The relationship between Mathematical Utility Theory and the Integrability Problem: some arguments in favour

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[*] The work of J. C. R. Alcantud and C. Rodríguez-Palmero has been partially supported by Ministerio de Ciencia y Tecnologa under the Research Project BEC2002-02456, and by Junta de Castilla y Len under the Research Project SA061/02.

Abstract.

The resort to utility-theoretical issues will permit us to propose a constructive procedure for deriving a homogeneous of degree one, continuous function that gives raise to a primitive demand function under suitably mild conditions. This constitutes the first elementary proof of a necessary and sufficient condition for an integrability problem to have a solution by continuous (subjective utility) functions.

Such achievement reinforces the relevance of a technique that was succesfully formalized in Alcantud and Rodríguez-Palmero [3]. The analysis of these two works exposes deep relationships between two apparently separate fields: mathematical utility theory and the revealed preference approach to the integrability problem.

JEL classification: D11

Key words: Strong Axiom of Homothetic Revelation; revealed preference; continuous homogeneous of degree one utility; integrability of demand.

1 Introduction

In this work we intend to defend the validity of a procedural proposal for solving problems in a realm with far-reaching implications, but whose development hinges on a number of varied and highly complex techniques. We purpose showing that there are close links between two apparently disconnected branches of the literature, that permit to derive a powerful yet simple way to approach a rather obscure though trascendental bulk of economic knowledge.

The general setting where we can frame our study is the search for answers to two main questions. In the first place: what does the neoclassical model of consumer behavior (consumers' choices are described as deriving from maximizing a utility, or perhaps only a preference, subject to a budget constraint) impose on observed, actual behavior? The approach followed by the authors above originates in classical works by Samuelson [25], [26] and involves algebraic conditions on the demand functions that relate to the maximizing pattern of behavior. A quite different approach is exemplified by the pioneering work of Antonelli [4] and Slutsky [28], where the derivatives of the demand function have a preeminent role.

Conversely, the integrability problem consists of explaining observed demand according to that model. The two approaches above provide different insights into that question, which are exemplified e.g. in, on the one hand -*revealed preference*-, Uzawa [31], Stigum [30] or Mas-Colell [22], and, on the other hand -*integrability theory*-, Hurwicz and Uzawa [16]. The (partial) answers provided by a number of well-known classical studies are highly ellaborated, technical, and lengthy when (semi)continuous utilities are involved, which caused a significant number of corrections and arguments in the specialized literature. Perhaps the only elementary approaches to the problem are Sondermann [29], which inspires on Hurwicz and Richter [15], as well as Alcantud and Rodríguez-Palmero [3], that inspires on both works. Neither of them reaches full continuity of the subjective utilities that explain the demand; nor have they provided necessary and sufficient conditions. The novelty there was the use of techniques that arise from mathematical utility theory. That possibility was fully exploited in Alcantud and Rodríguez-Palmero [3], where a procedural proposal is explicited in order to benefit most from the appeal to arguments from utility theory. That method consists of (a) transforming the integrability question into one of representability of the revealed preference by weak utilities, and then (b) study this naturally associated binary relation in order to check that it does admit adequate (e.g. semicontinuous) utilities. This systematized approach permitted to give simple and general answers to the integrability problem under research, gaining insight into the nature of the topic and obtaining Sondermann's result as a corollary, as well as other new sufficient conditions for integrability of the demand.

The success of that proposal is not restricted to those achievements regarding the general problem, and of course it might eventually lead to full characterizations in specific realms. That possibility is essentially vetoed in the initial problem, as the negative example in the appendix of Mas-Colell [22] shows; nonetheless, necessary and sufficient conditions for integrability by continuous functions for a class of demand functions that satisfy a list of widely-used axioms is achieved in Fuchs-Seliger [14].

In particular, in recent years there has been a renewed interest in, on the one hand, the structure of homothetic preferences -Dow and Werlang [12], Maccheroni [20]- and their relationship with homogeneous representations of them -Candeal and Induráin [9], Bosi, Candeal and Induráin [7], Liu and Wong [19, Appendix A]-; and, on the other hand, different issues around the representability of demand functions by homothetic preferences or homogeneous (subjective) utilities -Knoblauch [18], Liu and Wong [19]-. Applications justify the interest in the homothetic case. Castagnoli and Maccheroni [10] digress on the relevance of homothetic preferences in non-expected utility theories. The fact that we can deduce the agent's entire preference from a single indifference class simplifies the analysis of a good number of problems (e.g. of econometric nature) and justifies the wide use of special types of homothetic preferences (e.g. in the theories of price and quantity indices). The demand functions that derive from homothetic preferences are homothetic of degree one in income, which are those demand funtions for which the strong axiom of revealed preference aggregates by a fixed distribution of income (cf. Shafer [27, pp. 1177-

8]) when they satisfy the budget equality. With respect to representability of demand by homogeneous functions or, more generally, by homothetic preferences, a key reference is Liu and Wong [19]. It solves the problem quite in full, being especially remarkable as the authors realize that, whithin an axiomatic framework where all utilities are homogeneous of degree one, necessary and sufficient conditions for continuous integrability can be given under very mild assumptions -something virtually unreachable in the general setting, as we have argued above. However, their technique is extremely ellaborated, involving first-order predicate languages and extensions of fields, which are not in the toolkit of the average researcher in Economics. The Strong Axiom of Homothetic Revelation (SHA) that Liu and Wong introduce is their main analytical tool, and probably deserves further attention. Previously, a quite similar solution to the problem had been achieved in Fuchs-Seliger [13] along her study of aggregation properties of demand functions under a fixed distribution of income.

