ADDITIVE AND MULTIPLICATIVE UNCERTAINTY REVISITED:
What explains the contrasting biases?

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ABSTRACT

This note addresses the reasons why additive and multiplicative demand uncertainty produce differently signed biases in output price as compared to the certainty case in two-period monopoly models. With multiplicative uncertainty the price should be set *above* the certainty level while for additive uncertainty the price should be *lower* than the certainty level. This note gives an intuitive explanation for the result after first presenting a parsimonious review of the two models. We also discuss which, if either, of the two models is more realistic.

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1. Introduction

Investment under uncertainty has recently been dominated by real option models that stress the value of waiting or the value of flexibility. This contrasts with more traditional two-period models where convexity of the marginal profit under risk was argued to bias decisions (Abel 1983); or where the bias arose from a discontinuity in the profit function in the fix-price capacity-rationing case (Nickell 1978; Aiginger 1987; Lambert and Mulkay 1990). In our view, two-period models involving bets rather than options are still important in many economic applications.

To motivate the paper, imagine that you are the organiser of a concert or exhibition where the floor space or capacity has to be decided in advance of the show and where the price is also set in advance. The demand is only known stochastically. How should price and capacity be set? Should the price of tickets be higher or lower than under certainty?[1]

It turns out that the nature of the bias that is introduced depends radically on the type of demand uncertainty that is assumed. With multiplicative uncertainty the price should be set *above* the certainty level while for additive uncertainty the price should be *lower* than the certainty level. Most authors assume one or the other form of uncertainty in their models, e.g. additive (Pennings 2001) or multiplicative (Driver et al 1993; 1996). And although the result may be traced back to the classic edited collection of Arrow et al (1962), it has never been satisfactorily explained why the apparently minor change in specification of stochastic demand can have such radical implications for the bias to price under uncertainty.[2] Thus this note also serves as a warning to those working on uncertainty that results may be very sensitive to exact specifications.

[1] Other contexts where the model applies include manufactured items where capacity is decided in advance and for marketing reasons the price has to be announced and held whatever the demand (new cars, seasonal toys and products that become obsolete at the end of a period are sometimes cited as examples). This context - where both price and quantity or capacity have to be set *ex-ante* - is regarded by firms as highly relevant. See the discussion of “P-Q models” in Aiginger 1987, p.163-7; Karlin and Carr 1962; Driver et al 1993; 1996. See also Dana and Petruzzi (2001) for a recent contribution that extends the classic newsvendor problem by introducing an exogenous outside option for consumers.

[2] We focus on the price results rather than the bias that is caused to capacity under uncertainty. This is because the results for price are quite general whereas the result for capacity in the multiplicative case requires some restrictions on the demand curve (Driver et al 1993).

2. Formal derivation of the results

Let us denote by $D_0(p)$ the certainty demand curve. In the multiplicative case, certainty demand is multiplied by a stochastic shift term; in the additive case, a stochastic term is added to the certainty demand curve. Realised demand is in the two cases:

Additive (a-model)

$$D(p) = D_0(p) + a$$

$$a \in [\underline{a}, \overline{a}] \text{ with c.d.f. } F(a)$$

$$\int_{\underline{a}}^{\overline{a}} a \mathrm{d}F(a) = 0$$

Multiplicative (k-model)

$$D(p) = kD_0(p)$$

$$k \in [\underline{k}, \overline{k}] \text{ with c.d.f. } F(k)$$

$$\int_{\underline{k}}^{\overline{k}} k \mathrm{d}F(k) = 1$$

A risk-neutral monopolist sets the price p and the capacity Y before uncertain demand is realised. Capacity costs c per unit. Denoting by X expected sales, we have:

Additive

$$X(p,Y) = \int_{\underline{a}}^{Y-D_0(p)} [D_0(p) + a] \mathrm{d}F(a) +$$

$$Y[1 - F(Y - D_0(p))] = Y - \int_{\underline{a}}^{Y-D_0(p)} F(a) \mathrm{d}a$$

Multiplicative

$$X(p,Y) = \int_{\underline{k}}^{Y/D_0(p)} kD_0(p) \mathrm{d}F(k) +$$

$$Y[1 - F(Y/D_0(p))] = Y - D_0(p) \int_{\underline{k}}^{Y/D_0(p)} F(k) \mathrm{d}k$$

