. Those where posterior beliefs induce an SIPV environment in the resale market, and those where posterior beliefs are such that the highest value in the market is known. In both cases, the equilibrium in the optimal resale mechanism remains valid when the resale seller cannot prevent further resale.

Resale via Standard Auctions

Our equilibria remain valid if the resale seller is restricted to a standard auction with reserve price. Given the posterior beliefs that are relevant for our equilibrium constructions, a standard auction with optimal reserve *is* an optimal auction.

One Buyer and One Speculator

If only a single buyer is present in the market ($n = 1$), the first-price auction is never speculation-proof because in the absence of resale the buyer would bid 0. The lack of bidding competition among buyers also makes it easier than in the case of multiple buyers to construct an equilibrium (see Tröger, 2003). In second-price auctions, the structure of the equilibria with a single buyer is the same as with multiple buyers, but the equilibrium existence proof is simpler because it is obvious that the buyer will not offer the good for resale (see Garratt and Tröger, 2003). In both the first-price auction and the second-price auction, the initial seller's revenue equals 0 if resale is not possible. Hence, the resale opportunity can only increase the revenue.

7 Appendix: Proofs

LEMMA 2 *Consider any* $\theta \in [0,1]$. *For all* $\hat{\theta} \in [0, r^*(\theta))$, *we have* $P_\theta(\hat{\theta}) = 0$ *and* $Q_\theta(\hat{\theta}) = 0$. *For all* $\hat{\theta} \in [r^*(\theta), \theta]$,

$$\frac{P_\theta(\hat{\theta})}{Q_\theta(\hat{\theta})} = \hat{\theta} - \int_{r^*(\theta)}^{\hat{\theta}} \frac{F^{n-1}(\theta')}{F^{n-1}(\hat{\theta})} d\theta', \tag{21}$$

$$\frac{P_\theta(\hat{\theta})}{Q_\theta(\hat{\theta})} \text{ is weakly increasing in } \hat{\theta} \text{ and } \theta, \tag{22}$$

$$Q_\theta(\hat{\theta}) = F^{n-1}(\hat{\theta})/F^{n-1}(\theta). \tag{23}$$

For all $\hat{\theta} \in [\theta, 1]$,

$$Q_\theta(\hat{\theta}) = 1, \quad P_\theta(\hat{\theta}) = P_\theta(\theta). \tag{24}$$

PROOF OF LEMMA 2. Formulas (21) and (23) are standard. Buyers with types $\hat{\theta} > \theta$ occur with probability 0 according to the posterior beliefs; for such buyers it is optimal in the resale mechanism to take the actions that are optimal for type θ. This implies (24). To see (22), use that r^* is increasing and the derivative of $P_\theta(\hat{\theta})/Q_\theta(\hat{\theta})$ with respect to $\hat{\theta}$ is non-negative. *QED*

LEMMA 3 *For all* $\theta \in (0,1]$,

$$M(\theta) = \int_{r^*(\theta)}^{\theta} \left(\hat{\theta} - \frac{F(\theta) - F(\hat{\theta})}{f(\hat{\theta})} \right) \frac{dF^n(\hat{\theta})}{F^n(\theta)} d\hat{\theta}, \tag{25}$$

$$M'(\theta) = n \frac{f(\theta)}{F(\theta)} \left(\theta - M(\theta) - \int_{r^*(\theta)}^{\theta} \frac{F^{n-1}(\hat{\theta})}{F^{n-1}(\theta)} d\hat{\theta} \right). \tag{26}$$

Moreover,

$$M'(0) < \frac{n-1}{n}. \tag{27}$$

The function M *is Lipschitz continuous on* $[0,1]$, *and* M' *is continuous.*

PROOF. Formula (25) is standard from Myerson (1981), while (26) follows from standard differentiation rules. By differentiability, $F(\hat{\theta}) = f(0)\hat{\theta} + e(\hat{\theta})$ for some function e such that $e(\hat{\theta})/\hat{\theta} \to 0$ as $\hat{\theta} \to 0$. Using this and $r^*(\theta)/\theta \to 1/2$ as $\theta \to 0$, (25) can be used to show

$$M'(0) = \lim_{\theta \to 0} \frac{M(\theta)}{\theta} = \frac{n-1}{n+1} + \frac{1}{2^n(n+1)}, \tag{28}$$

which implies (27). Using (26) and (28), it can be confirmed that $\lim_{\theta\to 0} M'(\theta) = M'(0)$. Therefore, M' is continuous on $[0,1]$. Hence, M' is bounded above on $[0,1]$, which implies Lipschitz continuity of M. *QED*

Proof of Lemma 1. Using Lemma 2, we find

$$\begin{aligned}
M(\theta) &= n\int_{r^*(\theta)}^{\theta} P_\theta(\hat\theta)\frac{\mathrm{d}F(\hat\theta)}{F(\theta)} \\
&= n\int_{r^*(\theta)}^{\theta} \frac{P_\theta(\hat\theta)}{Q_\theta(\hat\theta)}Q_\theta(\hat\theta)\frac{\mathrm{d}F(\hat\theta)}{F(\theta)} \\
&\overset{(23)}{=} \int_{r^*(\theta)}^{\theta} \frac{P_\theta(\hat\theta)}{Q_\theta(\hat\theta)}\frac{\mathrm{d}F^n(\hat\theta)}{F^n(\theta)} \\
&\overset{(22)}{\leq} \int_{r^*(\theta)}^{\theta} \frac{P_\theta(\theta)}{Q_\theta(\theta)}\frac{\mathrm{d}F^n(\hat\theta)}{F^n(\theta)} \\
&\overset{(24)}{=} P_\theta(\theta)\left(1-\frac{F^n(r^*(\theta))}{F^n(\theta)}\right) \\
&\leq P_\theta(\theta) - \eta(\epsilon),
\end{aligned} \tag{29}$$

where $\eta(\epsilon) \overset{\text{def}}{=} P_\epsilon(\epsilon)F^n(r^*(\epsilon)) > 0$. *QED*

Proof of Proposition 2.

Define δ_n by (12). As shown on page 13, $r(z,n) > e(z,n)$ for all sufficiently small $z > 0$, implying $\delta_n < 1$.

Because $r(z,n) \leq 1$ and $\min_{z\in[0,1]} e(z,n) \to 1$ as $n \to \infty$, we have $\delta_n \to 1$ as $n \to \infty$, which proves the "Moreover" part.

To prove the "if and only if" part, denote, for all $n \geq 2$ and $\delta < 1$, by $\pi(n,\delta)$ the maximum profit that is possible for the speculator for any distribution F that satisfies Assumption 1 and any bid $b_s \geq 0$, given that all buyers play b^{I}. We have to show that

$$\forall n\ \exists \delta_n < 1:\ \pi(n,\delta) > 0 \Leftrightarrow \delta > \delta_n. \tag{30}$$

For any distribution function D, let $D^{(k,l)}$ denote the distribution function for the lth-highest order statistic among k i.i.d. random variables ($k = 1,2\ldots;\ l = 1,\ldots,k$) that are distributed according to D.

Fix any F, n, and $\delta < 1$. Suppose that all buyers play b^{I}. If the speculator bids $b_s = b^{\mathrm{I}}(1)$ and all buyers play b^{I}, the speculator's payoff equals

$$\pi(F,n,\delta) \overset{\text{def}}{=} \delta M(1) - \int_0^1 \theta \mathrm{d}F^{(n-1,1)}(\theta),$$

by definition of b^{I}. If the speculator bids $b_s < b^{\mathrm{I}}(1)$ and we define $\hat{\theta} = b^{\mathrm{I}-1}(b_s)$, the speculator's payoff equals

$$F(\hat{\theta})^n \left(\delta M(\hat{\theta}) - \int_0^1 \theta \mathrm{d}\hat{F}_{[0,\hat{\theta}]}^{(n-1,1)}(\theta) \right) \quad = \quad \underbrace{F(\hat{\theta})^n}_{\leq 1} \; \pi(\hat{F}_{[0,\hat{\theta}]}, n, \delta).$$

Because the distribution $\hat{F}_{[0,\hat{\theta}]}$ satisfies Assumption 1,

$$\pi(n, \delta) = \max\{0, \sup_F \pi(F, n, \delta)\}. \tag{31}$$

Next we show that

$$\sup_F \pi(F, n, \delta) \leq \max\{0, \max_{z \in [0,1]} \delta r(z, n) - e(z, n)\}. \tag{32}$$

Because a second-price auction is an optimal resale mechanism,

$$\begin{aligned} \pi(F, n, \delta) \quad &= \quad \delta \left(n(1 - F(r^*))F^{n-1}(r^*)r^* + \int_{r^*}^1 \theta \mathrm{d}F^{(n,2)}(\theta) \right) \\ &\quad - \int_0^1 \theta \mathrm{d}F^{(n-1,1)}(\theta), \end{aligned}$$

where r^* denotes the optimal reserve price. From Myerson (1981) we know that $r^* = 1/\lambda(r^*)$, where $\lambda(\theta) = f(\theta)/(1 - F(\theta))$, $\theta \in [0, 1)$, denotes the hazard-rate function for F. We can write

$$F(\theta) = 1 - e^{-\int_0^\theta \lambda(t)\mathrm{d}t}.$$

Define

$$\mu = \frac{1}{r^*} \int_0^{r^*} \lambda(t)\mathrm{d}t.$$

Then

$$\int_0^{r^*} (\lambda(t) - \mu)\mathrm{d}t = 0$$

and $\lambda(t) - \mu$ is weakly increasing in t. Therefore,

$$\forall \theta \in [0, r^*] : \int_0^\theta \lambda(t)\mathrm{d}t \leq \int_0^\theta \mu \mathrm{d}t = \theta\mu.$$

Therefore,

$$\forall \theta \in [0, r^*] : F(\theta) \leq G(\theta) \stackrel{\text{def}}{=} 1 - e^{-\theta\mu}.$$

Also define $G(\theta) = F(\theta)$ for $\theta \in (r^*, 1]$. Then $G(r^*) = F(r^*)$ and F stochastically dominates G. Therefore,

$$\begin{aligned}\pi(F, n, \delta) \quad &\leq \quad \hat{G} \stackrel{\text{def}}{=} \delta \left(n(1 - G(r^*))G^{n-1}(r^*)r^* + \int_{r^*}^{1} \theta \mathrm{d}G^{(n,2)}(\theta) \right) \\ &\qquad - \int_{0}^{1} \theta \mathrm{d}G^{(n-1,1)}(\theta). \qquad (33)\end{aligned}$$