The aim of this paper is twofold. Firstly, we propose an elementary approach to the problem of representing a consumer's demand by means of an homogeneous function. Our procedure is *constructive*, in contrast with the highly abstract technique available so far. In order to do so, we first need to put forward some technical, auxiliary results that concern the representability of homothetic preferences, and only after this preliminary analysis we will focus on the particular case of the preference revealed by a demand function. The final result will consist of a constructive procedure to derive a *homogeneous of degree one and continuous* function that gives raise to a given demand function under suitably mild new conditions.

Secondly, we value the conclusion that the resort to purely utility-theoretical issues in order to implement new technical results in the research into the integrability problem is proving to be quite succesful and promising. A different token in support of this research scheme is the application of the results in Alcantud and Rodríguez-Palmero [2] into Alcantud and Rodríguez-Palmero [3]. These successes might well boost the interest in fundamental questions involving integrability of demand functions and their relationship with utility-oriented problems, raising new answers from a different perspective. The success in tackling those questions depends, in a high proportion, on the structure of the original problem: we stress that the homogeneous case displays a very rich collection of properties and therefore leaves room to a quite elegant and appealing solution.

We distribute this research as follows. Section 2 focuses on the technical results that will be needed in order to solve the integrability question under inspection. Some relevant issues on the properties of the revealed homothetic preference, as well as the implications of the SHA, will be an intermediate stage from which there will be much to gain in the final part of the paper; we accomplish that task in Section 3. Finally, Section 4 integrates all these different steps, furnishing a constructive solution to the representability questions we have announced above; Theorems 2 and 3 culminate the technical research. In our view, their intrinsic relevance consists in that this is the first necessary and sufficient condition for an integrability problem to have a solution by continuous functions whose proof is elementary. We summarize the consequences of our research and some of its possible implications in Section 5. An Appendix contains the proofs of the auxiliary results.

2 Homogeneous and continuous weak utilities

In this Section we are concerned with the existence of a homogeneous of degree one and continuous weak utility for an acyclic binary relation on a real cone in a topological real vector space. To the best of our knowledge, such a topic was not previously considered in the literature. However, Bosi, Candeal and Induráin [7] presented a characterization of the existence of a homogeneous of degree one and continuous utility function for a complete preorder on a real cone in a topological real vector space. Also recently, Bosi and Zuanon [8] provided an axiomatization of the existence of a nonnegative, homogeneous of degree one and continuous order-preserving function for a not necessarily complete preorder. Further, the representation of an interval order by means of two nonnegative, homogeneous of degree one and semicontinuous real-valued functions was considered in Bosi [6]. We recall that a full axiomatization of the existence of an upper semicontinuous weak utility for an acyclic binary relation on a topological space was presented by Alcantud and Rodríguez-Palmero [2].

Our main tool will be the following theorem, where we present a characterization of the existence of a homogeneous of degree one and (upper-semi)continuous weak utility for an acyclic binary relation $\succ$ on a real cone in a topological real vector space. Afterwards, a technical *ad-hoc* Corollary will yield the particular sufficient condition which will be the key to yield an application to representing demand functions. Before that, we shall explicit the definitions involved.

A binary relation $\succ$ on a set X is said to be *acyclic* if $x_1 \succ x_2 \succ ... \succ x_n$ implies $x_1 \neq x_n$ for all $x_1, ..., x_n \in X$. Given an acyclic binary relation $\succ$ on a set X, we shall denote by $\succ\!\!\succ$ the *transitive closure* of $\succ$ (i.e., $x \succ\!\!\succ y$ if and only if there exist $x_1, ..., x_n \in X$ such that $x = x_1 \succ x_2 \succ ... \succ x_n = y$), which is a *partial order* (i.e., it is irreflexive and transitive). A subset A of a set X endowed with an acyclic binary relation $\succ$ is said to be a *lower set* if $y \succ x$ and $y \in A$ imply $x \in A$. An acyclic binary relation $\succ$ on a topological space X is said to be *tc-upper semicontinuous* if $\{z \in X : x \succ\!\!\succ z\}$ is an open set for every $x \in X$, i.e. if its transitive closure is upper semicontinuos.

If A is any subset of a real vector space A, then define, for every real number t, $tA = \{tx : x \in A\}$. We recall that a real cone X in a topological real vector space E is a subset of E such that $tx \in X$ for every $x \in X$ and $t \in \mathbb{R}^{++}$. An acyclic binary relation on a real cone X in a topological real vector space E is said to be *homothetic* if $x \succ y$ implies $tx \succ ty$ for every $x, y \in X$ and $t \in \mathbb{R}^{++}$. A real-valued function $\varphi : X \longrightarrow \mathbb{R}$ on the real cone X is *homogeneous of degree one* if $\varphi(\lambda x) = \lambda \varphi(x)$ for every $x \in X$ and $\lambda > 0$.