In each case we define profit and the two first-order conditions:

$$\Pi = pX(p,Y) - cY$$

$$\partial \Pi / \partial p = X + p \partial X / \partial p = 0 \qquad (1)$$

$$\partial \Pi / \partial Y = p \partial X / \partial Y - c = 0 \qquad (2)$$

2.1 Additive vs multiplicative: expected sales and the incidence of rationing

We now explain the contrasting results between the models.[3] It is useful to make a change of variable. In the additive case, define $z^a = Y - D_0(p)$ as the (additive) *capacity stance level*, which represents the planned margin of spare capacity or the difference between the set capacity and expected demand. In the multiplicative case, define $z^k = Y / D_0(p)$ as the multiplicative *capacity stance ratio*. Since actual capacity at a given price can be set below or above unconstrained expected demand, it follows that z^a can be either negative of positive, and z^k can be lower or greater than 1. Expected sales (X) is then transformed into:

Additive

$$X(p,Y)=\int_{\underline{a}}^{\bar{a}}\min[Y,D(p)]\mathrm{d}F(a)=$$

$$D_0(p)-\int_{Y-D_0(p)}^{\bar{a}}[D(p)-Y]\mathrm{d}F(a)$$

$$X(p,z^a)=D_0(p)-\int_{z^a}^{\bar{a}}(a-z^a)\mathrm{d}F(a)$$

Multiplicative

$$X(p,Y)=\int_{\underline{k}}^{\bar{k}}\min[Y,D(p)]\mathrm{d}F(k)=$$

$$D_0(p)-D_0(p)\int_{Y/D_0(p)}^{\bar{k}}[D(p)-Y]\mathrm{d}F(k)$$

$$X(p,z^k)=D_0(p)-D_0(p)\int_{z^k}^{\bar{k}}(k-z^k)\mathrm{d}F(k)$$

From the final twin expressions above it is obvious that expected sales will be lower than (unconstrained) certain demand. So there will be a demand loss due to rationing. But in the additive case the loss depends only on z^a, while, for a given z^k, in the multiplicative case the loss will be lower at higher prices. This gives a first hint: in a sense, the pricing decision may be used in the k-model to "reduce" uncertainty by setting high prices, while this does not happen with the a-model (for a given z^a). [4]

2.2 Additive vs multiplicative: profit maximisation

To derive formal results we focus on the profit functions. For now onwards, to save on notation, we simply refer to z with no superscript, knowing that it represents a level and a ratio in the two cases. Define expected profit:

Additive

$$\Pi(p,z)=pX(p,z)-c(z+D_0(p))=\underbrace{(p-c)D_0(p)}_{A}-\underbrace{cz}_{B}-\underbrace{p\int_{z}^{\bar{a}}(a-z)\mathrm{d}F(a)}_{C} \qquad (3)$$

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[3] For the reader who prefers to see a graphical presentation, the multiplicative case is drawn and discussed in Appendix 1.

[4] Alternatively, we may see this by contrasting the two expressions for expected sales X given before (1) above. We note that the level difference (Y - X) conditional on a given $F(\cdot)$ depends on p (via D_0) in the multiplicative case but not in the additive case. This (Y - X) term denotes the expected quantity of rationing for a given $F(\cdot)$. In the multiplicative case (but not the additive one) a higher p leads to a lower level of expected rationing for any given $F(\cdot)$.

Hence, the expected profit can be conveniently decomposed into the sum of different terms. The intuitive explanation for these terms may be simply explained. The first term (*A*) coincides with the unconstrained profit under certainty, that is maximised at the certainty price p^c, obtained from:

$$D_0(p^c) + (p^c - c)D_0'(p^c) = 0 \tag{4}$$

The second term (B) represents the loss from carrying precautionary capacity of quantity *z* (the difference between capacity and expected demand) at unit cost *c*. The third term (C) is the expected value of lost sales due to capacity rationing that occurs whenever $a > z$.

