A Possibility of Protracted Output Gaps in an Economy without Any Rigidity

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Abstract

The paper examines a new possibility of output gaps that can exist in an economy without any rigidity. The diving force behind the new possibility is non-cooperative behavior of consumers. The paper shows that there is a possibility that when a fundamental shock hits an economy this non-cooperative nature makes consumers deviate from the optimal path of the representative consumer, and thus agents do not jump consumption and proceed on the deviated transition path.

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1. Introduction

The concept of output gaps is widely used and seen essential particularly in monetary economics.[1] The output gap is defined as the difference between potential and actual output, and empirically is often estimated by the difference between trend and actual GDP. In many empirical works, estimates of output gaps often show relatively long duration of continuously deviated periods, namely for several years.[2] Recent empirical works such like Haltmaier (2001), Mc Morrow and Roeger (2001), Kamada and Masuda (2000), and Scott (2000) all indicate the possibility of relatively long duration of output gaps as well as many of past empirical researches.

There are basically two completely different views about output gaps.[3] One view stresses importance of rigidities or frictions in one or more markets, which delay adjustments particularly of prices or wages. The other view is represented by the early version of real business cycle models and asserts that, because most of shocks are productivity shocks, fluctuations of GDP are basically identical with fluctuations of potential GDP and thus estimated deviations of actual GDP from its trend should not be interpreted as output gaps.

The objective of the paper is to pursue a new completely different view such that even if there is no rigidity, protracted output gaps are theoretically possible. One reason to pursue the third possibility is a puzzle on the recent development of the Japanese economy. The Japanese economy has been mired in a persisting slump lasing over 10 years since 1990s, during which, continuing deflation, continuously rising unemployment rate and persisting low utilization of

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[1] See e.g. Svensson (1999).

[2] Although the definition of output gap is very clear, its estimation is very difficult mainly because it is hard to estimate the true potential output. Hence currently used several estimation methods generate very different estimates and leave wide room for different interpretations.

[3] See e.g. European Central Bank (2000).

capital have been observed.[4] These phenomena strongly suggest that the output has been far lower than its potential for long period, that is, there has been a protracted large output gap.[5] This over 10 years protracted large deviation from trend does not seem to be explained easily by some rigidities since the view based on rigidities assumes that a large deviation continues merely for a short period, namely a few years after a shock.

Without relying on rigidities, what kind of economic forces can generate protracted large output gaps?[6] The paper uncovers a new possibility that when a fundamental shock hits an economy, consumers do not proceed on the optimal path of the representative consumer during the transition to the new steady state, as a result of fully rational calculation. Proceeding on the deviated transition path from the optimal path of the representative consumer directly means protracted large output gap.

The driving force behind this strange phenomenon is non-cooperative consumers' behavior. When a fundamental shock fits an economy, all the agents are flown out of the optimal path they have proceeded. At that moment, each consumer must decide which direction to proceed hereafter. The new path is not naturally deterministic but is decided strategically based on the expected utility calculated considering other consumers' choices. Each consumer behaves for her own interest non-cooperatively considering the other consumers' strategies. One choice is to jump to the new saddle path of the representative consumer, however, it is not clear that the

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4 There are several estimates of the output gap in Japan. See e.g. Kamada and Masuda (2000) or Haltmaier (2001).

5 According to the reduced form Phillips curve, continuing deflation is a result of either monetary policies or a protracted large output gap. Since nominal short-term interest rate was nearly zero since mid-90s in Japan, theoretically it may be hard to blame only monetary policy as the cause of continuing deflation. If the monetary policy has had little influence on deflation in Japan, a continuing large output gap would be the only factor that makes deflation persist in Japan.

6 One simple possibility is that an economy unfortunately faced several negative shocks consecutively over 10 years. This explanation implies that a long persisting slump of the economy is simply a result of "unlucky." However, the occurrence of consecutive negative shocks in over 10 years may be too rare to be believed *a priori*.

path is also optimal for each consumer, since each consumer is not sure whether the other consumers will jump to that path or not. Each consumer needs to calculate her expected utility and judge whether a jump to the new optimal path of the representative consumer is also optimal for her or not.

For example, consider the case of an upward time preference shift depicted in Figure 1. To come back to the optimal path of the representative consumer, consumers must jump up their consumption at the time of the shock from the point of the old steady state to the point Z and afterwards they must gradually reduce their consumption to the much lower new steady state consumption. This zigzag course of the optimal path of the representative consumer appears intuitively unnatural. In actuality consumers may reduce their consumption from the old steady state to the new steady state straightly. Intuitively it is expected that all the consumers will chose the strategy of straight reduction of their consumption, but there remains a question whether this deviation from the optimal path of the representative consumer is optimal for each consumer. One of the motifs of the paper is to probe this question that may be answered intuitively yes but be thought theoretically problematic.

The paper shows that if the prior probability of deviation of most consumers when a shock hits an economy exceeds some critical point, all the consumers expect that the deviation gives them higher expected utilities and then they deviate. The reason why the expected utility of a consumer who tries to deviate is higher in case that most of the consumers are expected to deviate is that deviated consumers can reach the new steady state in a finite period.

The new possibility of output gaps in the paper has two important advantages; 1) it is not necessary to present a micro foundation for a difficult question why price adjustments are far slower than quantity adjustments,[7] and 2) it is not necessary to solely depend on productivity shocks as the early version of real business cycles model that has well-known shortcomings such as the weak internal-propagation mechanism that needs large and persisting productivity

[7] As for criticisms to the new Keynesian theory, see e.g. Mankiw (2000).

shocks or necessity of unimaginably frequent negative productivity shocks.[8]

The paper is organized as follows. In section 2, it is proved that there is a possibility that deviated transition paths that are different from the optimal path for the representative consumer are rationally chosen. Then the relation between the scale of shocks and deviated transition paths, as well as the characteristics of the optimal deviated transition path, is examined. In section 3, some concluding remarks are offered in section 3.

2. A Mechanism of Protracted Output Gaps in a Smooth Environment

2.1 An intuitive explanation of output gaps in an economy without any rigidity

If consumers are cooperative, they will proceed on the optimal path of the representative consumer and will not deviate from the path. However, it cannot be imagined that all the consumers are coordinated to behave as a person. Each consumer behaves for her own interest considering the other consumers' strategies. If consumers are non-cooperative, the story will not be so simple. The optimal choice of consumption for a consumer cannot be determined by only her own independent intention but will change according to the other consumers' choices. Each consumer must act strategically. If a consumer calculates that the expected utility when she deviated from the optimal path of the representative consumer is higher than that when she proceeds on the optimal path, she will deviate.

If most of the consumers are on the optimal path of the representative consumer, the number of consumers who dare to deviate will be expected to be small. In this situation, the expected utility of the consumer who tries to deviate may be smaller than that when she proceeds on the optimal path. Hence, all the consumers will proceed on the optimal path of the representative consumer even though they are non-cooperative.

However, consider a situation that a fundamental shock hits an economy and its steady state is changed. At the time of the shock, all the consumers are flown out of the optimal path they

[8] As for criticisms to real business cycle models, see e.g. King and Rebelo (2000) or Gali (1999).

have proceeded and must decide the direction in which to proceed hereafter while wandering far from the new optimal path of the representative consumer. A jump of consumption to the new saddle path is the new optimal path for the representative consumer. Nevertheless, this path is not naturally the new optimal path for each non-cooperative consumer. Each consumer's new paths will not be determined only by her own independent intention but will change according to the choices of other consumers. Consumers must act strategically. Since all the consumers are not on the optimal path just after the shock, the number of consumers who dare to deviate will be much larger compared with that when they are all on the optimal path. It can be shown that if the prior expected probability of deviation of other consumers exceeds a critical point, the expected utility of a consumer who tries to deviate is higher than that when she does not deviate. The same calculation will be done by other consumers, thus all the consumers expect that their expected utilities are higher when they deviate than when they do not deviate. Facing these calculations of consumers, firms expect that most consumers will deviate and thus firms' capital must be adjusted corresponding to the deviation of consumers—that is, unused excessive capital must be destroyed. As a result, all the consumers and firms expect that all the consumers and firms deviate from the optimal path of the representative consumers, and they behave based on this expectation and therefore in actually they deviate.

The reason why the expected utility of a consumer who dares to deviate is higher in case that many of the consumers are expected to deviate is that deviated consumers can reach the new steady state in a finite period. The saddle path of the representative consumer approaches the new steady state as the limit of its consumption steam but never reach the steady state. This difference-whether consumers reach the steady state in a finite period or not- is the most crucial factor for the deviation. Imagine that all the consumers deviate and reach the new steady state in a finite period although capital is not adjusted fully yet and thus is excessive. It may be intuitively agreed that the most probable way for firms is to merely destroy the excessive capital that generates merely maintenance costs. If there is a possibility of partial destruction of capital, the optimal path of the representative consumer is not naturally the optimal path for each consumer anymore.

2.2 The basic framework

(1) The model

It is assumed that all the households (consumers) are identical and infinitely living, and each of them maximizes the expected utility

$$E\int_0^{\infty}(1+\theta)^{-t}u(c_t)\,dt\,, \tag{1}$$

where E is the expectations operator, c_t is consumption per capita at time t, $u(\cdot)$ is utility function, and θ (> 0) is the rate of time preference.[9] It is assumed that $u' > 0$, $u'' < 0$, $u'(0) = \infty$. The representative firm is assumed to have a production function of $y = f(k)$, where k is labor-capital ratio and is described on a per-capita basis. Technological progress and depreciation of capital are not assumed. Given that $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$, $f'(0) = \infty$, $f'(\infty) = 0$ and labor supply increases at a constant rate of n, the growth rate of k is:

$$\frac{dk_t}{dt} = f(k_t) - c_t - nk_t\,. \tag{2}$$

c_t and k_t are continuous and differentiable, and u and f are continuous functions. It is assumed that $n = 0$.

The rational expectation is assumed, and markets are complete and neither rigidity nor productivity shock exists in the model. Hence, the economy is perfectly smoothly adjusted after shocks.

[9] θ is assumed to be small as usually supposed and thus approximately $\ln(1+\theta) = \theta$.

(2) Shocks

Any fundamental shock that moves the existing steady state may cause deviated transition paths. However, in the paper particularly a shock on time preference is examined. The reason why this shock is chosen is that by using this shock it is easy to explain the mechanism of deviation because of a unique nature of the shock such that when an upward time preference shock occurs, consumption must jump up to restore equilibrium although consumption at the new steady state must be much lower than that before the shock. The effects of an upward time preference shift are depicted in Figure 1. When an economy is at the steady state and then an upward shift of time preference rate that moves the vertical line $\frac{dc_t}{dt} = 0$ to a lower labor-capital ratio occurs, the consumption of the representative consumer needs to jump immediately from the old steady state to the point Z in order to restore equilibrium, and then proceeds on the new saddle path to the much lower new steady state. This zigzag course of the optimal path of the representative consumer appears unnatural intuitively, and in actuality consumers may reduce their consumption from the old steady state to the new steady state straightly.

Since the era of Böhm-Bawerk (1889) and Fisher (1930), time preference has been naturally supposed and observed to be time-variable although the rate of time preference in many economic analyses has been assumed to be constant for the purpose of simplicity and tractability since Samuelson (1937). Time-variable time preference is not merely a theoretical possibility. Parkin (1988), e.g., showed that the rate of time preference was as volatile as the technology and leisure preferences in the U.S. Uzawa (1968) presents the well known endogenous time preference model and Harashima (2004) extends it to a model that has a more realistic mechanism of endogenously changing time preference rate.[10]

[10] Harashima (2004) shows a possibility that the protracted slump in Japan was caused by an 2 % upward time preference shift.

For simplicity, I concentrate on the case of the upward time preference shift in the paper. The same argument can be applied for the case of the downward time preference shift.[11]

(3) Assumptions

If a deviated consumption path is chosen rationally, it must be a result of judging by individual consumers that the expected utility in the case of deviation is higher than that in the case of non-deviation. Hence it is necessary to compare the expected utilities in both cases to verify the possibility of a deviation. To simplify this comparison, it is assumed that there are two options concerning the jump of consumption in a situation that the rate of time preference shifts upward when all the consumers have been at the steady state. One is "*J*" where consumption of a consumer abruptly jumps to the new saddle path of the representative consumer and then the consumer proceeds on the new saddle path. The other is "*NJ*", where consumption of a consumer does not jump to the new saddle path of the representative consumer and then the consumer gradually reduces her consumption from the level at the old steady state to that at the new one. Those are illustrated in Figure 1. If "*J*", that is "jump" of consumption, is chosen, when the rate of time preference rises, consumption immediately jumps to point *Z* and after that the consumer proceeds on the new saddle path of the representative consumer. On the other hand, if "*NJ*", that is the "non-jump", is chosen, the consumer does not jump to *Z* but directly approaches the new steady state proceeding on the bold dashed line. The difference in consumption between the two options in each period t is b_t (> 0). Hence b_0 is difference of consumption between *Z* and the old steady state. It is assumed

[11] In the case of the downward time preference shift, if consumption does not jump, capital accumulation becomes somewhat difficult, which means that, for example, some kind of hoarding may be necessary. However, the effect of hoarding may be limited, and economic expansion will be restrained by its limitation. Hence, the business cycle may have an asymmetric feature between recovery and recession, that is, relatively slow progress during recovery in response to a downward time preference shift and a relatively sharp decrease during recession in response to an upward time preference shift.

that b_t diminishes gradually and continuously and at time s b_t becomes zero, that is, any consumer choosing *NJ* makes a very small jump to the new steady state consumption.