A *weak utility* for an acyclic binary relation $\succ$ on a set X is a real-valued function u on X such that $u(x) > u(y)$ for all $x, y \in X$ with $x \succ y$.

THEOREM 1 *Let $\succ$ be an acyclic binary relation on a real cone X in a topological real vector space E. Then the following conditions are equivalent:*

(i) *There exists a homogeneous of degree one and continuous weak utility u for $\succ$*

(ii) *There exists a countable family* $\{G_r\}_{r\in\mathbb{Q}}$ *of open lower subsets of* X *satisfying the following conditions:*

(a) $\bigcup_{r\in\mathbb{Q}} G_r = X$, $\bigcap_{r\in\mathbb{Q}} G_r = \varnothing$;

(b) $q < r$ *implies* $\overline{G_q} \subseteq G_r$ *for all* $q, r \in \mathbb{Q}$ *(*$\overline{C}$ *is the topological closure of* C*);*

(c) $qG_r = G_{qr}$ *for every* $r \in \mathbb{Q}$ *and* $q \in \mathbb{Q}^{++}$;

(d) *for every* $x, y \in X$ *such that* $x \succ\!\succ y$ *there exist* $r_1, r_2 \in \mathbb{Q}$ *such that* $r_1 > r_2$, $x \notin G_{r_1}$, $y \in G_{r_2}$.

Proof: see the Appendix.

REMARK 1 *If we seek upper semicontinuity only, then the conditions that characterize existence are (a), (b′)* $q < r$ *implies* $G_q \subseteq G_r$ *for all* $q, r \in \mathbb{Q}$*, (c) and (d). This is quite immediate from the proof above.*

The importance of semicontinuity in replacement of full continuity has been enhanced e.g. by Alcantud [1], among others, as an alternative to permitting to obtain optimality results under less restrictive conditions; the theoretical basis of this statement is enunciated e.g. in Berge [5], p. 76.

In subsequent Sections we will show that certain demand functions have an intrinsic structure that permits us to derive them from homogeneous of degree one and continuous functions. In order to ease this task, we intend to provide first a technical result that will be of application to that context. That preliminary result is the next Corollary.

COROLLARY 1 *Let* $\succ$ *be a tc-upper semicontinuous and homothetic acyclic binary relation on a real cone* X *in a topological real vector vector space* E*, and assume that the following conditions hold:*

(i) *for each non-minimal element* $x \in X$*, and for each* $y \in X$*, there is* $r > 0$ *rational such that* $rx \succ\!\succ y$*;*

(ii) *whenever* $x \succ\!\succ y$*, and for each* $\lambda \in (0, 1)$*, then* $x \succ\!\succ \lambda y$*.*

Then there is a homogeneous of degree one, upper semicontinuous weak utility u for $\succ$.

Suppose further

(iii) *there is $\overline{x} \in X$ that is not minimal for $\succ$ and such that $G_1 = \{a \in X : \overline{x} \succ\!\!\succ a\}$ satisfies: for each $q > 1$ rational, $\overline{G_1} \subseteq qG_1$.*

Then, there exists a homogeneous of degree one, continuous weak utility u for $\succ$.

Proof: see the Appendix.

REMARK 2 *Note that, provided that the primitive sets G_r are known, the proofs above are constructive and yield a computable utility $u(x)$. This observation will be key in order to justify that our final result on homogeneous representability of demand is not only an existence proof but it is also constructive.*

3 Revealed preference and homotheticity

In this Section we will perform an *ad-hoc* study of some questions related to concepts involved in the search of a solution to the homogeneous representablity problem. Formally, this is a preliminary stage, since we necessarily must have homothetic preferences that induce a given demand if a homogeneous of degree one function is to explain that demand function. In order to separate what depends on properties specific of the demand from what corresponds to general properties of more abstract models, we place ourselves in the general framework of choice structures first.

Therefore, let us fix a *choice structure* on a given set X , that is, a pair $(\mathcal{B}, c)$ where $\mathcal{B}$ is a collection of nonempty subsets of X and $c : \mathcal{B} \longrightarrow X$ is a correspondence such that $\varnothing \neq c(B) \subseteq B$ for all $B \in \mathcal{B}$. We say that the structure has univalued choices if $c(B)$ is a singleton for each $B \in \mathcal{B}$. Also, X will be a cone of a real vector space. This context permits to define the *homotheticity* of $\mathcal{R}$ binary relation on X. That concept means that $x \ \mathcal{R} \ y$ implies $(\lambda x) \ \mathcal{R} \ (\lambda y)$ for each $\lambda > 0$. *Unless otherwise stated, all choice structures will be univalued along this Section.*

If $y \neq x$ and there is a $B \in \mathcal{B}$ such that $y \in B$ and $x \in c(B)$ then we say that x is *directly revealed preferred to* y, and we write $x\,S\,y$ (cf. Samuelson (1938, 1950)). If there is a $B \in \mathcal{B}$ such that $y \in B$ and $x \in c(B)$ then we say that x is *(weakly) revealed preferred to* y, and we write $x\,V\,y$ (cf. Clark [11], Hurwicz and Richter [15], p. 60). Let F be defined by: xFy if and only if there is $\lambda > 0$ with $\lambda x\,V\,\lambda y$. Then, F will be called the *homothetic closure* of the (weak) revealed preference V (the homothetic closure H of S is defined in Liu and Wong [19], p. 291).