For arbitrary i.i.d. random variables with densities, the highest of $n-1$ dominates the 2nd highest of n in terms of the likelihood ratio. Therefore, the expectation of the highest of $n-1$ conditional on being greater or equal to r^* is greater or equal to the respective conditional expectation of the 2nd highest of n,

$$\frac{1}{1 - G^{(n,2)}(r^*)} \int_{r^*}^{1} \theta \mathrm{d}G^{(n,2)}(\theta) \leq \underbrace{\frac{1}{1 - G^{(n-1,1)}(r^*)} \int_{r^*}^{1} \theta \mathrm{d}G^{(n-1,1)}(\theta)}_{\stackrel{\text{def}}{=} G^*}.$$

Therefore,

$$\begin{aligned}\hat{G} \quad &\leq \quad \delta \left(n(1 - G(r^*))G^{n-1}(r^*)r^* + (1 - G^{(n,2)}(r^*))G^* \right) \\ &\qquad - \int_{0}^{r^*} \theta \mathrm{d}G^{(n-1,1)}(\theta) - (1 - G^{(n-1,1)}(r^*))G^*.\end{aligned}$$

Because $G^{(n-1,1)}(r^*) \leq G^{(n,2)}(r^*)$ and $G^* \geq r^*$, it follows that

$$\begin{aligned}\hat{G} \quad &\leq \quad \delta \left(n(1 - G(r^*))G^{n-1}(r^*)r^* + (1 - G^{(n,2)}(r^*))r^* \right) \\ &\qquad - \int_{0}^{r^*} \theta \mathrm{d}G^{(n-1,1)}(\theta) - (1 - G^{(n-1,1)}(r^*))r^* \\ &= \quad \delta \left(1 - (1 - e^{-z})^n\right) r^* \\ &\qquad - r^*(n-1)\frac{1}{z} \int_{0}^{z} \tau e^{-\tau}(1 - e^{-\tau})^{n-2} \mathrm{d}\tau - (1 - (1 - e^{-z})^{n-1})r^*,\end{aligned}$$

where $z \stackrel{\text{def}}{=} \mu r^* = \mu/\lambda(r^*) \leq 1$ and we have made the substitution $\tau = \theta\mu$ in the integral. Because $r^* \leq 1$, it follows that

$$\hat{G} \leq \max\{0, \delta r(z, n) - e(z, n)\},$$

which together with (33) shows (32).

Next we show that

$$\sup_F \pi(F, n, \delta) \geq \max_{z\in[0,1]} \delta r(z, n) - e(z, n). \tag{34}$$

For all $r^* \in (0, 1)$ and $\mu \in (0, 1/r^*)$, define $G_{r^*,\mu}$ as the distribution function on $[0, 1]$ with density

$$g_{r^*,\mu}(\theta) = \begin{cases} \mu e^{-\mu\theta} & \text{if } \theta \in [0, \overline{r}], \\ \mu e^{-\mu\overline{r}} + \alpha_{\overline{r},\mu}(\theta - \overline{r}) & \text{if } \theta \in [\overline{r}, 1], \end{cases}$$

where $\alpha_{\overline{r},\mu}$ is chosen such that $\int_0^1 g_{r^*,\mu}(\theta)\mathrm{d}\theta = 1$ and $\overline{r} < 1$ is so close to 1 that $\overline{r} > r^*$ and $\alpha_{\overline{r},\mu} > 0$. Given this construction, $G_{r^*,\mu}$ has an increasing hazard rate and the arguments given on page 13 show that

$$\sup_{r^*\in(0,1),\mu\in(0,1/r^*)} \pi(G_{r^*,\mu}, n, \delta) \geq \max_{z\in[0,1]} \delta r(z, n) - e(z, n).$$

This completes the proof of (34).

To prove (30), note that if and only if $\delta > \delta_n$, there exists $z \in [0, 1]$ such that $\delta > e(z, n)/r(z, n)$. Therefore, using (31), (32), and (34), one sees that $\pi(n, \delta) > 0$ if and only if $\delta > \delta_n$. *QED*

Proof of Proposition 3.

Let (b^*, b^*_s) be a perfect Bayesian equilibrium outcome. Define

$$\theta^* = \sup\{\theta \in [0, 1] \mid b^*(\theta) < b^*_s\},$$

where $\sup\emptyset = 0$. To prove (13), consider any buyer i with type $\theta_i \in [0, 1]$. Consider the case $\theta_i \in (\theta^*, 1)$ first. Then, $b^*(\theta_i) \geq b^*_s$. Buyer i's equilibrium payoff is

$$u_i(b^*(\theta_i), \theta_i) = E[(\theta_i - \tilde{b}^{(1)}_{-i})\mathbf{1}_{\tilde{b}^{(1)}_{-i} \leq b^*(\theta_i)}]. \tag{35}$$

Let us first show that $b^*(\theta_i) \leq \theta_i$. Suppose not. Define $\epsilon = (b^*(\theta_i) - \theta_i)/2 > 0$ and $\theta'' = \theta_i + \epsilon$. Then, for all $\theta \in [\theta_i, \theta'']$, using that b^* is weakly increasing,

$$b^*(\theta) \geq b^*(\theta_i) \geq \theta_i + 2\epsilon \geq \theta'' + \epsilon.$$

Therefore,

$$\forall\theta \in [\theta_i, \theta''] : \; b^*(\theta) \in [\theta'' + \epsilon, b^*(\theta'')].$$

Hence,

$$\Pr[\tilde{b}^{(1)}_{-i} \in [\theta'' + \epsilon, b^*(\theta'')]] > 0.$$

Therefore, the bid $b_i = \theta_i$ yields a higher expected payoff than (35)—contradiction. A similar argument shows $b^*(\theta_i) \geq \theta_i$, hence $b^*(\theta_i) = \theta_i$.

Now consider the case $\theta_i < \theta^*$. Then, $b^*(\theta_i) < b_s^*$ because otherwise, by Assumption 3, $b^*(\theta) \geq b_s^*$ for all $\theta > \theta_i$, implying $\theta^* \leq \theta_i$.

Using Assumption 3 again, $b^*(\theta_i) = 0$.

To prove (14), consider the case $\theta^* = 0$ first. By definition of θ^*, the speculator wins with probability 0. Hence, (13) implies $b_s^* = 0$, showing (14). Now consider the case $\theta^* > 0$.

Types $\theta_i < \theta^*$ prefer to bid 0 rather than make the bid θ^*. Therefore,

$$\forall \theta_i < \theta^* : \ \theta_i - b_s^* \leq \delta \left(\theta_i Q_{\theta^*}(\theta_i) - P_{\theta^*}(\theta_i) \right).$$

By (21) and (23), $P_{\theta^*}(\cdot)$ and $Q_{\theta^*}(\cdot)$ are continuous at θ^*. Therefore,

$$\theta^* - b_s^* \leq \delta(\theta^* \underbrace{Q_{\theta^*}(\theta^*)}_{=1} - P_{\theta^*}(\theta^*)).$$

Similarly, if $\theta^* < 1$,

$$\theta^* - b_s^* \geq \delta \left(\theta^* - P_{\theta^*}(\theta^*) \right),$$

which completes the proof of (14).

Formula (15) is straightforward from (14). Moreover, by (13), (14), and (15), in a θ^*-equilibrium the speculator wins in period 1 with probability $F^n(\theta^*)$, and her equilibrium payoff equals

$$u_s^* \stackrel{\text{def}}{=} F(\theta^*)^n \delta M(\theta^*). \tag{36}$$

Because F and M are continuous, any payoff $u_s^* \in [0, \delta M(1)]$ is obtained for some $\theta^* \in [0, 1]$.

Let us now construct the equilibria. Fix any $\theta^* \in (0, 1)$. (The case $\theta^* = 1$ is similar; the case $\theta^* = 0$ was treated in Proposition 1.) Let period-1 bids be defined by (13), $b^*(\theta^*) = 0$, $b^*(1) = 1$, and (14). Define on-path posterior beliefs by (1). Off-path beliefs are defined by (16). Given the posterior beliefs, define the resale mechanisms by (4).

We have to show that (6) and (7) hold. To prove (7), suppose that the buyers use the bid function b^*. If the speculator wins at a price $p^1 > \theta^*$ she believes that p^1 equals the highest value in the market. Therefore, her resale revenue equals p^1. Using this and definition (36), the speculator's expected payoff from any bid $b_s \geq 0$ is given by

$$u_s^{\mathrm{II}}(b_s) = \begin{cases} 0 & \text{if } b_s = 0, \\ u_s^* & \text{if } b_s \in (0, \theta^*], \\ u_s^* + E\left[(-\tilde{\theta}_{-i}^{(1)} + \delta\tilde{\theta}_{-i}^{(1)})\mathbf{1}_{\theta^* \leq \tilde{\theta}_{-i}^{(1)} \leq b_s}\right] & \text{if } b_s > \theta^*, \end{cases}$$

where $\tilde{\theta}_{-i}^{(1)}$ denotes the random variable for the highest value among buyers other than i. Because $u_s^* > 0$ and $-\tilde{\theta}_{-i}^{(1)} + \delta\tilde{\theta}_{-i}^{(1)} < 0$,

$$\arg\max_{b_s \geq 0} u_s^{\mathrm{II}}(b_s) = (0, \theta^*] \ni b_s^*,$$

which completes the proof of (7).

To prove (6), we first provide an upper bound for a winning buyer's period-2 payoff. For all $i \neq s$, $b_{-i} \in [0, \infty)^n$, and $\theta_i \in [0, 1]$, define

$$v(i, b_{-i}, \theta_i) = \theta_i Q^{\mathrm{II}}(i, b_{-i}, \theta_i) + P^{\mathrm{II}}(i, b_{-i}, \theta_i).$$

This payoff cannot exceed the total expected surplus available in the resale market, given buyer i's posterior beliefs. Hence,

$$v(i, b_{-i}, \theta_i) \leq \begin{cases} E\left[\max\{\theta_i, \tilde{\theta}_{-i}^{(1)}\} \mid \tilde{\theta}_{-i}^{(1)} \leq \theta^*\right] & \text{if } b_{-i-s}^{(1)} = 0, \\ \max\{\theta_i, b_{-i-s}^{(1)}\} & \text{if } b_{-i-s}^{(1)} \in (\theta^*, 1], \\ \max\{\theta_i, \theta^*\} & \text{if } b_{-i-s}^{(1)} \in (0, \theta^*]. \end{cases} \quad (37)$$

The next step is to show (*).

> (*) For all buyers j and all bids $b_j \geq 0$, buyer j does not expect to get a resale offer from another buyer.

Consider any bid vector $b_{-j} \in [0, 1]^n$ that is possible if all buyers other than j bid according to b^*, and the speculator bids b_s^*. Suppose that buyer $i \neq j$ wins. Then, $\theta_i > \theta^*$ because otherwise $b^*(\theta_i) = 0 < b_s^*$. The winning price satisfies $b_{-i-s}^{(1)} \in [b_s^*, \theta_i]$. This implies $\delta v(i, b_{-i}, \theta_i) \leq \delta\theta_i \leq \theta_i$ by (37). Therefore, it is optimal for buyer i to consume the good rather than offering it for resale, which completes the proof of (*).