Assumptions:

(A1) The deviated transition path of consumption c_t after an upward shift of time preference rate – that is *NJ* – is continuous and differentiable and $\frac{dc_t}{dt} < 0$ if $0 \leq t < s$.

(A2) $\bar{c} < c_t < \hat{c}_t$ if $0 \leq t < s$ and $c_t = \bar{c}$ if $s \leq t$, where $\hat{c}_t$ is the consumption on the new saddle path of the representative consumer and $\bar{c}$ is the new steady state consumption.

(A3) $b_t = \hat{c}_t - c_t > 0$ if $0 \leq t < s$ and $b_t = 0$ if $s \leq t$.

It should be noted that deviation b_t in the case of *NJ* does not simply mean the existence of continuous disequilibrium. b_t that was generated in period t will be eliminated quickly, for example, in a few quarters, by namely simply destroying excess capital by firms, and equilibrium will be recovered.[12] However, in the next period t + 1, a similar deviation from equilibrium b_{t+1} will occur since agents are not on the saddle path of the representative consumer. That is, each b_t does not mean the existence of a continuous disequilibrium but means successive generation of the same kind of temporary deviation from equilibrium and its successive elimination. Hence, the economy may be regarded basically as situating at equilibrium. However, successive deviations from equilibrium may make it appear as if there exists a continuous disequilibrium.

The paper examines a situation that consumers are not cooperative. For the convenience of analysis, it is assumed that, when a consumer chooses an option that is different from those of the other consumers, the difference of capital accumulation resulting from difference of

[12] Destroying capital may be achieved in practice by bankruptcy, renouncing credit and so on.

consumption b_t before time s between the consumer's and the other consumers' consumptions is reflected in the consumption after time s. That is, the return to this difference of capital accumulation (interest rate times difference of capital accumulation) a (> 0) is added to (or subtracted from) her consumption in each period after the time s.[13]

The expected utility is given in this situation by combining the cases of acting alone (*Jalone*: jumping alone or *NJalone*: non-jumping alone) and acting together (*Jtogether*: jumping together with other consumers or *NJtogether*: non-jumping together with other consumers). *Jalone* means that a consumer's consumption jumps while the other consumers' do not, and *NJalone* means that a consumer's consumption does not jump while the other consumers' do. *Jtogether* means that all consumers' consumptions jump in the same way, and *NJtogether* means none of the consumptions jump.

With p ($0 \leq p \leq 1$) that is subjective probability that the other consumers jump, expected utilities in cases of J and NJ are respectively, $E(J) = pE(Jtogether) + (1 - p)E(Jalone)$ and $E(NJ) = pE(NJalone) + (1 - p)E(NJtogether)$. Figure 2 shows an image of a deviated transition path.

In this situation, capital is assumed to be adjusted as follows.

Assumptions:

(A4) Firms adjust their capital to the new steady state level at s, if it is expected that $p = 0$ for most of the consumers.

(A5) If $E(J) - E(NJ) < 0$ for a consumer, $E(J) - E(NJ) < 0$ for all the consumers and thus $p =$

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[13] This assumption means that, if a consumer chooses non-jump while the others choose jump, she accumulates more savings than the others', and after time s, instead of choosing to consume it at once, she chooses to receive interest income from this extra saving and consumes the interest during each subsequent period. Also if a consumer chooses jump alone, her consumption is assumed to be smaller by a_t each period after time s in each subsequent period since her savings are smaller than the others.

0 for all the consumers. On the other hand If $E(J)$ - $E(NJ) > 0$ for a consumer, $E(J)$ - $E(NJ) > 0$ for all the consumers and thus $p = 1$ for all the consumers.

The assumption (A4) means that if all the consumers take a choice of non-jump, firms behave based on the expected realization of deviation of consumers, thus the expected excessive capital that is the difference between the existing capital and the new steady state capital is adjusted at s. Destroying the excessive capital is rational for firms because although all the consumers are at the new steady state at s, if firms continue to possess capital that is perfectly expected to be unnecessary, it will merely cost them.

The assumption (A5) indicates that a consumer expects that all the consumers expect their utility in the same manner. As a whole these two assumptions do not appear to be unnatural.

Then, the expected utility in case of J and NJ are respectively

$$E(J) = pE(Jtogether)+(1 - p)E(Jalone)$$

$$= pE\left[\int_0^s (1+\theta)^{-t} u(c_t + b_t)\, dt + \int_s^\infty (1+\theta)^{-t} u(\hat{c}_t)\, dt\right]$$
$$+ (1-p)E\left[\int_0^s (1+\theta)^{-t} u(c_t + b_t)\, dt + \int_s^\infty (1+\theta)^{-t} u(\bar{c} - \bar{a})\, dt\right], \qquad (3)$$

and

$$E(NJ) = pE(NJalone)+(1 - p)E(NJtogether)$$

$$= pE\left[\int_0^s (1+\theta)^{-t} u(c_t)\, dt + \int_s^\infty (1+\theta)^{-t} u(\hat{c}_t + a_t)\, dt\right]$$
$$+ (1-p)E\left[\int_0^s (1+\theta)^{-t} u(c_t)\, dt + \int_s^\infty (1+\theta)^{-t} u(\bar{c})\, dt\right], \qquad (4)$$

where

$$\bar{a} = \theta \int_0^s b_r \exp \int_r^s \ln\left(1 + i_q\right) dq \; dr \,, \qquad (5)$$

and

$$a_t = i_t \int_0^s b_r \exp \int_r^s \ln\left(1 + i_q\right) dq \; dr \qquad (6)$$

and $\bar{c}$ is consumption at the new steady state, i_t is interest rate and $E(Jtogether)$, $E(Jalone)$,

$E(NJalone)$ and $E(NJtogether)$ are the expected utilities in case of *Jtogether*, *Jalone*, *NJalone* and *NJtogether* respectively.[14] If the other consumers are expected to jump (this is the case of p) capital will not be adjusted at period s, thus consumption after s is $\hat{c}_t$. On the other hand if the other consumers are not expected to jump (this is the case of $1 - p$) capital will be adjusted at period s, thus consumption after s is $\bar{c}$.

In equation (3) consumption after s in the case of *Jalone* is $\bar{c} - \bar{a}$. This is because in the case of *Jalone*, the other consumers' aggregated b_t is destroyed and this destruction of capital─that is invested savings by consumers─is allocated equally to each consumer's savings including *Jalone* consumer's since destruction of capital is decided not by consumers but by firms. Hence, the *Jalone* consumer's savings is reduced by the same ratio as the other consumers, which results in the reduction of consumption of the *Jalone* consumer by $\bar{a}$ after s. Since the economy is basically at equilibrium, the interest rate equals the marginal productivity. If the other consumers are not expected to jump (this is the case of $1 - p$), after s, c_t, i_t and a_t are constant and i_s equals θ because the economy is at the new steady state where the marginal productivity is identical to the time preference. The interest rate i_q during transition changes from the old rate of time preference to the new one.

Assumptions:

(A6) $\frac{dk_t}{dt} = f(k_t) - \hat{c}_t$ if $t < s$, and $k_t = \bar{k}$ if $s \le t$ where $\bar{k}$ is the new steady state capital-labor ratio.

(A7) $i_t = f'(k_t)$. Therefore $\theta_1 \le i_t < \theta$ if $0 \le t < s$ and $i_t = \theta$ if $s \le t$ where θ_1 is the old rate of time preference and θ is the new rate of time preference.

[14] The equations (5) and (6) mean that the value of the added (or subtracted) saving by b_r will increase by compound interest between the time r to s.

(A8) k_t and thus i_t are continuous and differentiable if $0 \le t < s$.

The assumption (A6) means that unused resources b_t are eliminated by destruction of capital in each period, therefore capital is adjusted in each period by $f(k_t) - \hat{c}_t = f(k_t) - c_t - b_t$.

To analyze the mechanism of deviated transition path in detail, utility functions are specified to be usual geometric ones as follows.

Assumption:

(A9) Utility functions are constant relative risk aversion utility functions;

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \qquad \text{if } \gamma \neq 1$$

$$u(c_t) = \ln(c_t) \qquad \text{if } \gamma = 1 \tag{7}$$

where $\gamma > 0$.

(4) The timing

When a fundamental shock hits an economy, firstly, based on $\overline{p}$ that is the prior subjective probability that the other consumers jump, each consumer calculates her prior expected utility conditional on $\overline{p}$;

$$E(J, \overline{p}) - E(NJ, \overline{p})$$

$$= \overline{p}E(Jtogether) + (1-\overline{p})E(Jalone) - \{\overline{p}E(NJalone) + (1-\overline{p})E(NJtogether)\}$$

$$= \overline{p}\,\{E(Jtogether) - E(NJalone)\} + (1-\overline{p})\,\{E(Jalone) - E(NJtogether)\}$$

$$= \overline{p}E\left\{\int_0^s (1+\theta)^{-t}[u(c_t+b_t) - u(c_t)]\,dt + \int_s^\infty (1+\theta)^{-t}[u(\hat{c}_t) - u(\hat{c}_t + a_t)]\,dt\right\}$$
$$+ (1-\overline{p})\,E\left\{\int_0^s (1+\theta)^{-t}[u(c_t+b_t) - u(c_t)]\,dt + \int_s^\infty (1+\theta)^{-t}[u(\overline{c} - \overline{a}) - u(\overline{c})]\,dt\right\}. \tag{8}$$

Secondly, as a result of the above calculation, if $E(J, \overline{p}) - E(NJ, \overline{p}) > 0$, then each consumer expects that all the other consumers will jump and thus by the assumption (A5) it is expected that $p = 1$. Since it is expected that $p = 1$ for most of the consumers, by the assumption (A4), firms do not adjust capital at *s* and each consumer's expected utility becomes *E*(*J*) - *E*(*NJ*) = *E*(*Jtogether*) - *E*(*NJalone*). On the other hand if $E(J, \bar{p}) - E(NJ, \bar{p}) < 0$, then each consumer expects that all the consumers do not jump and thus by the assumption (A5) it is expected that $p = 0$ as a result. Since it is expected that $p = 0$ for most of the consumers, by the assumption (A4), firms adjust capital at *s* and each consumer's expected utility becomes *E*(*J*) - *E*(*NJ*) = *E*(*Jalone*) - *E*(*NJtogether*).

2.3 A possibility of a deviated transition path

(1) A possibility of non-jump

First, it can be shown that if consumers are cooperative and a representative consumer represents all the consumers, a jump will be chosen as the optimal path.

Proposition 1: If consumers are cooperative, that is, there is no possibility of *Jalone* and *NJalone*, the jump of consumption is the optimal path.

Proof: See Appendix.

The expected utility in the case of *J* is higher than in the case of *NJ*, since, in the case of *NJ*, the extra savings accumulated by b_t becomes excessive according to the new steady state labor-capital ratio and are not consumed or invested but simply destroyed. Hence, according to this result it is rational to choose *J*, that is, consumption jumps.

However, in reality consumers are non-cooperative, thus the opposite of the result may be possible. Firstly, a special case that the utility function is $u(c_t) = \lim_{\gamma \to 0} \frac{c_t^{1-\gamma}}{1-\gamma}$ is examined.

Lemma 1: If the utility function is $u(c_t) = \lim_{\gamma \to 0} \frac{c_t^{1-\gamma}}{1-\gamma}$, then *E*(*Jalone*) - *E*(*NJtogether*) > 0.

Proof: See Appendix:

Next, another special case of the extremely risk averse utility function is examined. Since utility functions are constant relative risk aversion utility functions, the new saddle path for the representative consumer is $\frac{d\hat{c}_t}{dt} = \frac{\hat{c}_t}{\gamma}\left[f'(k_t) - \theta\right]$, and thus as γ increases, $\hat{c}_0$ decreases to $\bar{c}$.

Lemma 2: If the utility function is $u(c_t) = \lim_{\gamma \to \infty} \frac{c_t^{1-\gamma}}{1-\gamma}$ and if $\bar{c} > \bar{a}$, then *E*(*Jalone*) - *E*(*NJtogether*) < 0.[15]

Proof: See Appendix:

Based on the above two Lemmas, a possibility of deviated transition path can be shown.

Lemma 3: If $\bar{c} > \bar{a}$, then there is γ^* such that if $\gamma = \gamma^*$ then *E*(*Jalone*) = *E*(*NJtogether*), and if $\gamma > \gamma^*$ then *E*(*Jalone*) - *E*(*NJtogether*) > 0.

Proof: See Appendix.

Proposition 2: If $\bar{c} > \bar{a}$ and $\gamma > \gamma^*$ then there is $\bar{p}^*$ such that if $\bar{p} < \bar{p}^*$ then non-jump of

[15] Given a Cobb-Douglas production function, the condition $\bar{c} > \bar{a}$ will be easily met for a several % shift of the rate of time preference and a few decades *s*.

consumption is the optimal path.