Multiplicative

$$\Pi(p,z) = pX(p,z) - czD_0(p) = \underbrace{(p-c)D_0(p)}_{A} - \underbrace{cD_0(p)(z-1)}_{B} - \underbrace{pD_0(p)\int_z^{\bar{k}}(k-z)\mathrm{d}F(k)}_{C} \tag{5}$$

Here the first term (A) is identical to the additive case. The remaining terms have parallel interpretations to the additive case. The second term (B) represents the expected cost of carrying precautionary capacity of amount $Y - D_0 = D_0(z - 1)$. The third term (C) represents, as in the additive case, the value of the lost sales due to rationing where the averaging is now over the interval where $k > z$. Using $z = 1 + \int_{\underline{k}}^{z}(z-k)\mathrm{d}F(k) - \int_z^{\bar{k}}(k-z)\mathrm{d}F(k)$, it is convenient to rewrite eq. (5) as:

$$\Pi(p,z) = \underbrace{(p-c)D_0(p)}_{A} - \underbrace{(p-c)D_0(p)\int_z^{\bar{k}}(k-z)\mathrm{d}F(k)}_{B'} - \underbrace{cD_0(p)\int_{\underline{k}}^{z}(z-k)\mathrm{d}F(k)}_{C'} \tag{5bis}$$

2.3 Optimal capacity stance, z

The problem that we now study is the maximisation of the expression of expected profit with respect to *p* and *z*. This is of course a simultaneous choice, but let us do it in two steps. Imagine that the price is somehow fixed, and consider the optimal choice of *z*. We have:

Additive | *Multiplicative*

$\partial\Pi / \partial z = -c + p(1 - F(z)) = 0$ $\qquad$ $\partial\Pi / \partial z = D_0(p)[-c + p(1 - F(z)] = 0$

Hence, in both cases the FOC w.r.t. the capacity stance can be re-written as:

$$pF(\cdot) = p - c \tag{6}$$

This is the solution to the traditional newsvendor problem, where price is exogenously set. There is *no* difference between the a-model and the k-model when the price is treated as an exogenous variable.

The FOC w.r.t. z (eq. (6)) has an intuitive interpretation (see also Fig. 1). The gain to revenue stemming from an increment to Y (or z) will only occur if the firm is already capacity constrained (probability = 1 – $F(\cdot)$) but the cost incurred (c) is unconditional. Expected value of the increment to Y should be zero at the optimal Y: $(p - c)(1 - F(\cdot)) - cF(\cdot) = 0$, which is the same as (6).

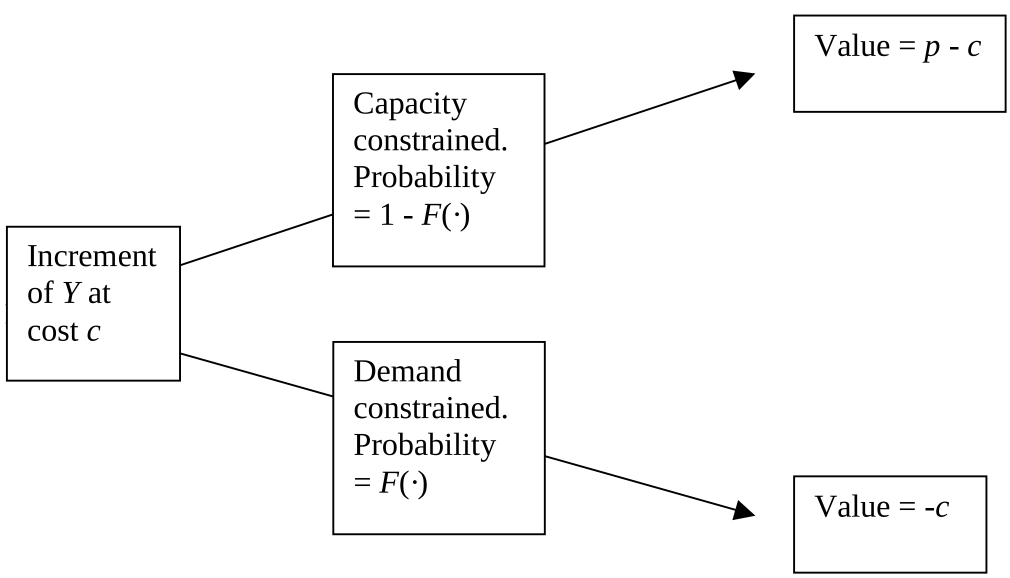

Figure 1 - Optimisation requires $(p - c)(1 - F(\cdot)) - cF(\cdot) = 0$ (eq. (6))

2.4 Optimal price, p

In contrast to the above, for a given *z*, the a-model and the k-model exhibit fundamental differences w.r.t. the choice of the price. This can be seen by direct inspection of eq. (3) and eq. (5bis).