For all $b_{-s} \in [0, \infty)^n$, and $\theta_i \in [0, 1]$, define

$$l_i(b_{-s}, \theta_i) = Q_i^{\mathrm{II}}(s, b_{-s}, 0, \theta_i)\theta_i - P_i^{\mathrm{II}}(s, b_{-s}, 0, \theta_i).$$

This is buyer i's expected period-2 payoff when the speculator wins in period 1, all buyers except i use the bid function b^*, and the buyers' actual bids are b_{-s}. Given the equilibrium construction,

$$l_i(b_{-s}, \theta_i) = \begin{cases} \theta_i Q_{\theta^*}(\theta_i) - P_{\theta^*}(\theta_i) & \text{if } b_{-s}^{(1)} = 0, \\ \max\{0, \theta_i - \theta^*\} & \text{if } b_i \in (0, b_s^*),\ b_{-s-i}^{(1)} = 0. \end{cases} \tag{38}$$

By (*), for all $i \neq s$, $\theta_i \in [0,1]$, and $b_i \geq 0$,

$$u_i(b_i, \theta_i) \quad = \quad \begin{cases} F^{n-1}(\theta^*)\ \delta\ l_i((b_i, 0, \ldots, 0), \theta_i) & \text{if } b_i \in [0, b_s^*), \\ E\left[(-\tilde{b}_{-i}^{(1)} + \max\{\theta_i, \delta v(i, \tilde{b}_{-i}, \theta_i)\})\mathbf{1}_{\tilde{b}_{-i}^{(1)} \leq b_i}\right] & \text{if } b_i \geq b_s^*, \end{cases} \tag{39}$$

where we use the shorthand $u_i \stackrel{\text{def}}{=} u_i^{\text{II}}$. We are now in a position to show that low-value buyers bid 0,

$$\forall \theta_i \in [0, \theta^*], b_i > 0:\ u_i(0, \theta_i) \geq u_i(b_i, \theta_i). \tag{40}$$

For all $b_i \in (0, b_s^*)$, we have $u_i(b_i, \theta_i) = 0 \leq u_i(0, \theta_i)$. For all $b_i \geq \theta^*$,

$$\begin{aligned} u_i(b_s^*, \theta_i) - u_i(b_i, \theta_i) \quad &= \quad -E\left[(-\tilde{b}_{-i}^{(1)} + \max\{\theta_i, \delta v(i, \tilde{b}_{-i}, \theta_i)\})\mathbf{1}_{\theta^* < \tilde{b}_{-i}^{(1)} \leq b_i}\right] \\ &\stackrel{(37)}{\geq} \quad E\left[(\tilde{b}_{-i}^{(1)} - \max\{\theta_i, \delta \tilde{b}_{-i}^{(1)}\})\mathbf{1}_{\theta^* < \tilde{b}_{-i}^{(1)} \leq b_i}\right] \\ &\geq \quad 0. \end{aligned}$$

Therefore, (40) follows once we show that

$$u_i(b_s^*, \theta_i) \leq u_i(0, \theta_i). \tag{41}$$

Note that

$$u_i(b_s^*, \theta_i) = (-b_s^* + \max\{\theta_i, \delta v^*\})F(\theta^*)^{n-1}, \tag{42}$$

where we use the shorthand $v^* = v(i, (0, \ldots, 0, b_s^*), \theta_i)$.

The envelope theorem or an explicit computation using Lemma 2, shows that

$$Q_{\theta^*}(\theta_i)\theta_i - P_{\theta^*}(\theta_i) = \int_0^{\theta_i} Q_{\theta^*}(\hat{\theta})\mathrm{d}\hat{\theta}. \tag{43}$$

Therefore,

$$\begin{aligned} u_i(0, \theta_i) \quad &= \quad \delta\left(\theta_i Q_{\theta^*}(\theta_i) - P_{\theta^*}(\theta_i)\right) F(\theta^*)^{n-1} \\ &\stackrel{(43)}{=} \quad \begin{cases} \delta \int_{r^*}^{\theta_i} F^{n-1}(\theta_i')\mathrm{d}\theta_i' & \text{if } \theta_i \geq r^*, \\ 0 & \text{if } \theta_i < r^*, \end{cases} \end{aligned} \tag{44}$$

where $r^* \stackrel{\text{def}}{=} r^*(\theta^*)$ denotes the speculator's optimal reserve price when she offers the good for resale after winning at price 0.

Because either $\theta_i \leq \delta v^*$ or $\theta_i \geq \delta v^*$, formulas (42) and (44) reveal that (41) follows once we show

$$\theta_i - b_s^* \leq \delta\left(\theta_i Q_{\theta^*}(\theta_i) - P_{\theta^*}(\theta_i)\right) \tag{45}$$

and

$$(\delta v^* - b_s^*)F(\theta^*)^{n-1} \leq \begin{cases} \delta \int_{r^*}^{\theta_i} F^{n-1}(\theta_i')\mathrm{d}\theta_i' & \text{if } \theta_i \geq r^*, \\ 0 & \text{if } \theta_i < r^*. \end{cases} \tag{46}$$

Define

$$\Delta(\theta_i) = \theta_i - b_s^* - (\theta_i Q_{\theta^*}(\theta_i) - P_{\theta^*}(\theta_i)). \tag{47}$$

By (14), $\Delta(\theta^*) = 0$. Moreover, using (43) we find

$$\Delta(\theta_i) = \theta_i - b_s^* - \int_0^{\theta_i} \delta Q_{\theta^*}(\theta_i')\mathrm{d}\theta_i' = \int_0^{\theta_i} (1 - \delta Q_{\theta^*}(\theta_i'))\mathrm{d}\theta_i' - b_s^*.$$

Hence, Δ is increasing and therefore $\Delta(\theta_i) \leq \Delta(\theta^*) = 0$, which implies (45).

To show (46), note that (14) implies

$$b_s^* = \theta^* - \delta \int_{r^*}^{\theta^*} \frac{F^{n-1}(\theta_i')}{F^{n-1}(\theta^*)}\mathrm{d}\theta_i'$$

and thus

$$\begin{aligned}
&(\delta v^* - b_s^*)F(\theta^*)^{n-1} - \begin{cases} \delta \int_{r^*}^{\theta_i} F^{n-1}(\theta_i')\mathrm{d}\theta_i' & \text{if } \theta_i \geq r^*, \\ 0 & \text{if } \theta_i < r^* \end{cases} \\
&= \delta v^* F(\theta^*)^{n-1} - \theta^* F^{n-1}(\theta^*) + \delta \int_{r^*}^{\theta^*} F^{n-1}(\theta_i')\mathrm{d}\theta_i' \\
&\qquad - \begin{cases} \delta \int_{r^*}^{\theta_i} F^{n-1}(\theta_i')\mathrm{d}\theta_i' & \text{if } \theta_i \geq r^*, \\ 0 & \text{if } \theta_i < r^* \end{cases} \\
&\leq v^* F(\theta^*)^{n-1} - \theta^* F^{n-1}(\theta^*) + \delta \int_{\theta_i}^{\theta^*} F^{n-1}(\theta_i')\mathrm{d}\theta_i' \\
&\stackrel{(37)}{\leq} \delta \int_0^{\theta^*} \max\{\theta_i', \theta_i\}\mathrm{d}F^{n-1}(\theta_i') - \theta^* F^{n-1}(\theta^*) + \delta \int_{\theta_i}^{\theta^*} F^{n-1}(\theta_i')\mathrm{d}\theta_i' \\
&= \delta F^{n-1}(\theta_i)\theta_i - \theta^* F^{n-1}(\theta^*) + \delta \left(\int_{\theta_i}^{\theta^*} \theta_i' \mathrm{d}F^{n-1}(\theta_i') + \int_{\theta_i}^{\theta^*} F^{n-1}(\theta_i')\mathrm{d}\theta_i' \right) \\
&= \delta F^{n-1}(\theta_i)\theta_i - \theta^* F^{n-1}(\theta^*) + \delta \left(F^{n-1}(\theta^*)\theta^* - F^{n-1}(\theta_i)\theta_i\right) \\
&= -(1-\delta)\theta^* F^{n-1}(\theta^*) \\
&\leq 0,
\end{aligned}$$

which completes the proof of (40).

The final step is showing that high-value buyers find it optimal to bid their value; i.e.,

$$\forall \theta_i \in [\theta^*, 1], b_i \geq 0: \; u_i(\theta_i, \theta_i) \geq u_i(b_i, \theta_i). \tag{48}$$

The payoff gain from any bid $b_i > \theta_i$ is given by

$$\begin{aligned} u_i(b_i, \theta_i) - u_i(\theta_i, \theta_i) &= E\left[(-\tilde{b}_{-i}^{(1)} + \max\{\theta_i, \delta v(\tilde{b}_{-i}, \theta_i)\})\mathbf{1}_{\theta_i \leq \tilde{b}_{-i}^{(1)} \leq b_i}\right] \\ &= E\left[(-\tilde{b}_{-i}^{(1)} + \max\{\theta_i, \delta \tilde{b}_{-i}^{(1)}\})\mathbf{1}_{\theta_i \leq \tilde{b}_{-i}^{(1)} \leq b_i}\right] \\ &\leq 0. \end{aligned}$$

The payoff gain from any bid $b_i \in [b_s^*, \theta_i)$ is given by

$$\begin{aligned} u_i(b_i, \theta_i) - u_i(\theta_i, \theta_i) &= -E\left[(-\tilde{b}_{-i}^{(1)} + \max\{\theta_i, \delta v(\tilde{b}_{-i}, \theta_i)\})\mathbf{1}_{b_i < \tilde{b}_{-i}^{(1)} \leq \theta_i}\right] \\ &= E\left[(\tilde{b}_{-i}^{(1)} - \theta_i)\mathbf{1}_{\theta^* \leq \tilde{b}_{-i}^{(1)} \leq b_i}\right] \\ &\leq 0. \end{aligned}$$

The payoff gain from any bid $b_i \in (0, b_s^*)$ is given by

$$\begin{aligned} u_i(b_i, \theta_i) - u_i(\theta_i, \theta_i) &= F(\theta^*)^{n-1}\delta(\theta_i - \theta^*) - E\left[(\theta_i - \tilde{b}_{-i}^{(1)})\mathbf{1}_{\tilde{b}_{-i}^{(1)} \leq \theta_i}\right] \\ &= F(\theta^*)^{n-1}\left(\delta(\theta_i - \theta^*) - (\theta_i - b_s^*)\right) \\ &\quad - E\left[(\theta_i - \tilde{b}_{-i}^{(1)})\mathbf{1}_{\theta^* \leq \tilde{b}_{-i}^{(1)} \leq \theta_i}\right] \\ &< 0. \end{aligned}$$

The payoff gain from bidding $b_i = b_s^*$ rather than $b_i = 0$ equals $\Delta(\theta_i)F(\theta^*)^{n-1}$, where $\Delta(\theta_i)$ is defined as in (47). Because $\Delta(\theta^*) = 0$ and Δ is increasing,

$$u_i(b_s^*, \theta_i) - u_i(0, \theta_i) = \Delta(\theta_i)F(\theta^*)^{n-1} \geq 0,$$

which completes the proof of (48). Condition (6) follows from (40) and (48). QED

PROOF OF PROPOSITION 4.