It is easy to check that $x\,H\,y$ if and only if ($x\,F\,y$ and $x \neq y$). By construction, H extends S and F extends V. Because V extends S -actually, $x\,S\,y$ if and only if ($x\,V\,y$ and $x \neq y$)-, F must extend H too.

For later use, we introduce:

$$WHA \qquad\qquad x\,F\,y \Rightarrow y\,S\,x \text{ false}$$

the weak axiom of revealed homothetic preference.

The choice structure $(\mathcal{B}, c)$ on X is *representable* if there exists a function $u : X \longrightarrow \mathbb{R}$ such that $c(B) = \{x \in B : \ u(x) \geqslant u(y) \text{ for all } y \in B\}$, for any $B \in \mathcal{B}$.

We say that the choice structure $(\mathcal{B}, c)$ on X is *rationalizable* if there exists a *preference* (i.e. complete, transitive binary relation) $\succcurlyeq$ such that $c(B) = \{x \in B : \ x \succcurlyeq y \text{ for all } y \in B\}$, for any $B \in \mathcal{B}$. Obviously, representability implies rationalizability. Observe that univaluedness of c yields (a) $x\,S\,y$ implies $x \succ y$, and therefore S is acyclic; and (b) $c(B) = \{x \in B : \ x \succ y \text{ for all } y \in B \setminus \{x\}\}$, for any $B \in \mathcal{B}$. Acyclicity of S constitutes Houthakker's *Strong Axiom of Revealed Preference* (SARP). Furthermore, this axiom implies Samuelson's *Weak Axiom of Revealed Preference* (WARP), which amounts to S being asymmetric. Moreover, it is well known that any of WARP or SARP implies that choices must be univalued, and that SARP is equivalent to rationalization (cf. Richter [23], also Richter [24], Corollary 1).

We next enunciate some fundamental properties that affect homothetic rationalizability. Given a multivalued choice structure on a cone of a real vector space X that is rationalizable by a homothetic preference $\succcurlyeq$:

Property 1. It is clear that $\succcurlyeq$ must extend V, F, and obviously the transitive closure F^* of F (which is transitive itself). The same can be said about S, H and the transitive closure H^* of H (which is transitive as well); the latter being a fact that is mentioned in Remark 1 (a) of Liu and Wong [19].

Property 2. The homothetic closure F of the (weak) revealed preference V rationalizes it. That is, for any fixed $B \in \mathcal{B}$, we show that $c(B) = \{x \in B : x\, F\, y \text{ for all } y \in B\}$. It is clear that $x \in c(B)$ yields $x\, V\, y$ and then $x\, F\, y$ for all $y \in B$. Let us take now $x \in B$ with $x\, F\, y$ for every $y \in B$. Assume, by way of contradiction, $x \notin c(B)$; equivalently, there is $z \in c(B)$ and thus $z \neq x$ obviously. Then $z \succ x$ while $x \succcurlyeq z$ -because $\succcurlyeq$ extends F and $x\, F\, z$-, a contradiction.

Turning back to the *univalued case*, we shall denote by x_B the only element in $c(B)$. Then, we have further consequences:

Property 3. The following SHA holds:

$$x\, H^*\, y \Rightarrow y\, S\, x \text{ false}$$

This latter condition holds because $x\, H^*\, y$ implies $x \succcurlyeq y$ (see Property 1) and this is not compatible with $y\, S\, x$ (and thus with $y \succ x$ due to the univaluedness of c).

For certain demand functions on $\mathbb{R}^l_+$, Theorem 1 of Liu and Wong [19] prove that SHA is sufficient for rationalizability by homothetic preferences too. The technique they use stems from Richter [24] and Kannai [17], and appeals to first-order predicate languages. That issue has been used by Liu and Wong to argue in favour of the empirical relevance of the H relation -cf. their Remark 1 (b)-, which is observable as long as choices are. The same kind of argument follows after the latter properties with regard to either V or F.

Observe that the SHA axiom is defined in Liu and Wong [19] as acyclicity of H, which is equivalent to the definitions above. Also, Property 3 is mentioned in Theorem 1 of Liu and Wong [19] for a particular context.

It is trivial that any of SHA or WHA forces choices to be univalued.

Moreover, SHA is stronger than WHA. For, under SHA, it is easy to show that any pair $x, y \in X$ such that $x\,F\,y$ can not satisfy $y\,S\,x$. If $x = y$, this is obvious. Otherwise, because $x\,H\,y$ and then $x\,H^*\,y$, the SHA says $y\,S\,x$ false.

It is also true that WHA implies WARP. For, under WHA, it is easy to show that S is asymmetric. Indeed, for any pair $x, y \in X$ such that $x\,S\,y$, because necessarily $x\,F\,y$ it follows that $y\,S\,x$ must be false.