In the a-model, expected sales is split into riskless demand minus a loss term that is not sensitive to price changes. When we turn to the profit, this is equal to the riskless profit (term A, maximized at p^c), minus a "sunk cost" to carry extra-capacity (term B, this comes with its algebraic sign), minus the *value* of the demand loss that is lower the lower is the price (term C). Overall, the firm will want for sure to set a price below the monopoly certainty price.

In the k-model, once the capacity stance is fixed, there is no first-order effect from the loss term given by term B' (starting from the certainty price), while term C' is minimised for very *high* prices. The effect of the high price here is to reduce the likelihood of rationed demand. Hence there is a natural tendency to push the price above the certainty level, for any value of the capacity stance. The effect here is due to lower precautionary capacity at higher prices in the multiplicative case as the expected level of planned excess capacity decreases while the ratio of planned excess capacity to expected demand remains constant.

The above intuitive account can be complemented by expressing the derivative of profit with respect to price. After some manipulation we obtain from eq. (3) and (5bis) the contrasting expressions below which show the opposite signed effects:

Additive

$$\left.\frac{\partial \Pi}{\partial p}\right|_{p=p^c} = -\int_z^{\bar{a}} [1 - F(a)]\mathrm{d}a < 0 .$$

Multiplicative

$$\left.\frac{\partial \Pi}{\partial p}\right|_{p=p^c} = -cD_0'(p)\int_{\underline{k}}^{z} F(z)\mathrm{d}a > 0 .$$

2.5 Alternative treatment of the first order condition w.r.t. p

It is also interesting, if somewhat circuitous, to contrast the FOC w.r.t. *p* in terms of expected sales for a given *Y*. From eq. (1), the derivative $\partial X/\partial p$ may be derived as follows:

Additive

$$\partial X/\partial p = D_0'(p)F(z) \tag{7}$$

Multiplicative

$$\partial X/\partial p = D_0'(p)F(z)z - D_0'(p)\int_{\underline{k}}^{z} F(k)\mathrm{d}k = D_0'(p)\int_{\underline{k}}^{z} k\mathrm{d}F(k) \tag{8}$$

In the multiplicative case, $\partial X/\partial p$ is lower (in absolute terms) than the additive corresponding term $D'_0F(\cdot)$. In particular, $D_0'(p)\int_{\underline{k}}^{z} k\mathrm{d}F(k)$ is a weighted average over the unconstrained (lower demand) regime, with mean (unconstrained) weight equal to unity. In the k-model, the unconstrained states are those states when demand has higher absolute elasticity than the average, which explains the tendency to charge higher prices.

For a formal proof for the k-model, we can manipulate (1) as follows:

$$\begin{aligned}
&\partial \Pi/\partial p = X + [YF(\cdot)-(Y-X)]D_0' p/D_0 = X + D_0'(p-c)Y/D_0 - (Y-X)D_0' p/D_0 = \\
&= X + (D_0' p/D_0)X - (D_0' c/D_0)Y > X + (D_0' p/D_0)X - (D_0' c/D_0)X = \\
&= [D_0 + D_0'(p-c)]X/D_0
\end{aligned} \tag{9}$$

Using the certainty condition for maximising profit (eq. (4)), it is immediate to see that the last term in the square bracket in eq. (9) is equal to zero when evaluated at the certainty price p^c. This implies that, as before, price is *higher* than the certainty price under multiplicative uncertainty.