For all $\theta^* \in [0, 1]$, let $\pi(\theta^*)$ denote the initial seller's expected revenue in a θ^*-equilibrium. In particular, $\pi(0)$ is the expected revenue of the initial seller when every agent bids her value in period 1. We show that $\pi(\theta^*) > \pi(0)$ for all θ^* sufficiently close to 0 and $\pi(\theta^*) < \pi(0)$ for all θ^* sufficiently close to 1.

Let $\tilde{\theta}^{(1)}$ and $\tilde{\theta}^{(2)}$ denote the highest and second highest value among the buyers.

$$\begin{aligned}\pi(\theta^*) - \pi(0) &= \Pr[\tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}](b_s^* - E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}]) \\ &\quad + \Pr[\tilde{\theta}^{(1)} < \theta^*](0 - E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(1)} < \theta^*]). \qquad (49)\end{aligned}$$

We have $\Pr[\tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}] = nF(\theta^*)^{n-1}(1 - F(\theta^*)) \to 0$ as $\theta^* \to 1$. Thus, $\pi(\theta^*) < \pi(0)$ for all θ^* sufficiently close to 1. Let us now consider the case where θ^* is close to 0.

Using (15), (49) implies

$$\begin{aligned}\pi(\theta^*) - \pi(0) &\geq \Pr[\tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}](P_{\theta^*}(\theta^*) - E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}]) \\ &\quad - \Pr[\tilde{\theta}^{(1)} < \theta^*]\ E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(1)} < \theta^*]. \qquad (50)\end{aligned}$$

By Lemma 2 and $r^{*\prime}(0) = 1/2$,[10]

$$\begin{aligned}\frac{P_{\theta^*}(\theta^*)}{\theta^*} &= 1 - \frac{1}{\theta^* F^{n-1}(\theta^*)} \int_{\theta^*/2}^{\theta^*} (f(0)\theta + \mathrm{o}(\theta))^{n-1}\,\mathrm{d}\theta \ + \frac{\mathrm{o}(\theta^*)}{\theta^*} \\ &= 1 - \frac{1}{\theta^* F^{n-1}(\theta^*)} \int_{\theta^*/2}^{\theta^*} \left(f(0)^{n-1}\theta^{n-1} + \mathrm{o}(\theta^{n-1})\right) \mathrm{d}\theta \ + \frac{\mathrm{o}(\theta^*)}{\theta^*} \\ &= 1 - \frac{f(0)^{n-1}}{\theta^* F^{n-1}(\theta^*)} \left(\frac{\theta^{*n}}{n}(1 - \frac{1}{2^n}) + \mathrm{o}(\theta^{*n})\right) \ + \frac{\mathrm{o}(\theta^*)}{\theta^*}.\end{aligned}$$

Therefore,

$$\lim_{\theta^* \to 0} \frac{P_{\theta^*}(\theta^*)}{\theta^*} = 1 - \frac{1}{n}(1 - \frac{1}{2^n}) = \frac{n-1}{n} + \frac{1}{n2^n} \qquad (51)$$

Similarly,

$$\lim_{\theta^* \to 0} \frac{E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}]}{\theta^*} = \frac{n-1}{n} \qquad (52)$$

and

$$\lim_{\theta^* \to 0} \frac{E[\tilde{\theta}^{(2)} \mid \tilde{\theta}^{(1)} < \theta^*]}{\theta^*} = \frac{n-1}{n+1}. \qquad (53)$$

Moreover,

$$\frac{\Pr[\tilde{\theta}^{(2)} < \theta^* < \tilde{\theta}^{(1)}]}{\Pr[\tilde{\theta}^{(1)} < \theta^*]} = n\frac{1 - F(\theta^*)}{F(\theta^*)} \to_{\theta^* \to 0} \infty. \qquad (54)$$

[10] In the following, $\mathrm{o}(\theta^l)$ stands for a function $k(\theta)$ with the property $k(\theta)/\theta^l \to 0$ as $\theta \to 0$.

Taking (50), (51), (52), (53), and (54) together implies that $\pi(\theta^*) > \pi(0)$ if θ^* is small. *QED*

We now turn to the first-price auction results. We will use the shorthand $u_i = u_i^{\mathrm{I}}$ throughout. Let b^* be any bid function. For all $b \in [0, b^*(1)]$, define

$$\phi(b) = \sup\{\theta_i \in [0,1] \mid b^*(\theta_i) < b\},$$

where $\sup \emptyset = 0$.

LEMMA 4 *Let (b^*, H) be an equilibrium outcome of the first-price auction with resale. Then, $b^*(\theta_i) \leq \theta_i$ for all $\theta_i \in [0,1]$. Moreover, $\phi(b) \geq b$.*

PROOF.

By Assumption 3, $b^*(0) = 0$. Now suppose that $b^*(\theta_i) > \theta_i$ for some $\theta_i > 0$. In particular, $b^*(\theta_i) > 0$ and thus $b^*(\theta_i)$ wins with positive probability by Assumption 3. Therefore, $H(b^*(\theta_i)) > 0$ and $F^{n-1}(\phi(b^*(\theta_i))) > 0$.

A deviation to the bid $b_i = \theta_i$ is profitable because

$$\begin{aligned} & u_i(\theta_i, \theta_i) - u_i(b^*(\theta_i), \theta_i) \\ & = H(b^*(\theta_i)) F^{n-1}(\phi(b^*(\theta_i)))(b^*(\theta_i) - \theta_i) \\ & \qquad + \delta E \left[\underbrace{\left(\theta_i Q_{\phi(\tilde{b}_s)}(\theta_i) - P_{\phi(\tilde{b}_s)}(\theta_i)\right)}_{\geq 0} \mathbf{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i},\ \theta_i < \tilde{b}_s \leq b^*(\theta_i)} \right] \\ & > 0. \end{aligned}$$

To prove the moreover-part, fix $b > 0$ and consider any $\theta_i < b$. Then $b^*(\theta_i) \leq \theta_i < b$. Because θ_i is arbitrary, $\phi(b) \geq b$. *QED*

LEMMA 5 *Let (b^*, H) be an equilibrium outcome of the first-price auction with resale. Then $\forall b > 0 : \Pr[\tilde{b}_s < b] > 0$.*

PROOF. Let $\underline{b}_s \leq 0$ be maximal with the property $\Pr[\tilde{b}_s < \underline{b}_s] = 0$. Suppose that $\underline{b}_s > 0$.

Let $u_s^* \geq 0$ denote the equilibrium expected payoff for the speculator; i.e., $\Pr[u_s(\tilde{b}_s) = u_s^*] = 1$.

First consider the case $H(\underline{b}_s) = 0$ (i.e., no atom at $\underline{b}_s$). By assumption, there exists a sequence $(b^m)_{m \in \mathbb{N}}$ such that $b^m \to \underline{b}_s$ as $m \to \infty$, and $u_s(b^m) = u_s^*$ and $b^m > \underline{b}_s$ for all m.

Denote by b^* the bid function used by the buyers. By Lemma 4, $\Pr[b^*(\tilde{\theta}_i) \leq \underline{b}_s] > 0$. Define $\theta^* = \inf_{b > \underline{b}_s} \phi(b)$. Because $\Pr[b^*(\tilde{\theta}_i) \leq \underline{b}_s] > 0$, we have $\theta^* > 0$. For all $b = b^m$,

$$u_s^* = u_s(b) = F(\phi(b))^n \left(\delta M(\phi(b)) - b\right),$$

where M is defined as in Proposition 5. Because $u_s^* \geq 0$, we have $M(\phi(b)) \geq \underline{b}_s/\delta$. Thus, (3) implies

$$P_{\theta^*}(\theta^*) \geq M(\theta^*) + \eta \geq \frac{\underline{b}_s}{\delta} + \eta, \tag{55}$$

for some $\eta > 0$. By Lemma 2, for $b > \underline{b}_s$ and $\theta_i < \theta^*$,

$$P_{\phi(b)}(\theta_i) \geq Q_{\phi(b)}(\theta_i) \frac{P_{\theta^*}(\theta_i)}{Q_{\theta^*}(\theta_i)} \underset{\substack{b \to \underline{b}_s \\ \theta_i \to \theta^*}}{\longrightarrow} P_{\theta^*}(\theta^*).$$

Therefore, there exists $\hat{b} > \underline{b}_s$ and $\hat{\theta} < \theta^*$ such that

$$\forall b \in (\underline{b}_s, \hat{b}) : \; P_{\phi(b)}(\hat{\theta}) \geq P_{\theta^*}(\theta^*) - \frac{\eta}{2}.$$

Combining this with (55), we find

$$\forall b \in (\underline{b}_s, \hat{b}) : \; P_{\phi(b)}(\hat{\theta}) \geq \frac{\underline{b}_s}{\delta} + \frac{\eta}{2}. \tag{56}$$

Buyer i with value $\theta_i = \hat{\theta}$ and bid $b_i = b^*(\hat{\theta})$ never wins. Moreover, if some buyer $j \neq i$ wins then $b_j = b^*(\theta_j) > \underline{b}_s \geq b_i$, implying $\theta_j > \hat{\theta}$. Therefore, $Q_i^{\mathrm{I}}(j, b_j, \theta_j) = 0$ and $P_i^{\mathrm{I}}(j, b_j, \theta_j) = 0$. Therefore,

$$u_i(b^*(\hat{\theta}), \hat{\theta}) = \delta E\left[\left(\hat{\theta} Q_{\phi(\tilde{b}_s)}(\hat{\theta}) - P_{\phi(\tilde{b}_s)}(\hat{\theta})\right) \mathbf{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i}}\right].$$

On the other hand, for all $b_i > \underline{b}_s$,

$$\begin{aligned} u_i(b_i, \hat{\theta}) \;\; \geq \;\; & \delta E\left[\left(\hat{\theta} Q_{\phi(\tilde{b}_s)}(\hat{\theta}) - P_{\phi(\tilde{b}_s)}(\hat{\theta})\right) \mathbf{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i},\; \tilde{b}_s > b_1}\right] \\ & + (\hat{\theta} - b_i) \Pr[\tilde{\theta}^{(1)}_{-i} < \theta^*, \; \tilde{b}_s \leq b_1]. \end{aligned}$$

Therefore, for all $b_i \in (\underline{b}_s, \hat{b})$,

$$\begin{aligned} u_i(b_i, \hat{\theta}) - u_i(b^*(\hat{\theta}), \hat{\theta}) \quad \geq \quad & -\delta E\left[\left(\hat{\theta} Q_{\phi(\tilde{b}_s)}(\hat{\theta}) - P_{\phi(\tilde{b}_s)}(\hat{\theta})\right) \mathbf{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i},\ \tilde{b}_s \leq b_i}\right] \\ & +(\hat{\theta} - b_i) \Pr[\tilde{b}_s > \tilde{b}^{(1)}_{-s-i},\ \tilde{b}_s \leq b_i] \\ & \geq E\left[\left(\delta P_{\phi(\tilde{b}_s)}(\hat{\theta}) - b_i\right) \mathbf{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i},\ \tilde{b}_s \leq b_i}\right] \\ & \overset{(56)}{\geq} E\left[\left(\underline{b}_s + \frac{\delta\eta}{2} - b_i\right) \mathbf{1}_{\tilde{b}_s > \tilde{b}^{(1)}_{-s-i},\ \tilde{b}_s \leq b_i}\right] \\ & \geq E\left[\left(\underline{b}_s + \frac{\delta\eta}{2} - b_i\right) \mathbf{1}_{\tilde{\theta}^{(1)}_{-i} < \theta^*,\ \tilde{b}_s \leq b_i}\right]. \end{aligned}$$

The last expression is strictly positive for all $b_i > \underline{b}_s$ that are sufficiently close to $\underline{b}_s$. This contradicts (8).