Besides, SHA implies SARP. For, under SHA, it is easy to show that S is acyclic. Indeed, for any possible cycle $x_1, ..., x_n \in X$ such that $x_1\,S\,....\,S\,x_n\,S\,x_1$, because necessarily $x_1\,H\,....\,H\,x_n$ (i.e. $x_1\,H^*\,x_n$) it follows that $x_n\,S\,x_1$ should be false, a contradiction.

Property 4. H rationalizes the structure, in the sense that $c(B) = \{x \in B : x\,H\,y \text{ for all } y \in B, y \neq x\}$. Let us fix $B \in \mathcal{B}$. It is plain that $c(B) = \{x \in B : x\,F\,y \text{ for all } y \in B\} \subseteq \{x \in B : x\,H\,y \text{ for all } y \in B, y \neq x\}$. Besides, given $x \in B$ such that $x\,H\,y$ whenever $y \in B \setminus \{x\}$, forcefully $x = x_B$. Otherwise $x\,H\,x_B\,H\,x$ and thus $x \sim x_B$ (because $\succcurlyeq$ extends H), contradicting the univaluedness of choice due to $x, x_B \in c(B) = \{x \in B : x \succcurlyeq y \text{ for all } y \in B\}$. In short, $c(B) = \{x \in B : x\,H\,y \text{ for all } y \in B, y \neq x\}$.

Yet another interesting theoretical result that builds on Property 3 is included in the Appendix for the topic's sake, since it is of little use when demand is studied.

4 Homogeneous representability of demand

In this final Section we apply the techniques we have developed before in order to find necessary and sufficient conditions for certain demand functions to be explained as the result of optimizing a continuous, homogeneous of degree one function (subjective utility). As we have announced, the basis to do with this approach will be Corollary 1 to our characterization of the existence of homogeneous and continuous weak utilities. First of all, we shall engage in a thorough study of the structure of such demand functions in order to be enabled to check that the corresponding assumptions hold.

The following context will be assumed in the remaining of this Section. X will denote a fixed consumption set (the non-negative orthant of $\mathbb{R}^n$ for some n). We adopt the usual notation for vector prices $p = (p^1, ..., p^n)$, income w and budget sets $B(p, w) = \{x \in X : p \cdot x \leqslant w\}$. $\mathcal{B}$ will be the collection of non-empty budget sets of X associated with the subset of price-income pairs $P \times M = \mathbb{R}^n_{++} \times (0, +\infty)$. Let h be a demand function on $\mathcal{B}$, that is, a function that selects exactly one element (denoted by $h(p, w)$ or by $h(B)$) for each $B = B(p, w) \in \mathcal{B}$.

Let us fix an arbitrary $B_0 = B(p_0, w_0)$, and denote by x_0 the only element in $h(B_0)$. Then, the following list of properties hold:

(1) For all $t \in (0, 1)$: $x_0 \, S \, (t \, x_0)$. Therefore, $q, r > 0$ and $r > q$ imply $(r \, x_0) \, H \, (q \, x_0)$.

Proof: The first assertion is immediate: $t \, x_0 \in B(p_0, w_0)$ and $\{x_0\} = h(B_0)$.

Concerning the second one: observe that $x_0 \, S \, (\frac{q}{r} \; x_0)$, and thus $x_0 \, H \, (\frac{q}{r} \, x_0)$ because H extends S. Since H is homothetic by construction, $(r \, x_0) \, H \, (q \, x_0)$.

(2) For all $x \in X$, there is $r \in \mathbb{Q}_{++} \setminus \{0\}$ with $(r \; x_0) \, H \, x$

Proof: If $x = x_0$, the prior property proves the assertion: $x_0 \, S \, \frac{x_0}{2}$, and now $(2 \, x_0) \, H \, x_0$ because H extends S and is homothetic.

If $x \neq x_0$, but $x \in B_0 = B(p_0, w_0)$, one has $x \, S \, x_0$ and so $x \, H \, x_0$.

Finally, if $x \neq x_0$ and also $x \notin B_0 = B(p_0, w_0)$, there is $r \in \mathbb{Q}$, $r \neq 0$, such that $p_0 \, \frac{x}{r} < w_0$ and $\frac{x}{r} \neq x_0$. By construction, $x_0 \, S \, \frac{x}{r}$, and now $(r \, x_0) \, H \, x$.

(3) Given $u : X \longrightarrow \mathbb{R}$ homogeneous of degree 1:

$$u \text{ generates } h \text{ if and only if } u \text{ is a weak utility for } H$$

Proof: This result follows easily; it parallels Lemma 1 in Alcantud and Rodríguez-Palmero [3].

In order to apply the technique exposed in the previous sections, we are also interested in studying the behavior of all sets with a particular form. We keep $p_0 \gg 0$ arbitrary but fixed and, for simplicity, x_w will denote the only element in $h(p_0, w)$ for each positive price w. Let us denote $G_w = \{y \in X : x_w H^* y\}$, for each $w > 0$. It is plain that $B(p_0, w) \setminus \{x_w\} \subseteq G_w$. We have:

(4) $G_{rw} = rG_w$ for each $r > 0$

Proof: We only need to prove $G_{rw} \subseteq rG_w$ for each $r > 0$. The converse inclusion is clearly immediate from that one (it requests that $G_w \subseteq \frac{1}{r} G_{rw}$ for each $r > 0$).