Proof: See Appendix.

This result appears amazing, because it asserts a possibility of protracted output gaps in an economy without any rigidity and a possibility of rational deviated transition paths. A rational non-jump choice means that successive deviations from equilibrium b_t occur in an economy until time s as a result of rational choices of agents. Thus, a sluggish situation with high unemployment rate and huge excess in production capacity will persist long after a huge upward time preference shift. This newly uncovered mechanism may shed new light on the view of equilibrium, or disequilibrium.

Proposition 3: If $s \to \infty$, then the jump of consumption is the optimal path.

Proof: See Appendix.

This proposition highlights importance of the finite period s. If s is set to be infinite, non-jump deviation is not optimal anymore. It implies that the finite period s is crucial for the possibility of deviation.

(2) The prior probability

The prior probability $\bar{p}$ plays a crucial role of determining deviations. However, how prior probability is formed is difficult to answer. One possibility is that it is formed based on a behavioral model induced by analyses of the past consumers' behaviors. Another possibility is that utilities have the feature of habit formation and thus $\bar{p}$ may be small by consumers' strong intention to keep consumption around the old steady state.

Third possibility is that, because *Jalone* is much riskier than *NJalone*, consumers set $\bar{p}$ very low to avert the risk. Because

$E(Jalone) - E(NJalone) =$

$$E\left[\int_0^s (1+\theta)^{-t} u(c_t + b_t)\, dt + \int_s^{\infty} (1+\theta)^{-t} u(\bar{c} - \bar{a})\, dt\right] - E\left[\int_0^s (1+\theta)^{-t} u(c_t)\, dt + \int_s^{\infty} (1+\theta)^{-t} u(\hat{c}_t + a_t)\, dt\right]$$

$$= E\left\{\int_0^s (1+\theta)^{-t} [u(c_t + b_t) - u(c_t)]\, dt + \int_s^{\infty} (1+\theta)^{-t} [u(\bar{c} - \bar{a}) - u(\hat{c}_t + a_t)]\, dt\right\}$$

$$< E\left\{\int_0^s (1+\theta)^{-t} [u(c_t + b_t) - u(c_t)]\, dt + \int_s^{\infty} (1+\theta)^{-t} [u(\bar{c} - \bar{a}) - u(\bar{c})]\, dt\right\} < 0, \qquad (9)$$

due to Lemma 3, then the case that a consumer mistakenly jumped since she thought the other consumers would jump although they in actuality did not jump is much riskier than the case that she mistakenly did not jump since she thought the other consumers would not jump although they in actuality jumped. Since consumers are risk aversive ($\gamma > 0$), then if there is no information about $\bar{p}$, a consumer will bet to the less risky latter case ($\bar{p} = 0$) rather than the riskier former case ($\bar{p} = 1$) and thus will set $\bar{p}$ very low.

If all the consumers are on the saddle path or at the steady state, the number of consumers who dare to deviate may be small and then $\bar{p}$ may be high. In this case, the best strategy of consumption for each consumer will be to stay on the optimal path of the representative consumer. However, in a situation that a fundamental shock hits an economy, consumers are flown out of and thus not on the optimal path of the representative consumer, it can not naturally be asserted that the number of consumers who try to deviate from the optimal path of the representative consumer is small. Because consumers are full of suspicion whether other consumers jump to the new saddle path or not, consumers may take a choice in the risk aversive manner described above, and thus may set $\bar{p}$ very low. Hence, when a fundamental shock hits an economy, a possibility of non-jump can not be denied naturally.

(3) The effect of the scale of shocks

A natural question may be how this possibility of deviation changes when the scale of shocks changes. In this sub-section this question is examined.

Lemma 4: If there is $\overline{p}^*$ and $\overline{p} < \overline{p}^*$, and if $i_t = \theta$ for any t, even though the utility function is $\lim_{\gamma \to 0} u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, the jump of consumption and the non-jump of consumption are indifferent.

Proof: See Appendix.

Lemma 5: If there is $\overline{p}^*$ and $\overline{p} < \overline{p}^*$, and if $i_t = \theta$ for any t, for any other utility function than $\lim_{\gamma \to 0} u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, the non-jump of consumption is the optimal path.

Proof: See Appendix.

The Lemma 5 indicates that how different i_t and θ are is an important determinant of the choice of deviation. The difference between i_t and θ will change according to the scale of shift. Since a change of the scale of shift changes streams of $\hat{c}_t$, b_t, i_t, and a, before examining the effect of scale of shocks on deviation, the relation among $\hat{c}_t$, b_t, i_t, and a, need be examined. Thereby how the equilibrium consumption changes according to changes of the rate of time preference is examined. Let $\hat{\bar{c}}(\theta)$ be the consumption at steady state corresponding to each θ. Since the model in the paper is a conventional Ramzey model, it is easily understood that $\frac{d\hat{\bar{c}}}{d\theta} < 0$ and $\frac{d^2\hat{\bar{c}}}{d\theta^2} > 0$ as shown in Panel (a) of Figure 3. Hence, if θ shifts upward from θ_1

to θ_2 and $\theta^* = \theta_1 - \theta_2 > 0$, then the function $\Psi(\theta^*) = -\int_{\theta_1}^{\theta_2} \frac{d\hat{c}}{d\theta} d\theta > 0$ has the features of $\frac{d\Omega(\theta^*)}{d\theta^*} > 0$ and $\frac{d^2\Omega(\theta^*)}{d\theta^{*2}} < 0$. On the other hand, the function $\Xi(\theta^*) = (1 + \theta_2)^{-1} = (1 + \theta_1 + \theta^*)^{-1} > 0$ has the features of $\frac{d\Xi(\theta^*)}{d\theta^*} < 0$ and $\frac{d^2\Xi(\theta^*)}{d\theta^{*2}} > 0$. Clearly, $\Xi(0) = 1$.

Assumptions:

(A10) b_t for any t ($< s$) and a, which differ for different θ*, is proportionate to $\Omega(\theta^*) = -\int_{\theta_1}^{\theta_2} \frac{d\hat{c}}{d\theta} d\theta$.

This assumption is introduced to specify how the stream of deviated consumption path c_t changes when θ* changes. The assumption, which asserts that the larger the difference between the old and new consumptions at steady states is, the larger the scales of b_t and a for any period are, does not appear to be unnatural but seems reasonable.

Remark: By the assumption (A10) and the features of $\frac{d\Omega(\theta^*)}{d\theta^*} > 0$ and $\frac{d^2\Omega(\theta^*)}{d\theta^{*2}} < 0$, for each given θ_1,

$$E\frac{d\left\{\frac{\int_0^s (1+\theta_1+\theta^*)^{-t}[u(c_t+b_t)-u(c_t)]\,dt}{\int_0^s (1+\theta_1+\theta^*)^{-t}\,dt}\right\}}{d\theta^*} > 0 \quad \text{and} \quad E\frac{d^2\left\{E\frac{\int_0^s (1+\theta_1+\theta^*)^{-t}[u(c_t+b_t)-u(c_t)]\,dt}{\int_0^s (1+\theta_1+\theta^*)^{-t}\,dt}\right\}}{d\theta^{*2}} < 0$$

and,

$$-E\frac{d\left\{\dfrac{\int_s^\infty (1+\theta_1+\theta^*)^{-t}[u(\bar{c}-\bar{a})-u(\bar{c})]\,dt}{\int_s^\infty (1+\theta_1+\theta^*)^{-t}\,dt}\right\}}{d\theta^*} > 0 \text{ and } -E\frac{d^2\left\{\dfrac{\int_s^\infty (1+\theta_1+\theta^*)^{-t}[u(\bar{c}-\bar{a})-u(\bar{c})]\,dt}{\int_s^\infty (1+\theta_1+\theta^*)^{-t}\,dt}\right\}}{d\theta^{*2}} > 0\cdot$$

Lemma 6: If there is $\bar{p}^*$ and $\bar{p} < \bar{p}^*$, for any given set of streams of c_t, and i_t, there is s^* such that if $s > s^*$ then

$$E\frac{\int_0^s (1+\theta)^{-t}[u(c_t+b_t)-u(c_t)]\,dt}{\int_0^s (1+\theta)^{-t}\,dt} < -E\frac{\int_s^\infty (1+\theta)^{-t}[u(\bar{c}-\bar{a})-u(\bar{c})]\,dt}{\int_s^\infty (1+\theta)^{-t}\,dt}\cdot$$

Proof: See Appendix.

This lemma asserts that the effect of a overrides that of b_t if $s > s^*$. Here, let

$$\Lambda = E\frac{\int_0^s (1+\theta_1+\theta^*)^{-t}[u(c_t+b_t)-u(c_t)]\,dt}{\int_0^s (1+\theta_1+\theta^*)^{-t}\,dt} \tag{10}$$

and

$$\Pi = -E\frac{\int_s^\infty (1+\theta_1+\theta^*)^{-t}[u(\bar{c}-\bar{a})-u(\bar{c})]\,dt}{\int_s^\infty (1+\theta_1+\theta^*)^{-t}\,dt}\cdot \tag{11}$$

By the Remark and the Lemma 6, the graphs of $\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\Lambda}{d\theta}\,d\theta$ and $\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\Pi}{d\theta}\,d\theta$ are those as shown in Panel (b) of Figure 3. Since $\frac{d^2\Lambda}{d\theta^{*2}} < 0$ and $\frac{d^2\Pi}{d\theta^{*2}} < 0$ by the Remark and thus

$$\frac{d^2\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\Lambda}{d\theta}\,d\theta}{d\theta^{*2}} = \frac{d\left(\left.\frac{d\Lambda}{d\theta}\right|_{\theta=\theta_1+\theta^*}\right)}{d\theta^*} = \frac{d\left(\left.\frac{d\Lambda}{d\theta^*}\right|_{\theta=\theta_1+\theta^*}\times\left.\frac{d\theta^*}{d\theta}\right|_{\theta=\theta_1+\theta^*}\right)}{d\theta^*} = \left.\frac{d^2\Lambda}{d\theta^{*2}}\right|_{\theta=\theta_1+\theta^*} < 0, \text{ and}$$

$$\frac{d^2\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\Pi}{d\theta}\,d\theta}{d\theta^{*2}} = \frac{d\left(\left.\frac{d\Pi}{d\theta}\right|_{\theta=\theta_1+\theta^*}\right)}{d\theta^*} = \frac{d\left(\left.\frac{d\Pi}{d\theta^*}\right|_{\theta=\theta_1+\theta^*}\times\left.\frac{d\theta^*}{d\theta}\right|_{\theta=\theta_1+\theta^*}\right)}{d\theta^*} = \left.\frac{d^2\Pi}{d\theta^{*2}}\right|_{\theta=\theta_1+\theta^*} < 0,$$ due to

$$\left.\frac{d\theta^*}{d\theta}\right|_{\theta=\theta_1+\theta^*} = \left.\frac{1}{\frac{d\theta}{d\theta^*}}\right|_{\theta=\theta_1+\theta^*} = \left.\frac{1}{\frac{d(\theta_1+\theta^*)}{d\theta^*}}\right|_{\theta=\theta_1+\theta^*} = 1,$$ both graphs increase as θ^* increases but their marginal increases decrease as θ^* increases.

In addition, let

$$\widetilde{\Lambda} = \Lambda\int_0^s(1+\theta_1+\theta^*)^{-t}\,dt = E\int_0^s(1+\theta_1+\theta^*)^{-t}[u(c_t+b_t)-u(c_t)]\,dt, \tag{12}$$

$$\widetilde{\Pi} = \Pi\int_0^s(1+\theta_1+\theta^*)^{-t}\,dt = -E\frac{\int_s^\infty(1+\theta_1+\theta^*)^{-t}[u(\bar{c}-\bar{a})-u(\bar{c})]\,dt}{\frac{\int_s^\infty(1+\theta_1+\theta^*)^{-t}\,dt}{\int_0^s(1+\theta_1+\theta^*)^{-t}\,dt}} \tag{13}$$

and

$$\overline{\Pi} = \Pi\int_s^\infty(1+\theta_1+\theta^*)^{-t}\,dt = -E\int_s^\infty(1+\theta_1+\theta^*)^{-t}[u(\bar{c}-\bar{a})-u(\bar{c})]\,dt. \tag{14}$$

Proposition 4: If there is $\bar{p}^*$ and $\bar{p} < \bar{p}^*$, and given a shift from θ_1 to $\theta_1+\hat{\theta}$ and given that $\gamma = \gamma^*$, if $2^{\frac{1}{s}}-1 > \theta_1$ and $s > s^*$ then there is $\bar{\theta}$ such that if $\theta^* < \bar{\theta}$ then the non-jump of consumption is the optimal path.

Proof: See Appendix.

Proposition 4 is important as Proposition 2. Proposition 2 asserts that for any θ^* if γ is large then the non-jump of consumption is optimal, and Proposition 4 asserts that for any γ^* if a shift of time preference rate θ^* is smaller than $\bar{\theta}$ then the non-jump of consumption is optimal.[16]

Proposition 5: If there is $\bar{p}^*$ and $\bar{p} < \bar{p}^*$, and given a shift from θ_1 to $\theta_1 + \hat{\theta}$ and given that $\gamma = \gamma^*$, if $\frac{1}{(1+\theta_1+\theta^*)^s - 1}\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\tilde{\Pi}}{d\theta}d\theta > \int_{\theta_1}^{\theta_1+\theta^*}\frac{d\tilde{\Lambda}}{d\theta}d\theta$ in case that $\theta^* < \hat{\theta}^{\#}$ for some $\hat{\theta}^{\#}(>0)$, then there is $\bar{\theta}$ such that if $\theta^* < \bar{\theta}$ then the non-jump of consumption is the optimal path, even though $s \leq s^*$.