In the case $H(\underline{b}_s) > 0$ the proof is similar. One defines $\theta^* = \phi(\underline{b}_s)$ and shows that the deviation $b_1 = \underline{b}_s$ is profitable for some type $\hat{\theta} < \theta^*$. *QED*

LEMMA 6 *Let (b^*, H) be an equilibrium outcome of the first-price auction with resale. Then*

$$\max_{b \geq 0} u_s(b) = 0.$$

PROOF. Let $u_s^* \geq 0$ denote the equilibrium expected payoff for the speculator; i.e., $\Pr[u_s(\tilde{b}_s) = u_s^*] = 1$. Suppose that $u_s^* > 0$.

We have $H(0) = 0$ because otherwise $u_s^* = 0$. By Lemma 5, there exists a sequence $(b^m)_{m \in \mathbb{N}}$ such that $b^m \to 0$ as $m \to \infty$, and $u_s(b^m) = u_s^*$ and $b^m > 0$ for all m.

Define $\theta^* = \sup\{\theta_i \in [0, 1] \mid b^*(\theta_i) = 0\}$. For all $b = b^m$,

$$u_s^* = u_s(b) \geq F(\theta^*)^n \left(\delta M(\phi(b)) - b\right).$$

Because $u_s^* > 0$, we have $\theta^* > 0$ and $M(\phi(b)) \geq u_s^*$ for all $b = b^m$. Therefore, $M(\theta^*) \geq u_s^*$. Using the same argument as in the proof of Lemma 5, this implies the existence of $\hat{b} > 0$ and $\hat{\theta} < \theta^*$ such that

$$\forall b \in (0, \hat{b}) : \ P_{\phi(b)}(\hat{\theta}) \geq \frac{u_s^*}{2}.$$

Buyer i with value $\theta_i = \hat{\theta}$ and bid $b_i = b^*(\hat{\theta}) = 0$ never wins. In the same manner as in the proof of Lemma 5, one obtains a contradiction by showing that a deviation to a small positive bid $b_i > 0$ is profitable. *QED*

LEMMA 7 *Let* (b^*, H) *be an equilibrium outcome of the first-price auction with resale. Then* H *is continuous on* $(0, \infty)$.

PROOF. Suppose that there exists $\hat{b} > 0$ where $H(\cdot)$ is not continuous; i.e., $\Pr[\tilde{b}_s = \hat{b}] > 0$.

Define $\theta^m = \phi(\hat{b}) - 1/m$ for all m large enough that $\theta^m > 0$. Let $\overline{b} = \lim_{m\to\infty} b^*(\theta^m)$. We have $\overline{b} = \hat{b}$ because otherwise $\Pr[b^*(\tilde{\theta}_i) \in (\overline{b}, \hat{b})] = 0$ which would imply $u_s((\hat{b} + \overline{b})/2) > u_s(\hat{b})$.

Using (3) we find $P_{\phi(\hat{b})}(\phi(\hat{b})) \geq M(\phi(\hat{b})) + \eta \geq \hat{b}/\delta + \eta$ because otherwise $u_s(\hat{b}) < 0$. By Lemma 2, the function $P_{\phi(\hat{b})}$ is continuous at $\phi(\hat{b})$ and thus

$$P_{\phi(\hat{b})}(\theta^m) > \hat{b}/\delta + \eta/2,$$

for all large m. Therefore, for large m,

$$\begin{aligned} &u_i(\hat{b}, \theta^m) - u_i(b^*(\theta^m), \theta^m) \\ &\geq H(b^*(\theta^m))F^{n-1}(\phi(b^*(\theta^m)))(b^*(\theta^m) - \hat{b}) + \Pr[\tilde{b}_s \in (b^*(\theta^m), \hat{b})](-1) \\ &\quad + \Pr[\tilde{b}_s = \hat{b}]F^{n-1}(\phi(\hat{b}))((1-\delta)\theta^m + (\delta P_{\phi(\hat{b})}(\theta^m) - \hat{b})). \end{aligned}$$

Therefore,

$$\liminf_{m\to\infty} u_1(\hat{b}, \theta^m) - u_1(b_1(\theta^m), \theta^m) \geq \Pr[\tilde{b}_s = \hat{b}]F^{n-1}(\phi(\hat{b}))(\delta P_{\phi(\hat{b})}(\theta^m) - \hat{b}) > 0.$$

I.e., for large m type θ^m has a profitable deviation. *QED*

LEMMA 8 *Let* (b^*, H) *be an equilibrium outcome of the first-price auction with resale. Then the bid function* b^* *is continuous. Moreover,* $\phi(b^*(\theta_i)) = \theta_i$ *for all* $\theta_i \in [0, 1]$. *The function* ϕ *is strictly increasing.*

PROOF. Standard.

LEMMA 9 *Let* (b^*, H) *be an equilibrium outcome of the first-price auction with resale. Then* $b^*(\theta_i) < \theta_i$ *for all* $\theta_i \in (0, 1]$. *Moreover,* $\phi(b) > b$ *for all* $b \in (0, b^*(1)]$.

PROOF. Suppose that $b^*(\theta_i) \geq \theta_i > 0$. Then $b^*(\theta_i) > 0$, implying $H(b^*(\theta_i)) > 0$ by Lemma 5. Now Lemma 4 shows that $b_1(\theta_1) = \theta_1$. Finally, a computation similar to that in the proof of Lemma 4 shows that a deviation to the bid $b_1 = \theta_1/2$ is profitable—contradiction.

The claim about ϕ now follows from Lemma 8. *QED*

For $\theta \in (0,1]$ and $b \in \mathbb{R}$, define

$$N(\theta,b) = (n-1)\frac{f(\theta)}{F(\theta)}(\theta - b),$$

$$K(\theta,b) = \begin{cases} N(\theta,b) & \text{if } b > \delta M(\theta), \\ \max\{\delta M'(\theta), N(\theta,b)\} & \text{if } b \leq \delta M(\theta). \end{cases} \tag{57}$$

For all $b \in (0, b^*(1)]$ where the derivative $\phi'(b)$ exists, define

$$\begin{aligned} R(b) &= F(\phi(b)) - (n-1)f(\phi(b))\phi'(b)(\phi(b) - b), \\ S(b) &= F(\phi(b))(\phi(b) - b - \delta(\phi(b) - P_{\phi(b)}(\phi(b)))), \end{aligned} \tag{58}$$

where $P_{\phi(b)}(\phi(b))$ denotes the expected payment of a bidder with value $\phi(b)$ in the mechanism $\mathcal{M}(\phi(b))$. Also define

$$L(b) = \begin{cases} \frac{R(b)}{S(b)} & \text{if } b = \delta M(\phi(b)), \\ 0 & \text{if } b > \delta M(\phi(b)). \end{cases} \tag{59}$$

(L is well-defined because $b = \delta M(\phi(b))$ implies $S(b) > 0$ by (3)).

Lemma 10 *Let (b^*, H) be an equilibrium outcome of the first-price auction with resale. Then ϕ is Lipschitz on $[0, b^*(1)]$.*

Moreover, b^* is Lipschitz on $[0,1]$, satisfies[11]

$$b^{*\prime}(\theta) = K(\theta, b^*(\theta)) \text{ a.e. } \theta \in (0,1), \quad b^*(0) = 0, \tag{60}$$

and is uniquely defined by these conditions.

The distribution H is locally Lipschitz on $(0,1]$, satisfies

$$h(b) \stackrel{def}{=} H'(b) = H(b)L(b) \text{ for a.e. } b \in (0, b^*(1)], \tag{61}$$

and is uniquely defined by these conditions.

Proof. For all $b_i \in (0, b^*(1)]$ and $\theta_i \in [0,1]$,

$$\begin{aligned} u_i(b_i, \theta_i) &= H(b_i)F(\phi(b_i))^{n-1}(\theta_i - b_i) \\ &\quad + \delta \int_{b_i}^{\infty} F(\phi(b_s))^{n-1}(\theta_i Q_{\phi(b_s)}(\theta_i) - P_{\phi(b_s)}(\theta_i))\mathrm{d}H(b_s). \end{aligned}$$

[11] "a.e." means "almost every, according to the Lebesgue measure."