If $y \in G_{rw}$, then $x_{rw} H^* y$ by construction, therefore $\frac{1}{r} x_{rw} H^* \frac{1}{r} y$ by homotheticity of H^*. Because $\frac{1}{r} x_{rw} \in B(p_0, w)$, it follows that $x_w \, S \, \frac{1}{r} x_{rw} \, H^* \, \frac{1}{r} y$ and so $x_w \, H^* \, \frac{1}{r} y$. Thus, $\frac{1}{r} y \in G_w$ and $y \in rG_w$.

(5) For every $w > 0$ income, $y \in G_w$ implies $\lambda y \in G_w$ for each $\lambda \in (0, 1)$

Proof: We first check the assertion that $y \in G_1$ implies $\lambda y \in G_1$ for each $\lambda \in (0, 1)$. Because $x_1 H^* y$ (x_1 is the only element in $h(p_0, 1)$), for some $x \in X$ forcefully $x_1 H^* x H y$ by definition of transitive closure (in case $x_1 H y$ we proceed with $x = x_1$). Then, $ax \, S \, ay$ for some $a > 0$ by construction, which means $ax \, S \, (a \lambda y)$ (because $\lambda(ay)$ is clearly available in the same demand situation for which ax is chosen, being ay affordable). In conclusion, $x_1 H^* x H (\lambda y)$ and so $x_1 H^* (\lambda y)$, which means by definition that $(\lambda y) \in G_1$.

Now, for every $w > 0$ income, if $y \in G_w$, then $\lambda y \in G_w$ for each $\lambda \in (0, 1)$. The reason is that $\frac{1}{w} y \in G_1$ because $y \in wG_1 = G_w$ by Property (4), and then $\frac{\lambda}{w} y \in G_1$ by the previous assertion, yielding $\lambda y \in wG_1 = G_w$.

(6) $w > w' > 0$ implies $G_{w'} \subseteq G_w$

Proof: We know $G_{w'} = w' G_1 \subseteq w G_1 = G_w$.

The SHA is necessary for a demand function to be represented by a homogeneous of degree 1 function (subjective utility), for in fact it is implied by homothetic rationalizability alone. We want to investigate the implications of requesting that the demand behavior be consistent with that requirement. Therefore, and *assuming SHA henceforth*, we have the following further properties:

(7) $\lambda x_w \notin G_w$ for each $\lambda > 1$

Proof: Should there exist $\lambda > 1$ for which $\lambda x_w \in G_w$, by (5) we would also have $x_w \in G_w$, that is, $x_w \, H^* \, x_w$. This contradicts acyclicity of H, i.e. SHA.

(8) h is exhaustive

Proof: Assume that, for the prices p_0 (which were fixed but arbitrary), it is true that there is an income w such that the demanded bundle x_w satisfies $p_0 \cdot x_w < w$.

We would then derive the existence of a $\lambda > 1$ for which $\lambda x_w \in B(p_0, w)$, and therefore $x_w \, S \, y$ by definition of S. But this means $\lambda x_w \in G_w$, against (7).

(9) $S = H$

Proof: Observe that, according to Remark 8 in Liu and Wong [19], because SHA holds then the demand function h satisfies $h(p, \lambda w) = \lambda h(p, w)$ for each $\lambda > 0$ and every $(p, w) \in P \times M$. This latter property easily yields homotheticity of S , that is to say, $S = H$.

(10) If $z \in G_w$ and $a \in X$ satisfies $z > a$ (by this we mean: $z_i > a_i$ for each component i) then $a \in G_w$.

Proof: There must be $y \in X$ with $x_w H^* y H z$; should we have $x_w H z$ then we proceed with $y = x_w$. Because $S = H$, $y = h(\overline{p}, \overline{w})$ for some $(\overline{p}, \overline{w}) \in P \times M$ that satisfies $\overline{p} z \leqslant \overline{w}$. But now $\overline{p} a < \overline{w}$, therefore $y \, S \, a$.

REMARK 3 *For a given demand function, Liu and Wong [19] prove in their Theorem 1 that it can be rationalized by a homothetic preference provided that SHA holds, and conversely. It is, therefore, to be presumed that additional axioms must be introduced in the model in order to obtain plain representability by homogeneous functions. We do not know of any result in this fashion. However, Property (3) permits to characterize de-*

mand behavior that derives from a homogeneous (of degreee 1) function -subjective utility- through SHA. We only need the purely numerical (i.e. without continuity) part of our Theorem 1, since H will be acyclic under the SHA axiom and conversely. In our view, this contribution is an anew result, but we leave this branch of the conclusions to the interested reader.

Finally, an answer to the homogeneous representability problem follows from the next property:

(11) Suppose that

$$x\,S\,y \;\Rightarrow\; \exists\; B' = B(\bar{p},\bar{w}) \in \mathcal{B} \text{ such that } \begin{cases} \bar{p}\cdot(tx+(1-t)y) \leqslant \bar{w} \text{ for some } \; t\in(0,1), \\ x\,S\,h(B'). \end{cases}$$

Then H^* is upper semicontinuous.