Proof: See Appendix.

This proposition enlarges the proposition 4 and shows that, even though $s \leq s^*$, deviated transition paths are possible under certain conditions.

(4) The optimal deviated transition path

In the previous sections, the deviated transition path is arbitrarily assumed. However, it should be determined by maxinization of each consumer's expected utility. In this sub-section the optimal deviated transition path is examined. Let

$\Psi = E(J) - E(NJ) = \{pE(Jtogether) + (1 - p)E(Jalone)\} - \{pE(NJalone) + (1 - p)E(NJtogether)\}$

$= p\{E(Jtogether) - E(NJalone)\} + (1 - p)\{E(Jalone) - E(NJtogether)\}$

$$= p\left\{E\int_0^s(1+\theta)^{-t}\left[u(c_t+b_t)-u(c_t)\right]dt + E\int_s^\infty(1+\theta)^{-t}\left[u(\hat{c}_t)-u(\hat{c}_t+a_t)\right]dt\right\}$$

$$+(1-p)\left\{E\int_0^s(1+\theta)^{-t}\left[u(c_t+b_t)-u(c_t)\right]dt + E\int_s^\infty(1+\theta)^{-t}\left[u(\bar{c}-\bar{a})-u(\bar{c})\right]dt\right\}, \quad (15)$$

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[16] If a shift of time preference rate $\hat{\theta}^*$ is larger than $\bar{\theta}$, the jump of consumption is the optimal path. This may imply that if people face a huge risk like a sever war in their neighborhood and thus their time preference rates go

where

$$a_t = i_t \int_0^s b_r \exp \int_r^s \ln\left(1 + i_q\right) dq \ dr \qquad (16)$$

and

$$\overline{a} = \theta \int_0^s b_r \exp \int_r^s \ln\left(1 + i_q\right) dq \ dr \ . \qquad (17)$$

If there is $\overline{p}^*$ and $\overline{p} < \overline{p}^*$, then $p = 0$ and thus

$$\Psi = E\int_0^s (1+\theta)^{-t}\left[u\left(c_t + b_t\right) - u\left(c_t\right)\right] dt + E\int_s^\infty (1+\theta)^{-t}\left[u\left(\overline{c} - \overline{a}\right) - u\left(\overline{c}\right)\right] dt \ . \qquad (18)$$

Since $\Psi = E(J) - E(NJ)$, a consumer expects that the deviated transition path that makes Ψ lowest will be most attractive in case of non-jump. Based on this conjecture, “the unconditional optimal deviated transition path” is defined as follows.

Definition: The unconditional optimal deviated transition path $\widetilde{c}_t$ is c_t such that $\widetilde{c}_t \in \arg\min_{c_t} \Psi$.

Proposition 6: If there is $\overline{p}^*$ and $\overline{p} < \overline{p}^*$,

$$Eu'\left(\widetilde{c}_t\right) = E \ \exp \int_t^s \left(i_q - \theta\right) dq \ u'\left(\overline{c} - \overline{a}\right), \text{ and thus}$$

$$E\widetilde{c}_t = E\left[\exp \int_t^s \left(i_q - \theta\right) dq\right]^{-\frac{1}{\gamma}} \left(\overline{c} - \overline{a}\right).$$

Proof: See Appendix.

Corollary 1: If there is $\overline{p}^*$ and $\overline{p} < \overline{p}^*$,

(1) $\widetilde{c}_0 > E\left(\overline{c} - \overline{a}\right)$

up significantly, they will jump up consumption and consume much of their savings.

(2) $E\tilde{c}_s = E(\bar{c} - \bar{a})$

(3) $E\dfrac{d\tilde{c}_t}{dt} < 0$

where $\tilde{c}_s = \lim_{t \to s} \hat{c}_t$.

Proof: See Appendix.

Proposition 7: If there is $\bar{p}^*$ and $\bar{p} < \bar{p}^*$,

(1) $\lim_{s \to 0} \tilde{c}_0 = \bar{c}$

(2) $\lim_{s \to \infty} \tilde{c}_0 = \hat{c}_0$

Proof: See Appendix.

Proposition 8: If there is $\bar{p}^*$ and $\bar{p} < \bar{p}^*$, then $\lim_{s \to \infty} E\tilde{c}_t = E\hat{c}_t$.

Proof: See Appendix.

Hence, in case of $s \to \infty$, the deviated transition path $\tilde{c}_t$ becomes identical to the new saddle path $\hat{c}_t$ of the representative consumer. This result seems natural because $s \to \infty$ means that firms will not destroy capital infinitely, which is the assumption in usually used models. This result, in reverse, implies importance of the finite period *s*, in which period all the agents return to the optimal paths of the representative consumer and firm.

Corollary 2: If there is $\bar{p}^*$ and $\bar{p} < \bar{p}^*$, then $\hat{c}_0 > \tilde{c}_0$.

Proof: See Appendix.

Corollary 3: If there is $\bar{p}^*$ and $\bar{p} < \bar{p}^*$, then s that makes $\tilde{c}_0 = \bar{c}_0$ where $\bar{c}_0$ is the consumption just before the shock, is $\bar{s}$ such that $E\int_0^{\bar{s}} \left(i_q - \theta\right) dq = E\gamma \ln\left(\frac{\bar{c} - \bar{a}}{\bar{c}_0}\right)$.

Proof: See Appendix.

The unconditional optimal deviated transition path has no boundary condition, thereby $\tilde{c}_0$ and $\tilde{c}_s$ do not necessarily equal $\bar{c}_0$, that is consumption just before a shock hits the economy, and $\bar{c}$. Thus at *s*, consumers need to jump upwards from $\bar{c} - \bar{a}$ to $\bar{c}$. In actuality, however, deviated transition paths may start from $\bar{c}_0$ and reach $\bar{c}$. Based on this conjecture, the following "conditional optimal deviated transition path" can be defined.

Definition: The conditional optimal deviated transition path $\tilde{c}^*_t$ is c_t such that $\tilde{c}^*_t \in \arg\min_{c_t} \Psi$ s.t. $\tilde{c}^*_0 = \bar{c}_0$, $\lim_{t \to s} \tilde{c}^*_t = \tilde{c}^*_s = \bar{c}$, $\frac{d\tilde{c}^*_t}{dt} < 0$ and $\frac{d^2\tilde{c}^*_t}{dt^2} > 0$.

Let

$$\Phi(c, \hat{c}, \Theta, t) = E\int_0^t \left(\hat{c}_r - c_r\right) \Theta(r)\, dr \tag{19}$$

and

$$\Theta(t) = E \exp \int_t^s \ln\left(1 + i_q\right) dq\,. \tag{20}$$

Thereby,

$$\Psi = E\int_0^s (1 + \theta)^{-t}\left[u(\hat{c}_t) - u(c_t)\right] dt + E\,\frac{(1 + \theta)^{-s}}{\theta}\left\{u\left[\bar{c} - \theta\Phi\left(c_s, \hat{c}_s, \Theta(s), s\right)\right] - u(\bar{c})\right\} \tag{21}$$

where $\frac{\partial \Phi}{\partial t} = E(\hat{c}_t - c_t)\,\Theta(t)$. There is a control variable c_t and three environmental parameters $\hat{c}, \Theta$ and Φ . Hence, if $\frac{\partial \hat{c}_t}{\partial t}$ and $\frac{\partial \Theta}{\partial t}$ are functions of the control variable and/or other environmental parameters, the maximum principle for a Bolza type payoff function can be applied for the above minimization problem without conditions $\frac{d\tilde{c}^*_t}{dt} < 0$ and $\frac{d^2\tilde{c}^*_t}{dt^2} > 0$. However, $\frac{\partial \hat{c}_t}{\partial t}$ and $\frac{\partial \Theta}{\partial t}$ are independent from the control variable and/or other environmental parameters, and thus the above optimization problem including the conditions $\frac{d\tilde{c}^*_t}{dt} < 0$ and $\frac{d^2\tilde{c}^*_t}{dt^2} > 0$ has to be solved numerically.

(5) Inventory

In the previous sections, it is assumed that unconsumed capital b_t is simply destroyed. However, there is another possibility of b_t —that is, inventory. If unconsumed capital b_t is temporally stored as inventories and consumed later, there will be no loss of resources. However, it is only possible if inventories are assumed to be productive and their marginal productivities are assumed more elastic. This assumption is adopted in e.g. Kydland and Prescott (1982, 1988) or Cooley and Hansen (1995) mainly not by its plausibility but by necessity of making RBC models workable. If inventories are productive it will be rational for firms to change some of more productive capital to less productive inventories in case they need to reduce capital.

The crucial point of this assumption is whether inventories are productive in actuality. Kydland and Prescott (1982) present some rationalizations of the assumption, but they may be insufficient to believe the assumption *a priori*.[17] It seems more natural to assume that inventories are stored not because they are productive but mainly because they work as the buffer for uncertainties to smooth production. Most firms think inventories, in reverse, are sources of additional costs and thus pursue to develop new technologies that reduce inventories to the minimum requirement level. Anyway, this is not a problem of whether b_t exist or not, but a problem of how to interpret b_t ─ that is, whether b_t can be interpreted as real disequilibria or not.

3. Concluding remarks

There are basically two different views of output gaps. One view stresses importance of rigidities or frictions in one or more markets, which delay adjustments particularly of prices or wages. The other view is represented by the early version of real business cycle models and asserts that, because most of the shocks are productivity shocks, fluctuations of GDP are basically identical with fluctuations of potential GDP and thus estimated deviations of actual GDP from its trend should not be interpreted as output gaps.

The objective of the paper is to pursue the third completely different view such that even if there is no rigidity, large protracted output gaps are theoretically possible to exist. The paper presents a new model that when a fundamental shock that moves steady state is given, there is a possibility that agents do not proceed on the optimal path for the representative consumer during the transition to the new steady state as a result of rational calculation. The driving force behind this strange phenomenon is non-cooperative behavior of consumers. When a shock is given, consumers need to jump to the new optimal path of the representative consumer to restore equilibrium without loss of resources. However, if the expected probability of deviation

[17] They assert that with large inventories, stores can economize on labor resources allocated to restocking, and firms, by making larger production runs, reduce equipment down-time associated with shifting from producing one type of goods to another.

of other consumers exceeds a critical point, the expected utility of a consumer who dares to deviate is higher than that when she does not deviate. The same calculation will be done by other consumers, thus all the consumers expect that their expected utilities are higher when they deviate than when they do not deviate. Facing these calculations of consumers, firms expect that most consumers will deviate and thus firms' capital must be adjusted corresponding to the deviation of consumers—that is, unused excessive capital must be destroyed. As a result, all the consumers and firms expect that all the consumers and firms deviate from the optimal path of the representative consumers, and they behave based on the expectation and therefore in actuality they deviate. As a result, it is not guaranteed that a jump to the new optimal path of the representative consumer is optimal for each consumer.

The novelty of the paper is that it uncovered a new possibility of protracted output gaps such that consumers do not jump consumption and proceed on a deviated transition path, as a result of strategic calculation of expected utility. This new possibility has two important advantages; 1) it is not necessary to present a micro foundation for a difficult question why price adjustments are far slower than quantity adjustments, and 2) it is not necessary to solely depend on productivity shocks as the standard real business cycle model that has well-known shortcomings such as the weak internal-propagation mechanism that needs large and persisting productivity shocks or necessity of unimaginable frequent negative productivity shocks.