Now consider $b_i, b_i' \in (0, b^*(1)]$ with $b_i' < b_i$, and $\theta_i' = \phi(b_i')$. First note that

$$\delta \int_{b_i'}^{b_i} F(\phi(b_s))^{n-1} \underbrace{Q_{\phi(b_s)}(\theta_i')}_{\text{decreasing in } b_s} \underbrace{\left(\theta_i' - \frac{P_{\phi(b_s)}(\theta_i')}{Q_{\phi(b_s)}(\theta_i')}\right)}_{\text{decreasing in } b_s \text{ by (21)}} \mathrm{d}H(b_s)$$

$$\leq \delta F(\phi(b_i))^{n-1} \left(\phi(b_i') - P_{\phi(b_i')}(\phi(b_i'))\right) (H(b_i) - H(b_i')), \tag{62}$$

because $Q_{\phi(b_i')}(\phi(b_i')) = 1$. Now,

$$\begin{aligned} 0 &\geq u_i(b_i, \theta_i') - u_i(b_i', \theta_i') \\ &= \left(H(b_i)F(\phi(b_i))^{n-1} - H(b_i')F(\phi(b_i'))^{n-1}\right)\theta_i' \\ &\quad - \left(H(b_i)F(\phi(b_i))^{n-1}b_i - H(b_i')F(\phi(b_i'))^{n-1}b_i'\right) \\ &\quad -\delta \int_{b_i'}^{b_i} F(\phi(b_s))^{n-1}(Q_{\phi(b_s)}(\theta_i')\theta_i' - P_{\phi(b_s)}(\theta_i'))\mathrm{d}H(b_s) \\ &\overset{(62)}{\geq} -H(b_i')k_1(b_i, b_i')(b_i - b_i') + k_2(b_i, b_i')(H(b_i) - H(b_i')), \end{aligned} \tag{63}$$

where

$$\begin{aligned} k_1(b_i, b_i') &= F(\phi(b_i))^{n-1} - \frac{F(\phi(b_i))^{n-1} - F(\phi(b_i'))^{n-1}}{b_i - b_i'}(\phi(b_i') - b_i'), \\ k_2(b_i, b_i') &= F(\phi(b_i))^{n-1}\left(\phi(b_i')(1-\delta) + \delta P_{\phi(b_i')}(\phi(b_i')) - b_i\right). \end{aligned}$$

Now fix some $c_i \in (0, b^*(1)]$. We write $\Xi(c_i) = 0$ if

$$\exists \xi > 0 : \ \Pr[\tilde{b}_s \in (c_i - \xi, c_i + \xi)] = 0,$$

and $\Xi(c_i) = 1$ otherwise. By Lemma 6,

$$\text{if } \Xi(c_i) = 1 \text{ then } \exists (c^m), c^m \to c_i, c^m \neq c_i : \ \delta M(\phi(c_m)) = c_m. \tag{64}$$

Therefore, by continuity of M and ϕ,

$$\text{if } \Xi(c_i) = 1 \text{ then } M(\phi(c_i)) = c_i. \tag{65}$$

First consider the case $\Xi(c_i) = 0$. Then, H is constant in the neighborhood $N(c_i) = (c_i - \xi, c_i + \xi) \cap (0, b^*(1)]$ of c_i. In particular, H is Lipschitz in $N(c_i)$. Moreover, (63) implies that

$$\forall c_i, \Xi(c_i) = 0, b_i \in N(c_i), b_i' \in N(c_i), b_i' < b_i : \ k_1(b_i, b_i') \geq 0. \tag{66}$$

Now consider the case $\Xi(c_i) = 1$. Using (3) and (65), there exists $\eta > 0$ such that for all b_i, b_i' in some neighborhood of c_i, and $b_i' < b_i$,

$$\begin{aligned} k_2(b_i, b_i') &\geq F(\phi(b_i))^{n-1}\left(\phi(b_i')(1-\delta) + \delta M(\phi(b_i')) + \delta\eta - b_i\right) \\ &\geq F(\phi(b_i))^{n-1}\frac{\delta\eta}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \forall c_i, \Xi(c_i) = 1 \qquad & \exists \underline{N}(c_i) > 0 \text{ and a neighborhood } N(c_i) \ni c_i : \\ & \forall b_i, b_i' \in N(c_i), b_i' < b_i : \; k_2(b_i, b_i') \geq \underline{N}(c_i). \end{aligned} \tag{67}$$

Together with (63) and $k_1(b_i, b_i') \leq 1$, this implies that H is Lipschitz in $N(c_i)$. We therefore conclude that in either case, $\Xi(c_i) = 0$ or $\Xi(c_i) = 1$, the function H is Lipschitz in a neighborhood of c_i, proving that H is locally Lipschitz in $(0, 1]$. Therefore, H is differentiable almost everywhere on $(0, 1]$.

Note also that if $\Xi(c_i) = 1$, formula (67) together with (63) implies that (66) holds for some neighborhood $N(c_i)$ of c_i. We conclude that

$$\forall b_i \in (0, b^*(1)], b_i' \in (0, b^*(1)], b_i' < b_i : \; k_1(b_i, b_i') \geq 0. \tag{68}$$

Moreover, (68) implies

$$\forall b_i, \; \phi \text{ differentiable at } b_i : \; \phi'(b_i) \leq \frac{1}{N(\phi(b_i), b_i)}. \tag{69}$$

Note that for all $b_i, b_i' \in (0, b^*(1)]$ with $b_i' < b_i$, and $\theta_i = \phi(b_i)$,

$$\begin{aligned} 0 &\leq u_i(b_i, \theta_i) - u_i(b_i', \theta_i) \\ &= \left(H(b_i)F(\phi(b_i))^{n-1} - H(b_i')F(\phi(b_i'))^{n-1}\right)\theta_i \\ &\quad - \left(H(b_i)F(\phi(b_i))^{n-1}b_i - H(b_i')F(\phi(b_i'))^{n-1}b_i'\right) \\ &\quad -\delta\int_{b_i'}^{b_i} F(\phi(b_s))^{n-1}(\underbrace{Q_{\phi(b_s)}(\theta_i)}_{=1 \text{ by } (24)}\theta_i - \underbrace{P_{\phi(b_s)}(\theta_i)}_{=P_{\theta_i}(\theta_i) \text{ by } (24)})\mathrm{d}H(b_s) \\ &= -H(b_i)l_1(b_i, b_i')(b_i - b_i') + l_2(b_i, b_i')(H(b_i) - H(b_i')), \end{aligned} \tag{70}$$

where

$$\begin{aligned} l_1(b_i, b_i') &= F(\phi(b_i))^{n-1} - \frac{F(\phi(b_i))^{n-1} - F(\phi(b_i'))^{n-1}}{b_i - b_i'}(\phi(b_i) - b_i'), \\ l_2(b_i, b_i') &= F(\phi(b_i))^{n-1}\left(\phi(b_i)(1-\delta) + \delta P_{\phi(b_i)}(\phi(b_i)) - b_i'\right). \end{aligned}$$

Again fix some $c_i \in (0, b^*(1)]$.

First consider the case $\Xi(c_i) = 0$. Then, (70) implies the existence of a neighborhood $N(c_i)$ of c_i such that

$$\forall b_i \in N(c_i), b_i' \in N(c_i), b_i' < b_i : \ l_1(b_i, b_i') \leq 0.$$

Together with (68) this implies, for all $b_i \in N(c_i)$, that $F(\phi)^{n-1}$ is differentiable at b_i, and

$$\lim_{b_i' \to b_i} \frac{F(\phi(b_i))^{n-1} - F(\phi(b_i'))^{n-1}}{b_i - b_i'} = \frac{F(\phi(b_i))^{n-1}}{\phi(b_i) - b_i}.$$

Hence, ϕ is differentiable at $b_i \in N(c_i)$, and,

$$\forall c_i, \ \Xi(c_i) = 0, \ b_i \in N(c_i) : \ \phi'(b_i) = \frac{1}{N(\phi(b_i), b_i)}. \tag{71}$$

Note also,

$$\begin{aligned} &\forall c_i, \ \delta M(\phi(c_i)) = c_i, \ \phi \text{ differentiable at } c_i : \\ &\qquad \delta M'(\phi(c_i))\phi'(c_i) = \lim_{b_i \searrow c_i} \frac{\delta M(\phi(b_i)) - \delta M(\phi(c_i))}{b_i - c_i} \leq 1, \end{aligned} \tag{72}$$

because $\delta M(\phi(b_i)) \leq b_i$ for all b_i, by Lemma 6.

Now consider the case $\Xi(c_i) = 1$. By (64),

$$\forall c_i, \ \Xi(c_i) = 1, \ \phi \text{ differentiable at } c_i : \ \delta M'(\phi(c_i))\phi'(c_i) = 1. \tag{73}$$

In summary, for all $c_i \in (0, b^*(1)]$ such that ϕ is differentiable at c_i,

$$\phi'(c_i) = \begin{cases} \frac{1}{N(\phi(c_i), c_i)} & \text{if } \delta M(\phi(c_i)) < c_i, \\ \min\{\frac{1}{N(\phi(c_i), c_i)}, \frac{1}{\delta M'(\phi(c_i))}\} & \text{if } \delta M(\phi(c_i)) = c_i, \end{cases} \tag{74}$$

by (69), (71), (72), and (73).

Next we show that ϕ and b^* are Lipschitz continuous. By (68), $F(\phi)^{n-1}$ is locally Lipschitz in $(0, 1]$, implying that ϕ is locally Lipschitz in $(0, 1]$. Hence,

$$\forall c_i, d_i \in (0, b^*(1)] : \phi(d_i) - \phi(c_i) = \int_{c_i}^{d_i} \phi'(b_i) \mathrm{d}b_i, \tag{75}$$

where ϕ' is given almost everywhere by the right-hand side in (74).

The mapping $b_i \mapsto N(\phi(b_i), b_i)$ is bounded above on $[0, b^*(1)]$ because $N(\phi(b_i), b_i) \leq (n-1)f(\phi(b_i))\phi(b_i)/F(\phi(b_i)) \to n - 1$ as $b_i \to 0$. Moreover,

M' is bounded above on $[0,1]$ by Lemma 3. Therefore, ϕ' is bounded below by a positive number. Hence, b^* is Lipschitz on $[0,1]$ by (75).

To see that ϕ is Lipschitz in a neighborhood of 0, we first prove that

$$\exists \hat{b} \in (0,1) \text{ a.e. } b \in (0,\hat{b}) : \ \phi'(b) = \frac{1}{N(\phi(b), b)}. \tag{76}$$

If not, there exists a sequence (b^m) with $b^m > 0$, $b^m \to 0$, $b^m = \delta M(\phi(b^m))$, and

$$\frac{1}{\delta M'(\phi(b^m))} < \frac{1}{N(\phi(b^m), b^m)}. \tag{77}$$

By (27),

$$\frac{b_m}{\phi(b_m)} = \frac{\delta M(\phi(b_m))}{\phi(b_m)} \to \delta M'(0) < \delta \frac{n-1}{n}.$$

Therefore,

$$\lim_{m\to\infty} N(\phi(b^m), b^m) = (n-1)(1 - \delta M'(0)) > \delta M'(0),$$

contradicting (77).

From (76) it follows that $\phi(b) = \phi^{\mathrm{I}}(b)$ for all $b \in [0, \hat{b}]$, where ϕ^{I} denotes the inverse of the no-resale equilibrium bid function b^{I}. It is well-known that ϕ^{I} is Lipschitz in a neighborhood of 0.

Because b^* is the inverse of ϕ, (60) follows from (74).

As for uniqueness of b^*, let b^* and c^* be two Lipschitz continuous functions that satisfy (60). By definition of K, if $b^*(\theta) > c^*(\theta)$ then $K(\theta, b^*(\theta)) < K(\theta, c^*(\theta))$, for all $\theta \in (0,1)$. Therefore, $b^* \le c^*$. The same argument shows $b^* \ge c^*$.