Proof: Observe that SARP was already assumed implicitely, for SHA holds by assumption now.

Now the transitive closure of S is upper semicontinuous; that fact is guaranteed by the proof of Proposition 2 in Alcantud and Rodríguez-Palmero [3]. By (9), the transitive closure of H, that is, H^*, is upper semicontinuous.

All the groundwork we have been performing before culminates in the following result and the comments following it:

THEOREM 2 *For any demand function that satisfies the condition given in (11): SHA is equivalent to the existence of a continuous, homogeneous of degree 1 function that generates the demand.*

Proof: Property 3 accounts for the necessary condition.

We now turn to the sufficient condition. Assume, therefore, that SHA holds. Due to (3), we need an homogeneous of degree 1, continuous weak utility for H, i.e. for S because of (9). By Corollary 1, we can assure the existence of the desired funtion. Note that S

is acyclic by SHA and homothetic due to (9). Property (11) ensures that its transitive closure H^* is upper semicontinuous. The requirement (i) holds by (2). Property (5) accounts for (ii): note that x not minimal means that $x = x_w = h(p, w)$ for some prices p and income w, and then $x\, H^*\, y$ means simply $y \in G_w$. So, we only need to check (iii), that is, $\overline{G_1} \subseteq qG_1$ for each $q > 1$ rational.

Fix $q > 1$ rational, and take $x \in \overline{G_1}$. Because $x > \frac{1}{q}x$, there is $\epsilon > 0$ such that every $z \in B(x, \epsilon) \cap X$ satisfies $z > \frac{1}{q}x$. But there must be $z \in B(x, \epsilon) \cap X \cap G_1$ because $x \in \overline{G_1}$, which forces $\frac{1}{q}x \in G_1$ by (10). Therefore, $x \in qG_1$. ☐

In comparison to Theorem 3 in Liu and Wong [19], and putting aside the much different techniques involved, Theorem 2 presumes an assumption weaker than full continuity in order to *check* that we can obtain continuous representability. The condition we have used in (11) was presented in Alcantud and Rodríguez-Palmero [3]. The relationship of that requirement with respect to others that have proven useful in the literature is explained there in detail. In particular, continuous demand functions display that behavior. Therefore, we get a sharpened consequence:

THEOREM 3 *Given h demand function on X: it is continuous and satisfies the SHA if and only if it can be represented by a continuous, homogeneous of degree 1 function.*

Proof: We only need to justify that, because homogeneous of degree 1 utilities induce locally nonsatiated preferences on X, Proposition 3.AA.1 in Mas-Colell et al. [21] applies and therefore the demand induced is continuous. ☐

5 Conclusion

Two related objectives of very different nature are addressed in this contribution.

On the one hand, the extensive use of utility-theoretical techniques, as well as a deep analysis of the structure of the homogeneous representability problem -all of them anew and with intrinsic interest-, have permitted us to propose a *constructive* procedure for deriving a homogeneous of degree one, continuous function giving raise to an original demand function under mild conditions. As far as we know, *this is the first elementary proof of a necessary and sufficient condition for an integrability problem to have a solution by continuous (subjective utility) functions.*

Our thorough analysis has given place to several side remarks. Observe that, unlike Liu and Wong, we made no assumption on the openness or finitiness of the range of h in our main theorems. Also, our demand functions were not initially requested to be exhaustive, but we have observed in (8) that *assuming SHA forces exhaustiveness*, a fact that seems to have gone unnoticed so far. Moreover, we have taken a further step by observing that continuity of the demand was also *necessary* to have a solution; this was due to the fact that continuous utilities inducing locally nonsatiated preferences on X give raise to a continuous demand, provided that this demand is a function.

The constructive nature of our proof may have implications for subsequent purposes. We must remark that the term *constructive* is used in the technical sense, and by no means tries to imply that applying our method in practice must be straightforward.

Still, and in order to boost this potential applicability, we end this first comment briefing the reader on the constructive procedure that computes a continuous, homogeneous of degree 1 function that induces h, from the raw revealed preference S. Through either S or its transitive clausure H^* we check SHA, and then define $G_1 = \{y \in X : x_w H^* y\}$ (associated with any fixed x_w); now let

$$u(x) = inf\{r \in \mathbb{Q} : x \in G_r\}$$

where $G_r = rG_1$ for each $r \in \mathbb{Q}$, $r > 0$, and $G_r = \varnothing$ for each $r \in \mathbb{Q}$, $r < 0$. Corollary 1 guarantees that this function is a continuous weak utility for $S = H$, and now (3) ensures that it induces the demand.

On the other hand, the most challenging contribution, in our view, leans upon the strong arguments we provided for an interesting debate. Indeed, we are of the opinion that this work adds to a preliminary bulk of evidence of a fruitful relationship -already hinted in the inspiring Alcantud and Rodríguez-Palmero [3]- between mathematical utility theory and revealed preference theory, yet to be explored in depth. The fact that this relationship has already provided elementary answers to very deep problems with highly complex solutions in the past literature may well motivate a growing interest in a very promising tool for research into the integrability problem.