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Appendix

1. Proof of Proposition 1:

Since there is no possibility of *Jalone* and *NJalone*, then $E(J) = E(Jtogether)$ and $E(NJ) = E(NJtogether)$. Hence,

$$E(J) - E(NJ) = E\left[\int_0^s (1+\theta)^{-t} u(c_t + b_t)\, dt + \int_s^\infty (1+\theta)^{-t} u(\hat{c}_t)\, dt\right]$$

$$- E\left[\int_0^s (1+\theta)^{-t} u(c_t)\, dt + \int_s^\infty (1+\theta)^{-t} u(\bar{c})\, dt\right]$$

$$= E\left\{\int_0^s (1+\theta)^{-t}\left[u(c_t + b_t) - u(c_t)\right] dt + \int_s^\infty (1+\theta)^{-t}\left[u(\hat{c}_t) - u(\bar{c})\right] dt\right\} > 0,$$

because $c_t + b_t > c_t$ and $\hat{c}_t > \bar{c}$. Therefore if consumers are cooperative, a jump of consumption is the optimal path. ■

2. Proof of Lemma 1:

Since $i_q < \theta$ $(q < s)$ during the transition period for any possible path in the case of upward time preference shift, we get, for any stream of c_t, i_t that satisfy the assumptions,

$E(Jalone)$ - $E(NJtogether)$

$$= E\int_0^s (1+\theta)^{-t} \lim_{\gamma\to 0}\left[u(c_t + b_t) - u(c_t)\right] dt + E\int_s^\infty (1+\theta)^{-t} \lim_{\gamma\to 0}\left[u(\bar{c} - \bar{a}) - u(\bar{c})\right] dt$$

$$= E\int_0^s (1+\theta)^{-t} b_t dt - E\int_s^\infty (1+\theta)^{-t} \bar{a}\, dt$$

$$= E\int_0^s (1+\theta)^{-t} b_t dt - E\theta\left\{\int_0^s \left[b_r \exp\int_r^s \ln(1+i_q)\, dq\right] dr\right\}\int_s^\infty (1+\theta)^{-t} dt$$

(since $i_t = \theta$ if $t \geq s$)

$$= E\int_0^s (1+\theta)^{-t} b_t dt - E(1+\theta)^{-s}\int_0^s \left[b_r \exp\int_r^s \ln(1+i_q)\, dq\right] dr$$

$$= E\int_0^s b_t\left[(1+\theta)^{-t} - (1+\theta)^{-s}\exp\int_t^s \ln(1+i_q)\, dq\right] dt > 0$$

$$= E(1+\theta)^s \int_0^s b_t \left[(1+\theta)^{s-t} - \exp \int_t^s \ln\left(1+i_q\right) dq \right] dt > 0,$$

since $(1+\theta)^{s-t} > \exp \int_t^s \ln\left(1+i_q\right) dq$ for any t ($0 \leqq t < s$), due to $\theta > i_q$. ■

3. Proof of Lemma 2:

It is assumed that a unit of consumption c_t as well as b_t and $\bar{a}$ is $\frac{[\varphi(\gamma-1)]^{\gamma-1}}{c_t}$ and $\gamma > 1$, where φ is a positive constant, thus $\frac{c_t^{1-\gamma}}{1-\gamma} = -\varphi < 0$.

Here, because in any deviated path $b_t > 0$, then for any period t ($< s$),

$$\lim_{\gamma\to\infty} u(c_t + b_t) - u(c_t) = \lim_{\gamma\to\infty} \frac{c_t^{1-\gamma}}{1-\gamma}\left(1+\frac{b_t}{c_t}\right)^{1-\gamma} + \varphi = \varphi \lim_{\gamma\to\infty}\left[\left(1+\frac{b_t}{c_t}\right)^{1-\gamma} + 1\right] = \varphi.$$

On the other hand, because in any deviated path $b_t > 0$ and $\bar{a} > 0$, then for any period t ($> s$), if $\bar{c} > \bar{a}$

$$\lim_{\gamma\to\infty} u(\bar{c}-\bar{a}) - u(\bar{c}) = \lim_{\gamma\to\infty} u[c_0 - (c_0-\bar{c}) - \bar{a}] - u[c_0 - (c_0 - \bar{c})]$$

$$= \lim_{\gamma\to\infty} \frac{c_0^{1-\gamma}\left(1 - \frac{c_0 - \bar{c} + \bar{a}}{c_0}\right)^{1-\gamma}}{1-\gamma} - \frac{c_0^{1-\gamma}\left(1-\frac{c_0-\bar{c}}{c_0}\right)^{1-\gamma}}{1-\gamma}$$

$$= \lim_{\gamma\to\infty} \frac{c_0^{1-\gamma}}{1-\gamma}\left[\left(1 - \frac{c_0 - \bar{c} + \bar{a}}{c_0}\right)^{1-\gamma} - \left(1-\frac{c_0-\bar{c}}{c_0}\right)^{1-\gamma}\right]$$

$$= -\varphi \lim_{\gamma\to\infty}\left(1 - \frac{c_0 - \bar{c} + \bar{a}}{c_0}\right)^{1-\gamma}\left[1 - \left(\frac{\bar{c}}{\bar{c}-\bar{a}}\right)^{1-\gamma}\right] = -\infty,$$

since $\lim_{\gamma\to\infty}\left(1 - \frac{c_0 - \bar{c} + \bar{a}}{c_0}\right)^{1-\gamma} = \infty$ and $\lim_{\gamma\to\infty}\left(\frac{\bar{c}}{\bar{c}-\bar{a}}\right)^{1-\gamma} = 0$ due to $\bar{\gamma} > 1$.

Hence, if the utility function is $u(c_t) = \lim_{\gamma\to\infty} \frac{c_t^{1-\gamma}}{1-\gamma}$ and if $\bar{c} > \bar{a}$, then

$E(Jalone)$ - $E(NJtogether)$ =

$$\int_0^s (1+\theta)^{-t} \lim_{\gamma\to\infty}[u(c_t + b_t) - u(c_t)]\, dt + \int_s^\infty (1+\theta)^{-t} \lim_{\gamma\to\infty}[u(\bar{c} - \bar{a}) - u(\bar{c})]\, dt = -\infty < 0. \blacksquare$$

4. Proof of Lemma 3:

For any set of streams of b_t and i_t, if $\gamma \rightarrow 0$ then $E(Jalone)$ - $E(NJtogether) > 0$ by the lemma 1 and if $\gamma \rightarrow \bar{\gamma}$ then $E(Jalone)$ - $E(NJtogether) < 0$ by the lemma 2. Hence, because c_t, b_t and i_t are continuous and u is a continuous function, there is a certain γ^* where $E(Jalone)$ - $E(NJtogether) = 0$ by the intermediate value theorem. Because c_t, a_t and b_t are monotonious and u is a monotonically continuous function, and because if $\gamma \rightarrow \bar{\gamma}$ then $E(Jalone)$ - $E(NJtogether) < 0$, if $\gamma > \gamma^*$. ■

5. Proof of Proposition 2:

By the Lemma 3, if $\gamma > \gamma^*$ then $E(Jalone)$ - $E(NJtogether) < 0$ for a consumer. On the other hand $E(Jtogether)$ - $E(NJalone) > 0$ since capital is not adjusted at s and thus the jump is optimal for all the consumers. Hence, for the consumer,

$\lim_{\bar{p}\to 0} E(J)$ - $E(NJ) = E(Jalone) - E(NJtogether) < 0$, and

$\lim_{\bar{p}\to\infty} E(J)$ - $E(NJ) = E(Jtogether) - E(NJalone) > 0$.

Thereby, by the intermediate value theorem, there is $\bar{p}^*$ such that if $\bar{p} = \bar{p}^*$ then $E(J)$ - $E(NJ) = 0$ and if $\bar{p} < \bar{p}^*$ then $E(J)$ - $E(NJ) < 0$. If $E(J)$ - $E(NJ) < 0$ for the consumer, then for all the consumers $E(J)$ - $E(NJ) < 0$ and $p = 0$ by the assumption (A5). By the assumption (A4) firms adjust their capital to the new steady state level at s. All the consumers do not jump since $E(J)$ - $E(NJ) = E(Jalone) - E(NJtogether) < 0$. Hence, if $\gamma > \gamma^*$ then there is $\bar{p}^*$ such that if $\bar{p} < \bar{p}^*$ then non-jump of consumption is the optimal path. ■

6. Proof of Proposition 3:

For any $\gamma\,(>\,0)$, $E\lim_{s\to\infty}\int_0^s (1+\theta)^{-t}\left[u(c_t+b_t)-u(c_t)\right]dt > 0$. On the other hand for, any $\gamma\,(>\,0)$, $E\lim_{s\to\infty}\int_s^\infty (1+\theta)^{-t}\left[u(\bar{c}-\bar{a})-u(\bar{c})\right]dt = 0$. Hence for any $\gamma\,(>\,0)$, $E(Jalone)$ - $E(NJtogether) > 0$, and thus for any $\bar{p}$ and p, $E(J)$ - $E(NJ) > 0$. ■

7. Proof of Lemma 4:

Since $\bar{p} < \bar{p}^*$ then $E(J)$ - $E(NJ)$ = $E(Jalone)$ - $E(NJtogether)$ by the proposition 2. For any stream of b_t and i_t,

$E(Jalone) - E(NJtogether)$

$$= E\int_0^s (1+\theta)^{-t}\lim_{\gamma\to 0}\left[u(c_t+b_t)-u(c_t)\right]dt + E\int_s^\infty (1+\theta)^{-t}\lim_{\gamma\to 0}\left[u(\bar{c}+\bar{a})-u(\bar{c})\right]dt$$

$$= E\int_0^s (1+\theta)^{-t} b_t dt - E\int_s^\infty (1+\theta)^{-t}\bar{a}\,dt$$

$$= E\int_0^s (1+\theta)^{-t} b_t\,dt - E\theta\left\{\int_0^s\left[b_r \exp\int_r^s \ln(1+i_q)\,dq\right]dr\right\}\int_s^\infty (1+\theta)^{-t}dt$$

(since $i_t = \theta$ if $t \geq s$)

$$= E\int_0^s (1+\theta)^{-t} b_t\,dt - E(1+\theta)^{-s}\int_0^s\left[b_r \exp\int_r^s \ln(1+i_q)\,dq\right]dr$$

$$= E\int_0^s b_t\left[(1+\theta)^{-t} - (1+\theta)^{-s}\exp\int_t^s \ln(1+i_q)\,dq\right]dt$$

$$= E(1+\theta)^{s}\int_0^s b_t\left[(1+\theta)^{s-t} - \exp\int_t^s (1+i_q)\,dq\right]dt = 0,$$

since $(1+\theta)^{s-t} = \exp\int_t^s \ln(1+i_q)\,dq$ for any t $(0 \leqq t < s)$, due to $\theta = i_q$.

Thereby,

$E(J)$ - $E(NJ)$ = $E(Jalone)$ - $E(NJtogether) = 0$.

Hence, the expected utility when J is chosen is identical with that when NJ is chosen, and thus the jump of consumption and the non-jump of consumption are indifferent. ■

8. Proof of Lemma 5:

Since $i_t = \theta$, by the lemma 4, in the case of the utility function with $\gamma \to 0$, $E(J)$ - $E(NJ) = 0$. There is no utility function with γ whose value is less than that of γ in the case of the utility function with $\gamma \to 0$. On the other hand, since $i_t = \theta$, by the Lemma 3, for the γ^* where $E(J)$ - $E(NJ) = 0$, if $\gamma > \gamma^*$ then the non-jump of consumption is the optimal path.

Hence, for any other utility function than $\lim_{\gamma \to 0} u(c_t) = \dfrac{c_t^{1-\gamma}}{1-\gamma}$, the non-jump of consumption is the optimal path. ■

9. Proof of Lemma 6:

Since $\bar{p} < \bar{p}^*$ then $E(J)$ - $E(NJ)$ = $E(Jalone)$ - $E(NJtogether)$ by the proposition 2.

Step 1.

If the utility function is $u(c_t) = \lim_{\gamma \to 0} \dfrac{c_t^{1-\gamma}}{1-\gamma}$, since $i_q < \theta$ $(q < s)$ during the transition period for any possible path in the case of an upward time preference shift, we get, for any given set of streams of c_t, and i_t,

$$E \frac{\int_0^s (1+\theta)^{-t} \lim_{\gamma \to 0} [u(c_t + b_t) - u(c_t)]\, dt}{\int_0^s (1+\theta)^{-t}\, dt} - \left\{ - E \frac{\int_s^\infty (1+\theta)^{-t} \lim_{\gamma \to 0} [u(\bar{c} - \bar{a}) - u(\bar{c})]\, dt}{\int_s^\infty (1+\theta)^{-t}\, dt} \right\}$$

$$= E \frac{\int_0^s (1+\theta)^{-t} b_t\, dt}{\int_0^s (1+\theta)^{-t} dt} - E \frac{\int_s^\infty (1+\theta)^{-t} \bar{a}\, dt}{\int_s^\infty (1+\theta)^{-t} dt}$$

$$= E \frac{\int_0^s (1+\theta)^{-t} b_t dt}{\int_0^s (1+\theta)^{-t} dt} - E\theta \int_0^s \left[b_r \exp \int_r^s \ln(1 + i_q)\, dq \right] dr$$

(since $i_t = \theta$ if $t \geq s$)

$$= E\theta\int_0^s b_t\left[\frac{(1+\theta)^{-t}}{1-(1+\theta)^{-s}} - \exp\int_t^s \ln\left(1+i_q\right)dq\right]dt\,.$$

Here, for any s and t ($< s$), $1 < \exp\int_t^s \ln\left(1+i_q\right)dq < \infty$ since $0 < i_q < \theta$. Thus since,

$$\lim_{s\to\infty}\lim_{t\to s}\frac{(1+\theta)^{-t}}{1-(1+\theta)^{-s}} = \lim_{s\to\infty}\lim_{t\to s}\frac{1}{(1+\theta)^t}\left[\frac{1}{(1+\theta)^s-1}+1\right] = 0$$

$$\lim_{s\to\infty}\lim_{t\to 0}\frac{(1+\theta)^{-t}}{1-(1+\theta)^{-s}} = \lim_{s\to\infty}\lim_{t\to 0}\frac{1}{(1+\theta)^t}\left[\frac{1}{(1+\theta)^s-1}+1\right] = 1,$$ thus

if $s\to\infty$, then for any t ($<s$)

$$\frac{(1+\theta)^{-t}}{1-(1+\theta)^{-s}} - \exp\int_t^s \ln\left(1+i_q\right)dq < 0,$$

and thus

$$E\frac{\int_0^s (1+\theta)^{-t}\lim_{\gamma\to 0}\left[u(c_t+b_t)-u(c_t)\right]dt}{\int_0^s (1+\theta)^{-t}\,dt} - \left\{-E\frac{\int_s^\infty (1+\theta)^{-t}\lim_{\gamma\to 0}\left[u(\bar{c}-\bar{a})-u(\bar{c})\right]dt}{\int_s^\infty (1+\theta)^{-t}\,dt}\right\} < 0$$