To show that H satisfies (61), consider any $b \in (0, b^*(1)]$. If $\Xi[b] = 0$ and $b = \delta M(\phi(b))$, we have $h(b) = 0$ and, by (71),

$$\frac{R(b)}{F(\phi(b))} = 1 - \phi'(b) N(\phi(b), b) = 0.$$

Hence, $R(b) = 0 = h(b) = H(b)L(b)$, as was to be shown.

If $\Xi[b] = 0$ and $b > \delta M(\phi(b))$, we have $h(b) = 0 = L(b) = H(b)L(b)$ by definition of L.

Now consider the case $\Xi[b] = 1$. The function r^* is continuous. Hence, using Lemma 2, the function $\hat{b} \mapsto P_{\phi(\hat{b})}(\phi(\hat{b}))$ is continuous on $(0,1]$. Therefore,

$$\lim_{b' \nearrow b} k_2(b, b') = S(b) = \lim_{c \searrow b} l_2(c, b) \qquad \forall\, b \in (0, b^*(1)]. \tag{78}$$

Because ϕ is differentiable almost everywhere,

$$\lim_{b' \nearrow b} k_1(b,b') = R(b) = \lim_{c \searrow b} l_1(c,b) \qquad \text{a.e. } b \in (0, b^*(1)]. \tag{79}$$

From (63), (70), (78), and (79) we get (61).

It remains to be shown that H is unique. Firstly, the boundary condition $H(b^*(1)) = 1$ holds by optimality of the speculator's bid. Secondly, for any $\epsilon > 0$, L is bounded above on $[\epsilon, 1]$ by (3). Therefore, applying the Picard-Lindelöf Theorem to the differential equation (61) implies that H is unique on $[\epsilon, 1]$. As a distribution function, H is right-continuous at 0 and thus uniquely determined at 0 as well. *QED*

LEMMA 11 *Let* $\underline{\theta} \in (0,1)$ *and consider a bounded and continuous function* $\hat{N} : [\underline{\theta}, 1] \times \mathbb{R} \to \mathbb{R}$. *Define*

$$\hat{K}(\theta, b) = \begin{cases} \hat{N}(\theta, b) & \text{if } b > 0, \\ \max\left\{0, \hat{N}(\theta, b)\right\} & \text{if } b \leq 0. \end{cases} \tag{80}$$

Then the initial value problem

$$\forall \theta \in [\underline{\theta}, 1) : \; f'_+(\theta) = \hat{K}(\theta, f(\theta)), \quad f(\underline{\theta}) = 0, \tag{81}$$

where f'_+ *denotes the derivative from the right, has a Lipschitz continuous solution* f *on* $[\underline{\theta}, 1]$.

PROOF. Following Aubin-Cellina (1984, p. 101), define

$$\overline{K}(\theta, b) = \bigcap_{\epsilon > 0} \overline{\text{co}}\; \hat{K}(B_\epsilon(\theta, b)),$$

where $B_\epsilon(\theta, b)$ denotes the ϵ-ball around (θ, b) according to any norm in $\mathbb{R}^2$, and $\overline{\text{co}}$ denotes the closed-convex-hull operator. Then, $\overline{K}$ is an upper hemi-continuous (or, in Aubin-Cellina's (1984) terminology, upper semi-continuous) correspondence, and its values are closed and convex. Moreover, $\overline{K}$ is globally bounded. Therefore, the differential inclusion problem

$$f'(\theta) \in \overline{K}(\theta, f(\theta)) \text{ a.e. } \theta \in (\underline{\theta}, 1), \quad f(\underline{\theta}) = 0, \tag{82}$$

has an absolutely continuous solution f (see Aubin-Cellina, 1984, Theorem 4, p.101). Note that, by (82), for a.e. $\theta \in [\underline{\theta}, 1]$: if $f(\theta) < 0$ then $f'(\theta) \geq 0$. Hence,

$$\forall \theta \in [\underline{\theta}, 1] : \; f(\theta) \geq 0. \tag{83}$$

We will now show that f satisfies

$$f'(\theta) = \hat{K}(\theta, f(\theta)) \text{ a.e. } \theta \in (\underline{\theta}, 1). \tag{84}$$

First, consider $\theta \in [\underline{\theta}, 1]$ with $f(\theta) > 0$, or $f(\theta) = 0$ and $\hat{N}(\theta, 0) \geq 0$. Then, $\overline{K}(\theta, f(\theta) = \hat{K}(\theta, f(\theta))$.

Second, consider $\theta \in [\underline{\theta}, 1]$ with $f(\theta) = 0$ and $\hat{N}(\theta, 0) < 0$. For a.e. such θ, (82) implies $f'(\theta) \leq 0$. On the other hand, $f'(\theta) \geq 0$ by (83). Therefore, $f'(\theta) = 0 = \hat{K}(\theta, f(\theta))$, completing the proof of (84).

Because f is absolutely continuous,

$$\forall \theta \in [\underline{\theta}, 1] : \ f(\theta) = \int_{\underline{\theta}}^{\theta} \hat{K}(\hat{\theta}, f(\hat{\theta})) \mathrm{d}\hat{\theta}. \tag{85}$$

This together with the fact that $\hat{K}$ is bounded, implies that f is Lipschitz continuous.

To show (81), it is sufficient that, for all $\theta \in [\underline{\theta}, 1)$,

$$\Delta(\theta, \theta') \stackrel{\text{def}}{=} |\hat{K}(\theta', f(\theta')) - \hat{K}(\theta, f(\theta))| \rightarrow_{\theta' \searrow \theta} 0. \tag{86}$$

If $f(\theta) > 0$, or $f(\theta) = 0$ and $\hat{N}(\theta, 0) > 0$, we have $\hat{K}(\theta, f(\theta)) = \hat{N}(\theta, f(\theta))$ and $\hat{K}(\theta', f(\theta')) = \hat{N}(\theta', f(\theta'))$ for $\theta' > \theta$ close to θ. Thus, (86) follows from continuity of $\hat{N}$.

Now consider the case where $f(\theta) = 0$ and $\hat{N}(\theta, 0) = 0$. Then, $\hat{K}(\theta, f(\theta)) = 0$. Hence, (86) follows from

$$\Delta(\theta, \theta') \leq |\hat{N}(\theta', f(\theta'))| \rightarrow_{\theta' \searrow \theta} |\hat{N}(\theta, f(\theta))| = 0.$$

Finally, consider the case where $f(\theta) = 0$ and $\hat{N}(\theta, 0) < 0$. Then, $\hat{N}(\hat{\theta}, f(\hat{\theta})) < 0$ for all $\hat{\theta} \in [\theta, \theta']$, for all $\theta' > \theta$ close to θ. Therefore, $\hat{K}(\hat{\theta}, f(\hat{\theta})) \leq 0$ and thus $f(\theta') = 0$ by (85) and (83). Therefore, $\hat{K}(\theta', f(\theta')) = 0$, implying $\Delta(\theta, \theta') = 0$ and thus (86). *QED*

LEMMA 12 *The initial value problem*

$$\forall \theta \in (0, 1) : \ {b^*}'_+(\theta) = K(\theta, b^*(\theta)), \quad b^*(0) = 0, \tag{87}$$

where ${b^*}'_+$ *denotes the derivative from the right, has a solution* b^* *on* $[0, 1]$,

$$b^* \text{ is Lipschitz continuous,} \tag{88}$$

$$\forall \theta \in (0, 1] : \ b^*(\theta) < \theta, \tag{89}$$

$$b^* \text{ is strictly increasing,} \tag{90}$$

$$b^* \geq b^I, \tag{91}$$

$$b^* \geq \delta M, \tag{92}$$

$$\phi \stackrel{\text{def}}{=} b^{*-1} \text{ is Lipschitz continuous.} \tag{93}$$

PROOF. Define $\underline{\theta}$ to be the smallest $\theta \in [0,1]$ with $b^{\mathrm{I}}(\theta) = \delta M(\theta)$ (let $\underline{\theta} = 1$ if no such θ exists). Because $b^{\mathrm{I}\prime}(0) = (n-1)/n > \delta M'(0)$ by (27), we have $\underline{\theta} > 0$. Defining $b^*(\theta) = b^{\mathrm{I}}(\theta)$ for $\theta \in [0, \underline{\theta}]$, we have $b^{\mathrm{I}\prime}(\theta) = N(\theta, b^{\mathrm{I}}(\theta))$ and $b^{\mathrm{I}}(0) = 0$. Thus, (87) is satisfied, and it is standard from the theory of first-price auctions without resale that b^* has the desired properties (88) to (93) on $[0, \underline{\theta}]$.

Define $\hat{N} : [\underline{\theta}, 1] \times \mathbb{R} \to \mathbb{R}$ by

$$\hat{N}(\theta, b) = \begin{cases} N(\theta, b + \delta M(\theta)) - \delta M'(\theta), & \text{if } b \in [0, \theta - \delta M(\theta)], \\ N(\theta, \theta) - \delta M'(\theta), & \text{if } b > \theta - \delta M(\theta), \\ N(\theta, \delta M(\theta)) - \delta M'(\theta), & \text{if } b < 0, \end{cases}$$

and define $\hat{K}$ as in (80). Then, Lemma 11 implies that there exists a Lipschitz continuous f such that (81) holds. By definition of $\hat{K}$,

$$\forall \theta \in [\underline{\theta}, 1] : \; f(\theta) \le \theta - \delta M(\theta), \; f(\theta) \ge 0.$$

Therefore, $b^*(\theta) \stackrel{\text{def}}{=} f(\theta) + \delta M(\theta)$, $\theta \in [\underline{\theta}, 1]$, yields a Lipschitz continuous solution for (87).

To prove (89), suppose that $b^*(\theta) \ge \theta$ for some $\theta > \underline{\theta}$. Let $\hat{\theta}$ be minimal with that property. Then, $b^*(\hat{\theta}) = \hat{\theta} > \delta M(\hat{\theta})$ and thus b^* is differentiable at $\hat{\theta}$ and $b^{*\prime}(\hat{\theta}) = 0$. Hence, $b^*(\theta) > \theta$ for $\theta < \hat{\theta}$, when θ is close to $\hat{\theta}$, a contradiction.

For all $\theta \ge \underline{\theta}$,

$$b^{*\prime}_{+}(\theta) = K(\theta, b^*(\theta)) \ge N(\theta, b^*(\theta)) > 0$$

by (89). This implies (90). It also implies that ϕ is Lipschitz on $[b^*(\underline{\theta}), b^*(1)]$. I.e., (93) follows.

Inequality (91) follows from $b^{\mathrm{I}\prime}(\theta) = N(\theta, b^{\mathrm{I}}(\theta))$, for all $\theta \in (0,1)$. Inequality (92) is immediate from the definition of K. *QED*

PROOF OF PROPOSITION 5

Uniqueness of equilibrium follows from Lemma 10.