We give below a parallel alternative proof for the additive case that the uncertainty price is lower than the certainty price. Using (7) and substituting (6) into (1) gives: $X(p,Y)+D_0'(p)(p-c)=0$ which, when evaluated at the certainty price, may be compared with the certainty condition for maximum profit, equation (4). As proved earlier, expected sales must be less than certainty unconstrained demand: $X(p^c,Y)-D_0(p^c)<0$. Thus, expected marginal profit w.r.t. *p* under additive uncertainty is less than zero at the certainty price p^c and price should accordingly be *lower* under this form of uncertainty.

3. Which model is more realistic?

Given the contrasting results of the two models it seems important to consider which, if either, has a better claim to realism. This question was considered in Aiginger (1987, p. 166) but he concluded that there was no rational economic basis to choose between the specifications. In this final section we offer a slightly different perspective. We consider constant elasticity demand curves and note that the conjectured price elasticity of demand may remain invariant to the realisation of high or low demand schedules only in the multiplicative case. In the additive case, by contrast, the (absolute) elasticity is lower in the case of high realised demand than under certainty.[5] This makes the elasticity anti-cyclical, even under monopoly. Some industrial economists have argued that in different market structures, thin-market effects or easier collusion in recessions should make the elasticity pro-cyclical, while others argue for an a-cyclical or pro-cyclical mark-up (for this debate, see Bloch and Olive 2001, Haskel et al 1995, Rotemberg and Saloner 1986). Even under monopoly there may be different views as to the cyclicality of the elasticity and one's modelling strategy in setting up the stochastic demand equations may be influenced by one's beliefs in respect of that. Those who want to hedge their bets can do so by choosing the multiplicative model where the effect can be neutral.

[5] To see this, consider the intersection of each parallel realised demand curve with the ray P = Q in a standard demand curve setting in P-Q space. The slopes of the demand curves at these intersections measure inverse demand elasticities. A conjectured translation to the right of the demand curve (say from central demand D_0 to high realized demand D_1) corresponds to conjectured upward cyclical movement. Since the derivative of P w.r.t. Q is higher at this intersection of the ray than at the central demand intersection, the absolute elasticity of demand is lower at high realised demand.

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Appendix 1. Graphical presentation

Both eq. (1) and eq. (2) in the text can be represented in a diagram. Fig. A1 plots a graphical solution for the multiplicative case.[6] The thick line represents expected sales *X* for a given *Y*. Clearly, *X* coincides with $D_0(p)$ when prices are so high that there is *never* rationing. *X* starts departing from certainty demand when prices are lowered such that $\bar{k}D_0(p) \geq Y$, where the limiting condition is obtained, in graphical terms, as the resulting price when *Y* hits the highest possible realization of demand (in the figure, this happens for prices below .705). We can also quite easily anticipate that *X* has to coincide with *Y* when prices are so low that demand is *always* rationed, in all possible states, i.e. $\underline{k}D_0(p) \geq Y$. In graphical terms, the limiting price is obtained as the resulting price when *Y* hits the lowest possible realized demand (prices below .114 in the figure).

Similarly, marginal revenue coincides first with certainty marginal revenue, then the two curves depart when sales start being capacity constrained (eq. (1)). Notice that for the parameter constellation chosen in Fig. A1, the plot of marginal revenue under certainty coincides with the realization in the lowest state $\underline{k}D_0(p)$. This was simply done to keep the number of curves drawn in the diagram at a minimum and to prevent cluttering the diagram.

As far as optimality w.r.t. *Y* is concerned, when the firm is not capacity constrained (which happens for high prices), an increase in *Y* would not cause a change in *X*. As the price is reduced, *X* becomes more and more sensitive to an increase in *Y*, which explains the increasing curve depicting eq. (2). At equilibrium, the marginal revenue w.r.t. price is set equal to zero (eq. (1)) and, simultaneously, capacity is optimally set where the marginal cost *c* is equal to the marginal benefit (eq. (2)). The corresponding optimal price for this multiplicative example is denoted by p^m. The graphical representation for the additive case would show similar behavior of the curves.

[Figure A1]

[6] Fig. 1 is drawn for the following specification: $D_0(p) = 1 - p$, $c = 0.2$, *k* uniformly distributed between 0.5 and 1.5. The certainty price is $p^c = 0.6$. The optimal price and capacity under uncertainty are respectively $p^m = 0.624346$ and $Y = 0.443145$.