On the other hand, since

$$\lim_{s\to 0}\lim_{t\to s}\frac{(1+\theta)^{-t}}{1-(1+\theta)^{-s}} = \lim_{s\to 0}\lim_{t\to s}\frac{1}{(1+\theta)^t}\left[\frac{1}{(1+\theta)^s-1}+1\right] = \infty$$

$$\lim_{s\to 0}\lim_{t\to 0}\frac{(1+\theta)^{-t}}{1-(1+\theta)^{-s}} = \lim_{s\to 0}\lim_{t\to 0}\frac{1}{(1+\theta)^t}\left[\frac{1}{(1+\theta)^s-1}+1\right] = \infty,$$

if $s\to 0$, then for any t ($<s$)

$$\frac{(1+\theta)^{-t}}{1-(1+\theta)^{-s}} - \exp\int_t^s \ln\left(1+i_q\right)dq > 0,$$

and thus

$$E\frac{\int_0^s (1+\theta)^{-t}\lim_{\gamma\to 0}\left[u(c_t+b_t)-u(c_t)\right]dt}{\int_0^s (1+\theta)^{-t}\,dt} - \left\{-E\frac{\int_s^\infty (1+\theta)^{-t}\lim_{\gamma\to 0}\left[u(\bar{c}-\bar{a})-u(\bar{c})\right]dt}{\int_s^\infty (1+\theta)^{-t}\,dt}\right\} > 0$$

Because c_t, b_t and i_q are continuous and u is a continuous function, then by the intermediate value theorem, for any set of streams of b_t and i_t, there is s^* such that if $s > s^*$ then

$$E\frac{\int_0^s (1+\theta)^{-t}\left[u(c_t + b_t) - u(c_t)\right]dt}{\int_0^s (1+\theta)^{-t}\,dt} < -E\frac{\int_s^\infty (1+\theta)^{-t}\left[u(\bar{c} - \bar{a}) - u(\bar{c})\right]dt}{\int_s^\infty (1+\theta)^{-t}\,dt}.$$ [18]

Step 2.

Since c_t, b_t and i_t, are monotonic and continuous and u is a monotonically continuous function, as explained in the proof of the Lemma 3, $E\int_0^s (1+\theta)^{-t}\left[u(c_t + b_t) - u(c_t)\right]dt + E\int_s^\infty (1+\theta)^{-t}\left[u(\bar{c} - \bar{a}) - u(\bar{c})\right]dt$ is a monotonically decreasing function of γ.

Hence, for the same θ and s and streams of c_t, b_t and i_q as those of the utility function $u(c_t) = \lim_{\gamma\to 0}\frac{c_t^{1-\gamma}}{1-\gamma}$, in case of utility functions with higher γ than that of the utility function $u(c_t) = \lim_{\gamma\to 0}\frac{c_t^{1-\gamma}}{1-\gamma}$, the ratio $-\frac{E\int_0^s (1+\theta)^{-t}\left[u(c_t + b_t) - u(c_t)\right]dt}{E\int_s^\infty (1+\theta)^{-t}\left[u(\bar{c} - \bar{a}) - u(\bar{c})\right]dt}$ is smaller than that in the case of the utility function $u(c_t) = \lim_{\gamma\to 0}\frac{c_t^{1-\gamma}}{1-\gamma}$.

Therefore, in case of utility functions with higher γ than that of the utility function $u(c_t) = \lim_{\gamma\to 0}\frac{c_t^{1-\gamma}}{1-\gamma}$, if $s > s^*$ then also

[18] Here, $\int_0^s \frac{(1+\theta)^{-t}}{1-(1+\theta)^{-s}}\,dt = \frac{1}{\theta}$. On the other hand, since $\int_t^{t+1} \exp\int_t^s \ln\left(1+i_q\right)dq\,dt = I_t > 1$, $\int_0^s \exp\int_t^s \ln\left(1+i_q\right)dq\,dt = \int_0^s I_t\,dt > s$. Hence, if b_t is constant, in the case that θ is e.g. 0.05, if s is at least larger than 20 years, then $\frac{1}{\theta} < s$ and thus $E\theta\int_0^s b_t\left[\frac{(1+\theta)^{-t}}{1-(1+\theta)^{-s}} - \exp\int_t^s \ln\left(1+i_q\right)dq\right]dt$ will be

$$E\frac{\int_0^s (1+\theta)^{-t}\left[u(c_t+b_t)-u(c_t)\right]dt}{\int_0^s (1+\theta)^{-t}\,dt} < -E\frac{\int_s^\infty (1+\theta)^{-t}\left[u(\bar{c}-\bar{a})-u(\bar{c})\right]dt}{\int_s^\infty (1+\theta)^{-t}\,dt}.$$

Since there is no utility function with γ whose value is less than that of γ in the case of the utility function with $\gamma \to 0$, for any utility function, if $s > s^*$ then the above inequality holds. Hence, there is s^* such that if $s > s^*$ then both inequalities hold. ■

10. Proof of Proposition 4:

Since $\bar{p} < \bar{p}^*$ then $E(J)$ - $E(NJ)$ = $E(Jalone)$ - $E(NJtogether)$ by the proposition 2.

Step 1.

Because $\int_0^s (1+\theta_1+\theta^*)^{-t}\,dt$ monotonically increases as θ^* increases, the graphs of $\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\tilde{\Lambda}}{d\theta}d\theta$ and $\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\tilde{\Pi}}{d\theta}d\theta$ are simple upper shifts of the graphs of $\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\Lambda}{d\theta}d\theta$ and $\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\Pi}{d\theta}d\theta$. The graphs of $\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\tilde{\Lambda}}{d\theta}d\theta$ and $\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\tilde{\Pi}}{d\theta}d\theta$ are shown in Panel (c) of Figure 3. Because of $\frac{d^2\int_0^s (1+\theta_1+\theta^*)^{-t}\,dt}{d\theta^*} < 0$, $\frac{d^2\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\Lambda}{d\theta}d\theta}{d\theta^{*2}} < 0$ and $\frac{d^2\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\Pi}{d\theta}d\theta}{d\theta^{*2}} < 0$, then $\frac{d^2\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\tilde{\Lambda}}{d\theta}d\theta}{d\theta^{*2}} < 0$ and $\frac{d^2\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\tilde{\Pi}}{d\theta}d\theta}{d\theta^{*2}} < 0$.[19]

As for the graph of $\int_{\theta_1}^{\theta_1+\theta^*}\frac{d\tilde{\Pi}}{d\theta}d\theta$, because

negative. Considering $\int_0^s I_t\,dt$ is larger than s e.g. 1.2 times s, s will be much shorter than 20 years.

[19] It is easily shown that $\frac{d^2\int_0^s (1+\theta_1+\theta^*)^{-t}\,dt}{d\theta^*} < 0$.

$$\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta}\, d\theta = \int_{\theta_1}^{\theta_1+\theta^*} \frac{d\left[\widetilde{\Pi}\, \dfrac{\int_s^{\infty} \left(1+\theta_1+\theta^*\right)^{-t}}{\int_0^s \left(1+\theta_1+\theta^*\right)^{-t}}\right]}{d\theta}\, d\theta = \frac{1}{\left(1+\theta_1+\theta^*\right)^s - 1} \int_{\theta_1}^{\theta_1+\theta^*} \frac{d\widetilde{\Pi}}{d\theta}\, d\theta,$$

and because if $2^{\frac{1}{s}} - 1 > \theta_1$ then while $2^{\frac{1}{s}} - 1 - \theta_1 > \theta^*$, $\dfrac{1}{\left(1+\theta_1+\theta^*\right)^s - 1} > 1$ and as θ^* increases form 0 to more than $2^{\frac{1}{s}} - 1 - \theta_1$ then $\dfrac{1}{\left(1+\theta_1+\theta^*\right)^s - 1}$ asymptotically approaches zero, therefore $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta}\, d\theta$ is larger than $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\widetilde{\Pi}}{d\theta}\, d\theta$ when θ^* is small, $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta}\, d\theta$ is smaller than $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\widetilde{\Pi}}{d\theta}\, d\theta$ when θ^* is large and $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta}\, d\theta$ approaches zero when θ^* becomes larger. Hence the graph of $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta}\, d\theta$ is as shown in Panel (c) of Figure 3.

Because of the feature of $\dfrac{d^2 \int_{\theta_1}^{\theta_1+\theta^*} \frac{d\widetilde{\Lambda}}{d\theta}\, d\theta}{d\theta^{*2}} < 0$ and $\dfrac{d^2 \int_{\theta_1}^{\theta_1+\theta^*} \frac{d\widetilde{\Pi}}{d\theta}\, d\theta}{d\theta^{*2}} < 0$, the curve $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta}\, d\theta$ and the curve $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\widetilde{\Lambda}}{d\theta}\, d\theta$ cross at $\overline{\theta}$.

Step 2.

Our concerns are $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\widetilde{\Lambda}}{d\theta}\, d\theta$ that indicates an increase of $\widetilde{\Lambda} = E\int_0^s \left(1+\theta_1+\theta^*\right)^{-t} \left[u\left(c_t+b_t\right) - u\left(c_t\right)\right] dt$ when θ increases from θ_1 to $\theta_1 + \theta^*$ and $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta}\, d\theta$ that indicates an increase of $\overline{\Pi} = -E\int_s^{\infty} \left(1+\theta_1+\theta^*\right)^{-t} \left[u\left(\overline{c}-\overline{a}\right) - u\left(\overline{c}\right)\right] dt$ when θ increases from θ_1 to $\theta_1 + \theta^*$. By Panel (c) of Figure 3, clearly, if θ^* is smaller than $\overline{\theta}$, $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\widetilde{\Lambda}}{d\theta}\, d\theta < \int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta}\, d\theta$. If $\theta^* = \overline{\theta}$, then $\int_{\theta_1}^{\theta_1+\overline{\theta}} \frac{d\widetilde{\Lambda}}{d\theta}\, d\theta = \int_{\theta_1}^{\theta_1+\overline{\theta}} \frac{d\overline{\Pi}}{d\theta}\, d\theta$.

Here, if $\gamma = \gamma^*$ and $E(Jalone)$ - $E(NJtogether) = 0$ when a shift is $\hat{\theta}$, then $\int_{\theta_1}^{\theta_1+\hat{\theta}} \frac{d\tilde{\Lambda}}{d\theta} d\theta$ = $\int_{\theta_1}^{\theta_1+\hat{\theta}} \frac{d\overline{\Pi}}{d\theta} d\theta$. Hence, if $\gamma = \gamma^*$ then $\overline{\theta} = \hat{\theta}$ and $\overline{\Pi}$ equals $\tilde{\Lambda}$ and thus $E(Jalone)$ - $E(NJtogether)$ = 0. On the other hand, if $\gamma = \gamma^*$ and $\theta^* < \overline{\theta} = \hat{\theta}$ then $\int_{\theta_1}^{\theta_1+\hat{\theta}} \frac{d\tilde{\Lambda}}{d\theta} d\theta < \int_{\theta_1}^{\theta_1+\hat{\theta}} \frac{d\overline{\Pi}}{d\theta} d\theta$ and thus $\overline{\Pi}$ is larger than $\tilde{\Lambda}$ and $E(Jalone)$ - $E(NJtogether) < 0$. if $\gamma = \gamma^*$ and $\theta^* > \overline{\theta} = \hat{\theta}$ then $\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\tilde{\Lambda}}{d\theta} d\theta > \int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta} d\theta$ and thus $\overline{\Pi}$ is smaller than $\tilde{\Lambda}$ and $E(Jalone)$ - $E(NJtogether)$ > 0. Hence if $\gamma = \gamma^*$ and $\theta^* < \overline{\theta}$ then, $E(J)$ - $E(NJ)$ = $E(Jalone)$ - $E(NJtogether) < 0$, therefore the non-jump of consumption is the optimal path. ■

11. Proof of Proposition 5:

Since $\overline{p} < \overline{p}^*$ then $E(J)$ - $E(NJ)$ = $E(Jalone)$ - $E(NJtogether)$ by the proposition 2. If $\frac{1}{(1+\theta_1+\theta^*)^s - 1}\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta} d\theta > \int_{\theta_1}^{\theta_1+\theta^*} \frac{d\tilde{\Lambda}}{d\theta} d\theta$ then $E(Jalone)$ - $E(NJtogether) < 0$, even though $s \le s^*$. Hence, there is $\overline{\theta}$ such that if $\theta^* < \overline{\theta}$ then,
$E(J)$ - $E(NJ)$ = $E(Jalone)$ - $E(NJtogether) < 0$ even though $s \le s^*$, that is, the non-jump of consumption is the optimal path, even though $s \le s^*$. ■

12. Proof of Proposition 6:

Since $\overline{p} < \overline{p}^*$ then,

$$\Psi = E\int_0^s (1+\theta)^{-t}\left[u(c_t + b_t) - u(c_t)\right] dt + E\int_s^\infty (1+\theta)^{-t}\left[u(\overline{c} - \overline{a}) - u(\overline{c})\right] dt \text{, and}$$

$$\frac{\partial \Psi}{\partial c_t} = -E(1+\theta)^{-t} u'(c_t) + E\theta\int_s^\infty (1+\theta)^{-t} dt \ \exp\int_t^s \ln(1+i_q)\, dq\ u'(\overline{c} - \overline{a})$$

$$= -E(1+\theta)^{-t} u'(c_t) + E(1+\theta)^{-s} \exp\int_t^s \ln(1+i_q)\, dq\ u'(\overline{c} - \overline{a}) \text{. In addition,}$$

$$\frac{\partial^2 \Psi}{\partial c_t^2} = -E(1+\theta)^{-t} u''(c_t) - E\theta(1+\theta)^{-s}\left[\exp\int_t^s \ln(1+i_q)\,dq\right]^2 u''(\bar{c}-\bar{a}) > 0$$

since $u'' < 0$.