To prove equilibrium existence, define b^* as a solution to (87) and define $\phi = b^{*-1}$. Inequality then (17) follows from (91) and (92). The next step is to construct H. For any given $\epsilon > 0$, (3) implies that

$$\forall b \in [\epsilon, b^*(1)] : \quad \text{if } b = \delta M(\phi(b)) \quad \text{then } \delta P_{\phi(b)}(\phi(b)) \ge b + \delta \eta(\phi(\epsilon)). \tag{94}$$

By definition of K, for all $b \in (0, b^*(1)]$,

$$b^{*\prime}_{+}(\phi(b)) \geq N(\phi(b), b^*(\phi(b))),$$

implying

$$R(b) \geq 0 \text{ a.e. } b \in (0, b^*(1)], , \tag{95}$$

where R is defined as in (58). Define L as in (59). From (95) and (94) it follows that L is a nonnegative function and is bounded above on any given interval $[\epsilon, 1]$ with $\epsilon > 0$. For all $b_s \in [0, b^*(1)]$, define

$$H(b_s) = e^{-\int_{b_s}^{b^*(1)} L(b) \ \mathrm{d}b}. \tag{96}$$

For every $\epsilon > 0$, the function H is Lipschitz on $[\epsilon, 1]$ because L is bounded above on $[\epsilon, 1]$. Moreover, the limit $H(0) = \lim_{b_s \searrow 0} H(b_s)$ exists because L is non-negative. Also because L is non-negative, H is weakly increasing. Therefore, H is a distribution function. Differentiating (96) yields (61).

Having constructed (b^*, H), define posterior beliefs and resale mechanisms by (2) and (5). We have to show that (8) and (9) hold.

To show (8), consider any deviating bid $b_i \in (b^*(\theta_i), b^*(1)]$ of a buyer with type $\theta_i \in (0, 1]$, and suppose the buyer offers the good for resale upon winning. (The other cases, where the buyer consumes the good upon winning or deviates to a bid $b_i \in (0, b^*(\theta_i))$, are similar. By continuity of u_i, type $\theta_i = 0$ has no incentive to deviate either.)

We obtain an upper bound $\overline{u}_i(b_i, \theta_i)$ for buyer i's payoff by assuming she gets the entire surplus that is available in the resale market,

$$\begin{aligned}\overline{u}_i(b_i, \theta_i) \quad = \quad & H(b_i)\left(F(\phi(b_i))^{n-1}(\delta\theta_i - b_i) + \delta \int_{\theta_i}^{\phi(b_i)} (\hat{\theta} - \theta_i)\mathrm{d}F^{n-1}(\hat{\theta})\right) \\ & +\delta \int_{b_i}^{\infty} F(\phi(b_s))^{n-1} U_{\phi(b_s)}(\theta_i)\mathrm{d}H(b_s),\end{aligned}$$

where

$$U_{\phi(b_s)}(\theta_i) \stackrel{\text{def}}{=} \theta_i Q_{\phi(b_s)}(\theta_i) - P_{\phi(b_s)}(\theta_i).$$

Therefore, for all b_i where $\overline{u}_i$ is differentiable with respect to b_i,

$$\begin{aligned}
&\frac{\partial \overline{u}_i}{\partial b_i}(b_i,\theta_i)\\
&= H(b_i)F^{n-2}(\phi(b_i))\left((n-1)f(\phi(b_i))\phi'(b_i)(\delta\phi(b_i)-b_i)-F(\phi(b_i))\right)\\
&+h(b_i)F^{n-1}(\phi(b_i))\left(-\delta U_{\phi(b_i)}(\theta_i)+\delta\theta_i-b_i+\delta\int_{\theta_i}^{\phi(b_i)}\frac{\hat{\theta}-\theta_i}{F^{n-1}(\phi(b_i))}\mathrm{d}F^{n-1}(\hat{\theta})\right)\\
&\overset{(61)}{=} H(b_i)F^{n-2}(\phi(b_i))\Bigg[(n-1)f(\phi(b_i))\phi'(b_i)(\delta\phi(b_i)-b_i)-F(\phi(b_i))\\
&+L(b_i)F(\phi(b_i))\left(-\delta U_{\phi(b_i)}(\theta_i)+\delta\theta_i-b_i+\delta\int_{\theta_i}^{\phi(b_i)}\frac{\hat{\theta}-\theta_i}{F^{n-1}(\phi(b_i))}\mathrm{d}F^{n-1}(\hat{\theta})\right)\Bigg].
\end{aligned}$$

This together with (95) implies

$$\text{if } L(b_i)=0 \text{ then } \frac{\partial \overline{u}_i}{\partial b_i}(b_i,\theta_i)\le 0. \tag{97}$$

From now on suppose that $L(b_i)>0$ and hence $b_i=\delta M(\phi(b_i))$. Plugging in $L(b_i)$ above yields

$$\begin{aligned}
&\frac{\partial \overline{u}_i}{\partial b_i}(b_i,\theta_i)\\
&\le H(b_i)F^{n-2}(\phi(b_i))\overbrace{\left((n-1)f(\phi(b_i))\phi'(b_i)(\phi(b_i)-b_i)-F(\phi(b_i))\right)}^{\le 0 \text{ by } (95)}\\
&\quad\cdot\left(1-\frac{-\delta U_{\phi(b_i)}(\theta_i)+\delta\theta_i-b_i+\delta\int_{\theta_i}^{\phi(b_i)}\frac{\hat{\theta}-\theta_i}{F^{n-1}(\phi(b_i))}\mathrm{d}F^{n-1}(\hat{\theta})}{\phi(b_i)-b_i-\delta(\phi(b_i)-P_{\phi(b_i)}(\phi(b_i)))}\right).
\end{aligned}$$

This implies

$$\text{if } L(b_i)>0 \text{ then } \frac{\partial \overline{u}_i}{\partial b_i}(b_i,\theta_i)\le 0, \tag{98}$$

because

$$
\begin{aligned}
&\phi(b_i) - b_i - \delta(\phi(b_i) - P_{\phi(b_i)}(\phi(b_i))) \\
&\qquad - \left(-\delta U_{\phi(b_i)}(\theta_i) + \delta\theta_i - b_i + \delta \int_{\theta_i}^{\phi(b_i)} \frac{\hat{\theta} - \theta_i}{F^{n-1}(\phi(b_i))} \mathrm{d}F^{n-1}(\hat{\theta}) \right) \\
&= \phi(b_i) - \delta\theta_i - \delta \left(U_{\phi(b_i)}(\phi(b_i)) - U_{\phi(b_i)}(\theta_i) \right) - \delta \int_{\theta_i}^{\phi(b_i)} \frac{\hat{\theta} - \theta_i}{F^{n-1}(\phi(b_i))} \mathrm{d}F^{n-1}(\hat{\theta}) \\
&= \phi(b_i) - \delta\theta_i - \delta \int_{\theta_i}^{\phi(b_i)} \frac{F^{n-1}(\hat{\theta})}{F^{n-1}(\phi(b_i))} \mathrm{d}\hat{\theta} - \delta \int_{\theta_i}^{\phi(b_i)} \frac{\hat{\theta} - \theta_i}{F^{n-1}(\phi(b_i))} \mathrm{d}F^{n-1}(\hat{\theta}) \\
&= \phi(b_i) - \delta\theta_i - \delta\, \hat{\theta} \frac{F^{n-1}(\hat{\theta})}{F^{n-1}(\phi(b_i))} \Bigg|_{\theta_i}^{\phi(b_i)} + \delta\theta_i \frac{F^{n-1}(\phi(b_i)) - F^{n-1}(\theta_i)}{F^{n-1}(\phi(b_i))} \\
&= (1 - \delta)\phi(b_i) \geq 0.
\end{aligned}
$$

Because ϕ and H are locally Lipschitz continuous on $(0, b^*(1)]$, the mapping $b_i \mapsto \overline{u}_i(b_i, \theta_i)$ has the same property and can thus be written as the integral over its derivative. Therefore, using $u_i(b^*(\theta_i), \theta_i) = \overline{u}_i(b^*(\theta_i), \theta_i)$, (97) and (98),

$$
\begin{aligned}
u_i(b_i, \theta_i) - u_i(b^*(\theta_i), \theta_i) \quad &\leq \quad \overline{u}_i(b_i, \theta_i) - \overline{u}_i(b^*(\theta_i), \theta_i) \\
&= \int_{b^*(\theta_i)}^{b_i} \frac{\partial \overline{u}_i}{\partial b_i}(b, \theta_i) \mathrm{d}b \ \leq 0.
\end{aligned}
$$

Hence, no type $\theta_i > 0$ has an incentive to deviate, which completes the proof of (8).

To complete the equilibrium existence proof, we have to show (9). From (17) it follows that $b_s \geq \delta M(\phi(b_s))$, and thus $u_s(b_s) \leq 0$, for all $b_s \in (0, b^*(1)]$. It remains to be shown that

$$
\Pr[u_s(\tilde{b}_s) < 0] = 0.
$$

Consider the event $u_s(\tilde{b}_s) < 0$. Then, $b^*(\phi(\tilde{b}_s)) > \delta M(\phi(\tilde{b}_s))$ and thus $L(\tilde{b}_s) = 0$. Using (61), the probability of that event is

$$
\int_{(0, b^*(1)]} \mathbf{1}_{L(b_s)=0} h(b_s) \mathrm{d}b_s = \int_{(0, b^*(1)]} \mathbf{1}_{L(b_s)=0} L(b_s) H(b_s) \mathrm{d}b_s = 0,
$$

thereby completing the equilibrium existence proof.

Finally, we prove equivalence of (18) to (20). First, suppose that $H(0) = 1$. Then $L(b) = 0$ for a.e. $b \in (0, b^*(1)]$. Therefore, by definition of L,

$$b > \delta M(\phi(b)) \text{ or } R(b) = 0 \text{ a.e. } b \in (0, b^*(1)]. \tag{99}$$

By definition of K, inequality $b > \delta M(\phi(b))$ implies

$$b^{*\prime}_{+}(\phi(b)) = N(\phi(b), b^*(\phi(b))).$$

Together with (99) this yields

$$F(\phi(b)) - (n-1)f(\phi(b))\phi'(b)(\phi(b) - b) = 0 \text{ a.e. } b \in (0, b^*(1)].$$

The unique solution to this differential equation with boundary condition $\phi(0) = 0$ is the inverse of b^{I}. Therefore, $b^* = b^{\mathrm{I}}$.

Second, suppose that $b^* = b^{\mathrm{I}}$. Then, (17) implies $\delta M(\theta) \leq b^{\mathrm{I}}(\theta)$ for all $\theta \in [0, 1]$.

Third, suppose that $\delta M(\theta) \leq b^{\mathrm{I}}(\theta)$ for all $\theta \in [0, 1]$. Then, (17) implies $b^* = b^{\mathrm{I}}$ for all $\theta \in [0, 1]$. Therefore, $L(b) = 0$ for all $b \in (0, b^*(1)]$, by definition of b^{I}. Hence, $H(0) = 1$. *QED*

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