Ψ has one control variable c_t and environmental parameters, derivatives of which are all not related to the control variable. Hence, if for any $c_t (0 \le t < s)$, $\frac{\partial \Psi}{\partial c_t} = 0$, then Ψ is minimized.

If $\frac{\partial \Psi}{\partial c_t} = 0$,

$$Eu'(c_t) = E\frac{\exp\int_t^s \ln(1+i_q)\,dq}{(1+\theta)^{s-t}} u'(\bar{c}-\bar{a}) = E\exp\int_t^s (i_q - \theta)\,dq\, u'(\bar{c}-\bar{a}).$$

Hence, if $\tilde{c}_t$ is c_t such that $\tilde{c}_t \in \arg\min_{c_t} \Psi$, then $Eu'(\tilde{c}_t) = E\exp\int_t^s (i_q - \theta)\,dq\, u'(\bar{c}-\bar{a})$.

In case of $\gamma \ne 1$, $Eu'(\tilde{c}_t) = E\tilde{c}_t^{-\gamma}$ and $E\exp\int_t^s (i_q-\theta)\,dq\, u'(\bar{c}-\bar{a}) = E\exp\int_t^s (i_q-\theta)\,dq(\bar{c}-\bar{a})^{-\gamma}$.

Thus, $E\tilde{c}_t = E\left[\exp\int_t^s (i_q-\theta)\,dq\right]^{-\frac{1}{\gamma}}(\bar{c}-\bar{a})$. Similarly, it is proved in case of $\gamma = 1$. ■

13. Proof of Corollary 1:

(1) Since, $Eu'(\tilde{c}_t) = E\exp\int_t^s (i_q-\theta)\,dq\, u'(\bar{c}-\bar{a})$, then

$$E\lim_{t\to 0} u'(\tilde{c}_t) = u'(\tilde{c}_0) = E\exp\int_0^s (i_q-\theta)\,dq\, u'(\bar{c}-\bar{a}).$$

Because $E\exp\int_0^s (i_q-\theta)\,dq < 1$ due to $i_q < \theta$, then $\tilde{c}_0 > E(\bar{c}-\bar{a})$.

(2) $E\lim_{t\to s} u'(\tilde{c}_t) = Eu'(\tilde{c}_s) = Eu'(\bar{c}-\bar{a})$, since $E\lim_{t\to s}\exp\int_t^s \ln(1+i_q)\,dq = 1$ and $E\lim_{t\to s}(1+\theta)^{s-t} = 1$. Hence, $E(\tilde{c}_s) = E(\bar{c}-\bar{a})$.

(3) $E\frac{du'(\tilde{c}_t)}{dt} = Eu''(\tilde{c}_t)\frac{d\tilde{c}_t}{dt}$

$$= E\theta \exp \int_t^s \left(i_q - \theta\right) dq \; u'(\bar{c} - \bar{a}) - E\, i_t \exp \int_t^s \left(i_q - \theta\right) dq \; u'(\bar{c} - \bar{a})$$

$$= E(\theta - i_t) \exp \int_t^s \left(i_q - \theta\right) dq \; u'(\bar{c} - \bar{a}) > 0 \text{ since, } \quad \theta - i_t > 0 .$$

Because $Eu''(\tilde{c}_t) < 0$, then $E\,\frac{d\tilde{c}_t}{dt} < 0$ ■

14. Proof of Proposition 7:

(1) $\lim_{s\to 0} u'(\tilde{c}_0) = E \lim_{s\to 0} \exp \int_0^s \left(i_q - \theta\right) dq \; u'(\bar{c} - \bar{a})$

$$= E \lim_{s\to 0} \exp \int_0^s \left(i_q - \theta\right) dq \; u'\left[\bar{c} - \theta \int_0^s b_r \exp \int_r^s \ln\left(1 + i_q\right) dq\, dr\right] = Eu'(\bar{c}).$$

Therefore $\lim_{s\to 0} \tilde{c}_0 = \bar{c}$.

(2) In case of $\gamma \neq 1$,

$$\lim_{s\to\infty} u'(\tilde{c}_0) = \lim_{s\to\infty} \tilde{c}_0^{\,-\gamma} = E \lim_{s\to\infty} \exp \int_0^s \left(i_q - \theta\right) dq \; (\bar{c} - \bar{a})^{-\gamma}$$

$$= E \lim_{s\to\infty} \exp \int_0^s \left(i_q - \theta\right) dq \left[\bar{c} - \theta \int_0^s (\hat{c}_r - \tilde{c}_r) \exp \int_r^s \ln\left(1 + i_q\right) dq\, dr\right]^{-\gamma} .$$

Thus,

$$E \lim_{s\to\infty}\left[\bar{c} - \theta \int_0^s (\hat{c}_r - \tilde{c}_r) \exp \int_r^s \ln\left(1 + i_q\right) dq\, dr\right] = E \lim_{s\to\infty}\left[\exp \int_0^s \left(i_q - \theta\right) dq\right]^{\frac{1}{\gamma}} \tilde{c}_0 = 0,$$

since $E \lim_{s\to\infty} \exp \int_0^s \left(i_q - \theta\right) dq = 0$ due to $E \lim_{t\to\infty} i_t = E \lim_{t\to\infty} f'(k_t) = \theta$.

Hence, $E\bar{c} = E \lim_{s\to\infty} \theta \int_0^s (\hat{c}_r - \tilde{c}_r) \exp \int_r^s \ln\left(1 + i_q\right) dq\, dr$. Since $\lim_{s\to\infty} \exp \int_0^s \ln\left(1 + i_q\right) dq = \infty$

and $E\,\frac{d\tilde{c}_t}{dt} < 0$ by the Corollary 1, then

if $E \lim_{s\to\infty}(\hat{c}_0 - \tilde{c}_0) > 0$, $E\bar{c} = \lim_{s\to\infty} \theta \int_0^s (\hat{c}_r - \tilde{c}_r) \exp \int_r^s \ln\left(1 + i_q\right) dq\, dr = \infty$, and

if $E \lim_{s\to\infty}(\hat{c}_0 - \tilde{c}_0) < 0$, $E\bar{c} = \lim_{s\to\infty} \theta \int_0^s (\hat{c}_r - \tilde{c}_r) \exp \int_r^s \ln\left(1 + i_q\right) dq\, dr = -\infty$. This contradicts $0 < E\overline{c} < \infty$ because $E\overline{c}$ is the new steady state consumption. Hence, $E \lim_{s\to\infty}(\hat{c}_0 - \widetilde{c}_0) = 0$ and thus $\lim_{s\to\infty} \widetilde{c}_0 = \hat{c}_0$.

Similarly, it is proved in case of $\gamma = 1$ ■

15. Proof of Proposition 8:

It is shown in the proof of the Proposition 7 that in case of $\gamma \neq 1$, $E\,\overline{c} = E \lim_{s\to\infty} \theta\int_0^s (\hat{c}_r - \widetilde{c}_r)\exp\int_r^s \ln(1 + i_q)\,dq\,dr$. Since for any period k $(<\infty)$

$\lim_{s\to\infty} \exp\int_k^s \ln(1 + i_q)\,dq = \infty$ and $E\,\dfrac{d\widetilde{c}_t}{dt} < 0$ by the Corollary 1, then

if $E \lim_{s\to\infty}(\hat{c}_k - \widetilde{c}_k) > 0$, $E\overline{c} = \lim_{s\to\infty} \theta\int_k^s (\hat{c}_r - \widetilde{c}_r)\exp\int_r^s \ln(1 + i_q)\,dq\,dr = \infty$, and

if $E \lim_{s\to\infty}(\hat{c}_k - \widetilde{c}_k) < 0$, $E\overline{c} = \lim_{s\to\infty} \theta\int_k^s (\hat{c}_r - \widetilde{c}_r)\exp\int_r^s \ln(1 + i_q)\,dq\,dr = -\infty$. This

contradicts $0 < E\overline{c} < \infty$ because $E\overline{c}$ is the new steady state consumption. Hence, for any period k $(<\infty)$ $E \lim_{s\to\infty}(\hat{c}_k - \widetilde{c}_k) = 0$ and thus $\lim_{s\to\infty} E(\widetilde{c}_t) = E(\hat{c}_t)$.

Similarly, it is proved in case of $\gamma = 1$. ■

16. Proof of Corollary 2:

By the proposition 6, $u'(\widetilde{c}_0) = E\,\exp\int_0^s (i_q - \theta)\,dq\,u'(\overline{c} - \overline{a})$.

$$\frac{du'(\widetilde{c}_0)}{ds} = E\,\frac{d\left[\exp\int_0^s (i_q - \theta)\,dq\,u'(\overline{c} - \overline{a})\right]}{ds}$$

$= E(i_s - \theta)\exp\int_0^s (i_q - \theta)\,dq\,u'(\overline{c} - \overline{a}) < 0$ since $E(i_s - \theta) < 0$, where $i_s = \lim_{t\to s} i_t$.

Hence, $\dfrac{d\widetilde{c}_0}{ds} > 0$.

Here, suppose that there is $s^{\#}$ such as $\hat{c}_0 < \widetilde{c}_0$ if $s = s^{\#}$, then in case of $s > s^{\#}$ there must be some periods where $\frac{d\tilde{c}_0}{ds} < 0$, because $\lim_{s \to \infty} \tilde{c}_0 = \hat{c}_0$ by the proposition 7. This contradicts $\frac{d\tilde{c}_0}{ds} > 0$ that is proved above. Hence, there is not $s^{\#}$ such as $\hat{c}_0 < \tilde{c}_0$ if $s = s^{\#}$, and thus $\hat{c}_0 > \tilde{c}_0$. ■

17. Proof of Corollary 3:

By the proposition 6, $u'(\tilde{c}_0) = E \exp \int_0^s (i_q - \theta)\, dq\; u'(\bar{c} - \bar{a})$. In case of $\gamma \neq 1$,

$u'(\tilde{c}_0) = u'(\bar{c}_0) = \bar{c}_0^{\,-\gamma} = E \exp \int_0^s (i_q - \theta)\, dq\; (\bar{c} - \bar{a})^{-\gamma}$. Thus

$$E\gamma \ln\left(\frac{\bar{c} - \bar{a}}{\bar{c}_0}\right) = E\int_0^s (i_q - \theta)\, dq \,.$$

Similarly, it is proved in case of $\gamma = 1$. ■

Figure 1

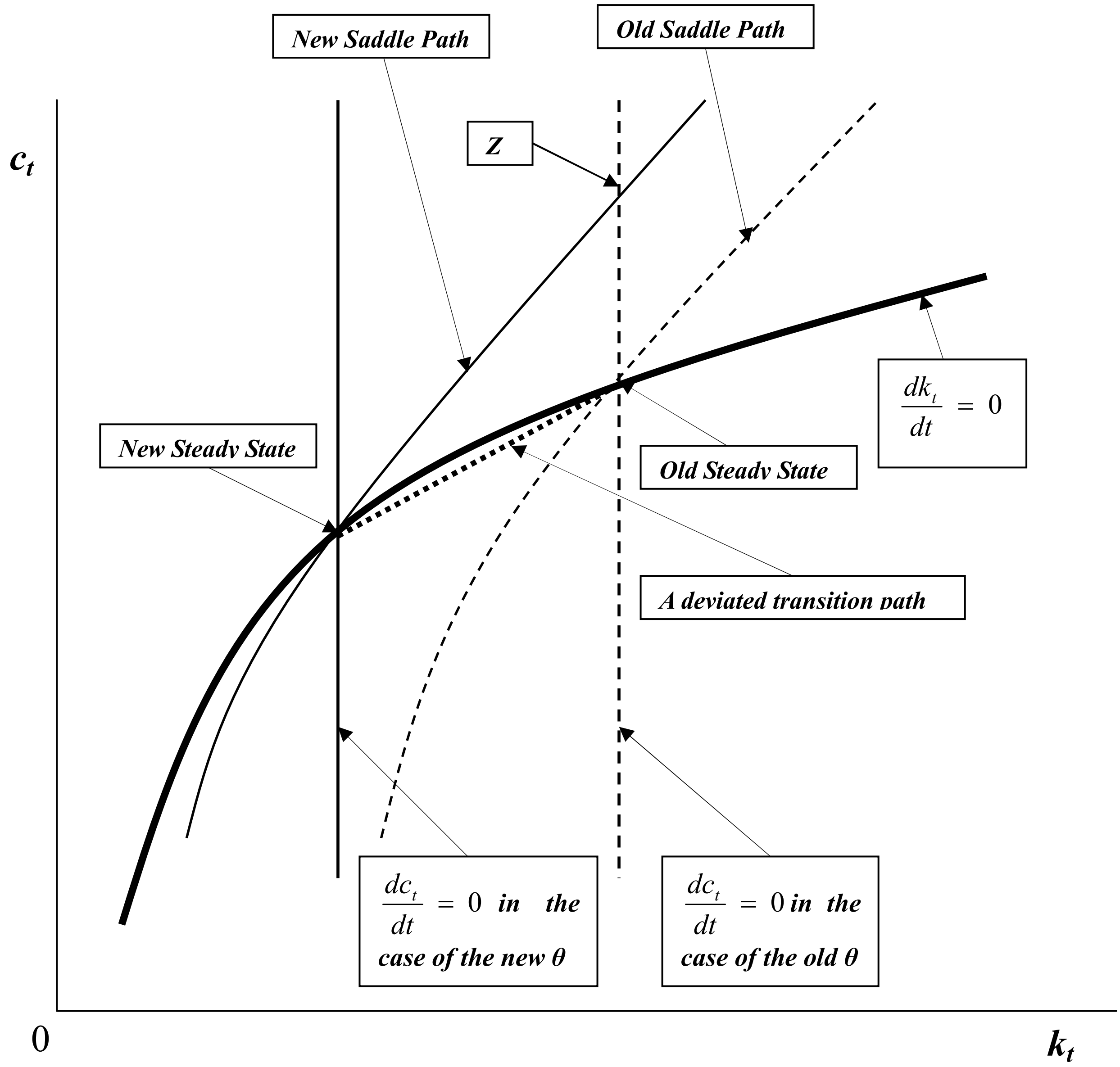
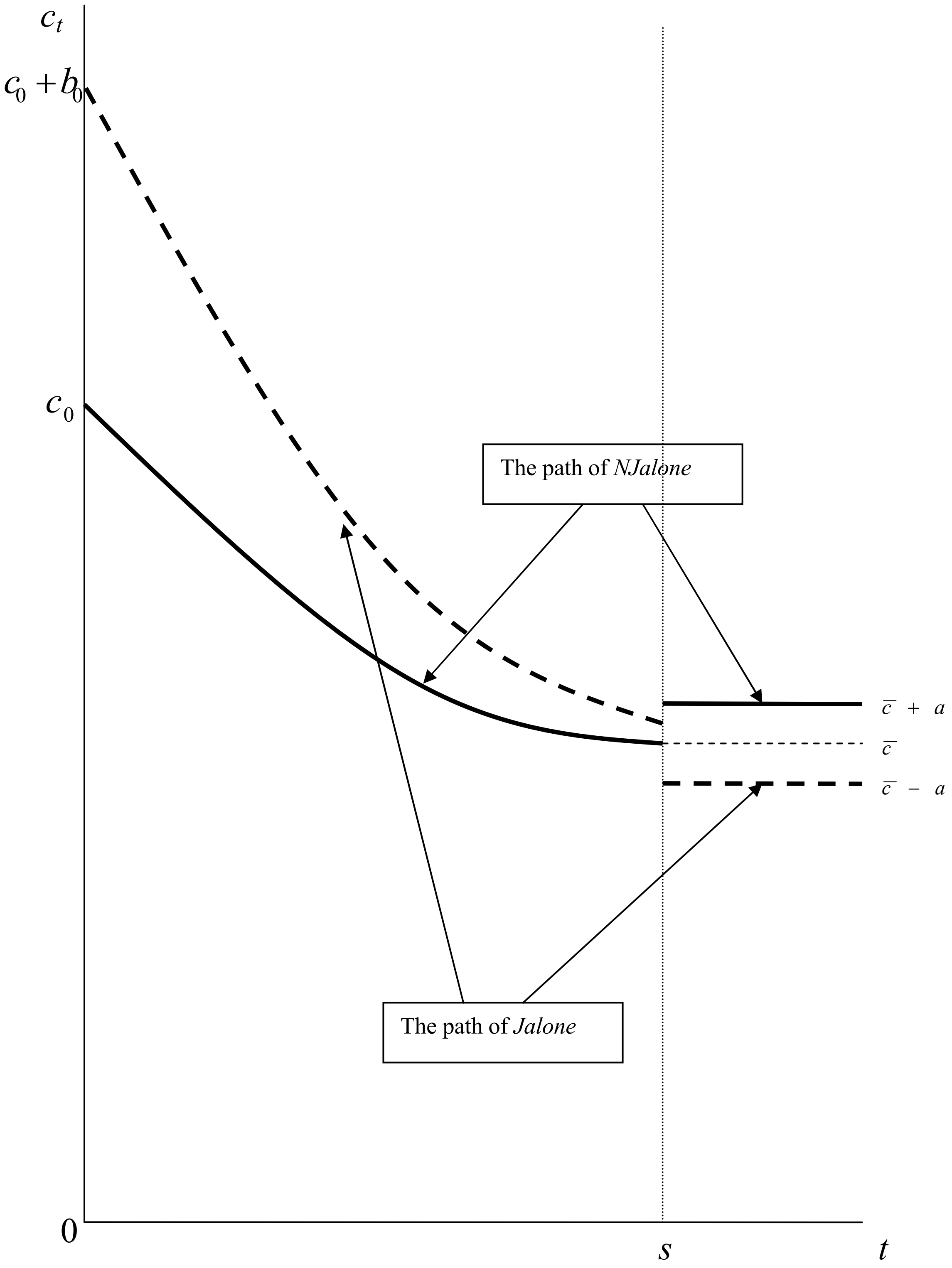

New Saddle Path
Old Saddle Path
c_t
Z
$\frac{dk_t}{dt} = 0$
New Steady State
Old Steady State
A deviated transition path
$\frac{dc_t}{dt} = 0$ in the case of the new θ
$\frac{dc_t}{dt} = 0$ in the case of the old θ
0
k_t

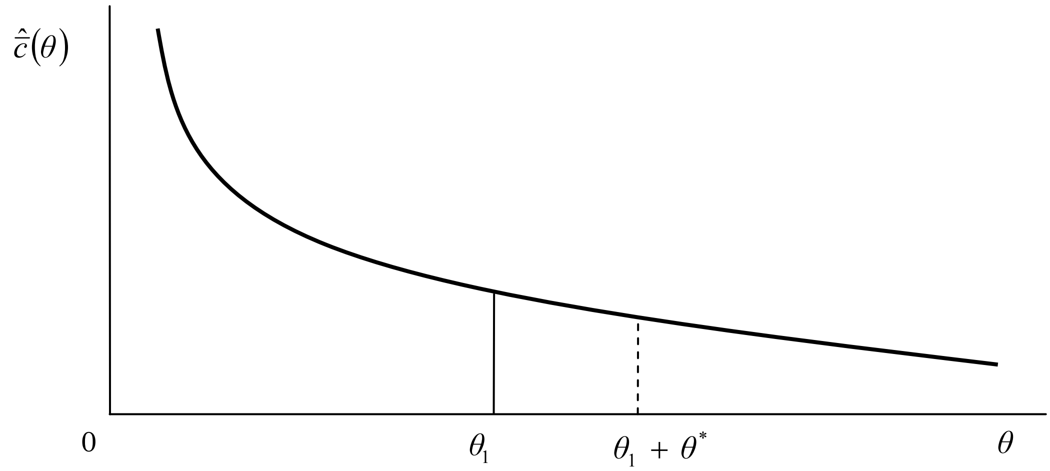
Figure 2



c_t
$c_0 + b_0$
c_0
The path of *NJalone*
The path of *Jalone*
$\bar{c} + a$
$\bar{c}$
$\bar{c} - a$
0
s
t

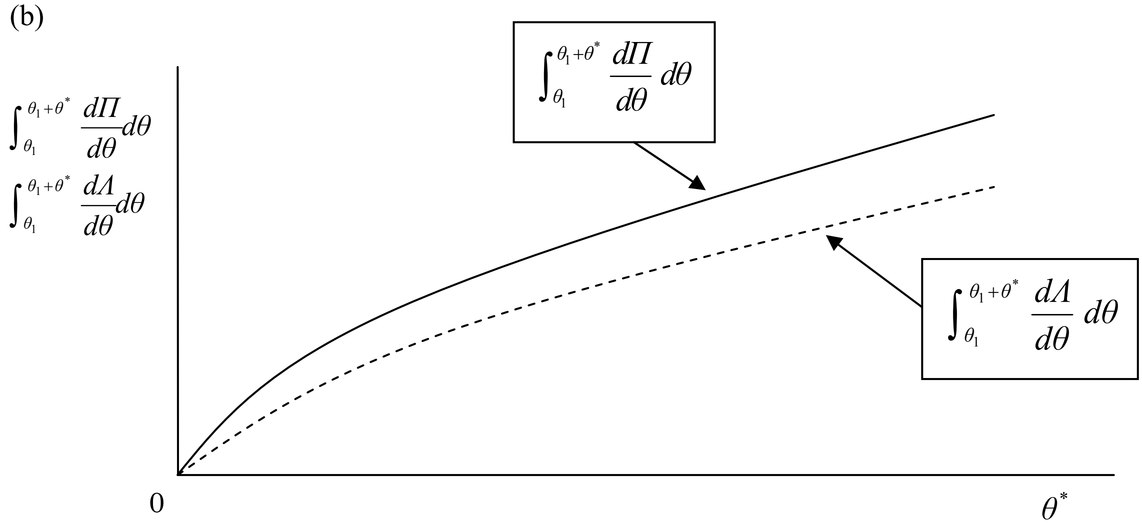
Figure 3

(a)



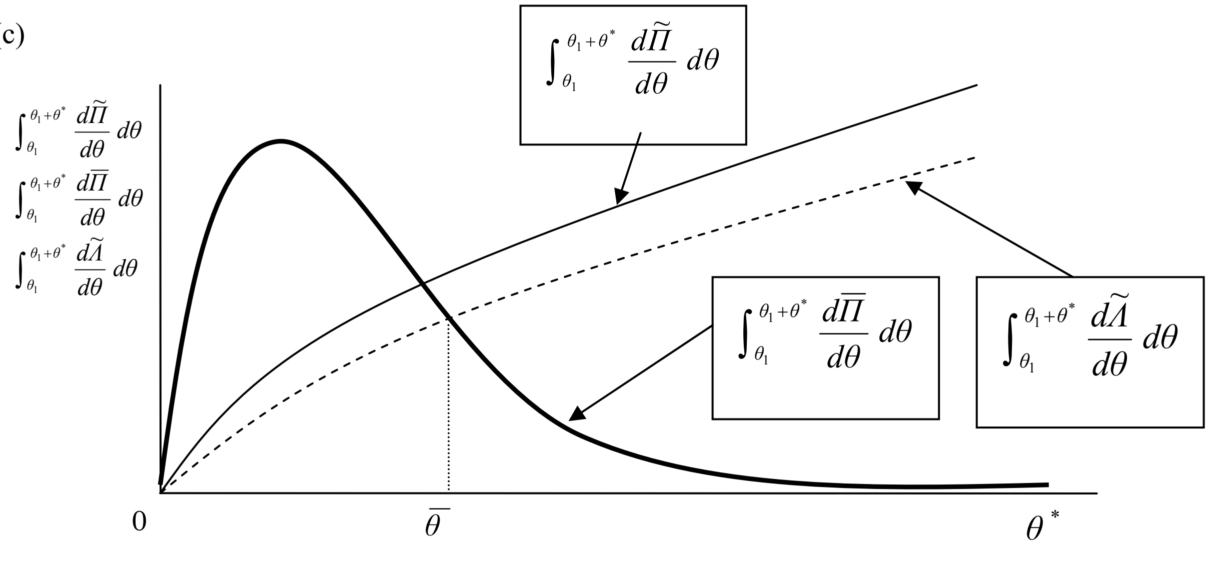
$\hat{c}(\theta)$
0
θ_1
$\theta_1 + \theta^*$
θ

(b)



$\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\Pi}{d\theta} d\theta$
$\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\Lambda}{d\theta} d\theta$
$\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\Pi}{d\theta} d\theta$
$\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\Lambda}{d\theta} d\theta$
0
θ^*

(c)



$\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\tilde{\Pi}}{d\theta} d\theta$
$\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta} d\theta$
$\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\tilde{\Lambda}}{d\theta} d\theta$
$\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\tilde{\Pi}}{d\theta} d\theta$
$\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\overline{\Pi}}{d\theta} d\theta$
$\int_{\theta_1}^{\theta_1+\theta^*} \frac{d\tilde{\Lambda}}{d\theta} d\theta$
0
$\overline{\theta}$
θ^*