6 Appendix: proofs and a side result

Proof of Theorem 1: (i) ⇒ (ii). Assume that there exists a homogeneous of degree one and upper semicontinuous weak utility u for $\succ$, and consider the countable family $\{G_r\}_{r\in\mathbb{Q}} = \{x \in X : u(x) < r\}_{r\in\mathbb{Q}}$. It is clear that every G_r is an open lower subset of X for every $r \in \mathbb{Q}$. Further, for every $r \in \mathbb{Q}$ and $q \in \mathbb{Q}^{++}$ we have that $qG_r = q\{x \in X : u(x) < r\} = \{x \in X : u(x) < qr\} = G_{qr}$ since u is homogeneous of degree one, and therefore condition (c) holds. Finally, it is easily seen that the family $\{G_r\}_{r\in\mathbb{Q}}$ satisfies conditions (a), (b) and (d).

(ii) ⇒ (i). Assume that there exists a countable family $\{G_r\}_{r\in\mathbb{Q}}$ of open lower subsets of X satisfying conditions (a), (b), (c) and (d). Define a real-valued function u on X as follows:

$$u(x) = inf\{r \in \mathbb{Q} : x \in G_r\}.$$

First observe that u is well defined by conditions (a) and (b) together. We claim that u is a homogeneous of degree one and continuous weak utility for $\succ$. Let us first show that that u is upper semicontinuous. Consider any element $x \in X$ and any real number α such that $u(x) < \alpha$. Then, from the definition of u, there exists $r \in \mathbb{Q}$ such that $u(x) < r < \alpha$, $x \in G_r$. Hence, G_r is an open lower set containing x such that $u(z) < \alpha$ for every $z \in G_r$. In order to show that u is lower semicontinuous, let us first prove that $u(x) = inf\{r \in \mathbb{Q} : x \in \overline{G}_r\}$ for every $x \in X$. It is clear that $inf\{r \in \mathbb{Q} : x \in \overline{G}_r\} \leqslant inf\{r \in \mathbb{Q} : x \in G_r\}$ for every $x \in X$, since $G_r \subseteq \overline{G}_r$ for every $r \in \mathbb{Q}$. Now assume that there exists $x \in X$ with $inf\{r \in \mathbb{Q} : x \in \overline{G}_r\} < inf\{r \in \mathbb{Q} : x \in G_r\}$. Consider $r_1, r_2 \in \mathbb{Q}$ such that $inf\{r \in \mathbb{Q} : x \in \overline{G}_r\} < r_1 < r_2 < inf\{r \in \mathbb{Q} : x \in G_r\}$. Then $x \in \overline{G}_{r_1}$, $x \notin G_{r_2}$, and this is contradictory, since $\overline{G}_{r_1} \subseteq G_{r_2}$ by the above condition (b). So it must be $inf\{r \in \mathbb{Q} : x \in \overline{G}_r\} = inf\{r \in \mathbb{Q} : x \in G_r\}$ for every $x \in X$. Now consider any $x \in X$, and any real number α such that $\alpha < u(x)$. Further, let $r_1, r_2 \in \mathbb{Q}^{++}$ be such that $\alpha < r_1 < r_2 < u(x)$. Then we have that $x \notin \overline{G_{r_1}}$ because otherwise $x \in \overline{G_{r_1}}$ implies $x \in G_{r_2}$ and this contradicts the fact that $u(x) > r_2$. Hence, $X \setminus \overline{G_{r_1}}$ is an open subset of

X containing x such that $\alpha < u(z)$ for every $z \in X \setminus \overline{G_{r_1}}$ (observe that $u(z) \leqslant \alpha$ implies $u(z) < r_1$ which in turn implies $z \in \overline{G_{r_1}}$ since $u(x) = inf\{r \in \mathbb{Q} : x \in \overline{G_r}\}$ for every $x \in X$).

Further, u is a weak utility for $\succ$ by condition (d). Indeed, for every $x, y \in X$ such that $x \succ y$ there exist $r_1, r_2 \in \mathbb{Q}$ with $r_1 > r_2$, $y \in G_{r_2}$, $x \notin G_{r_1}$ and therefore we have that $u(x) \geqslant r_1 > r_2 \geqslant u(y)$ from the definition of u. In order to show that u is homogeneous of degree one, assume that there exists $t \in \mathbb{R}^{++}$ such that $u(tx) < tu(x)$. Then, from the definition of u, there exists $r \in \mathbb{Q}$ such that $u(tx) < r < tu(x)$, $tx \in G_r$. By continuity of scalar multiplication and upper semicontinuity of u, there exists $q \in \mathbb{Q}^{++}$ such that $u(qx) < r < qu(x)$, $qx \in G_r$. From the definition of u, it follows that $x \notin G_{\frac{r}{q}} = \frac{1}{q}G_r$ since $u(x) > \frac{r}{q}$, and therefore we arrive at the contradiction $qx \notin G_r$. Analogously it can be shown that for no $t \in \mathbb{R}^{++}$, and $x \in X$ it is $tu(x) < u(tx)$. So the proof is complete. $\square$

Proof of Corollary 1: The case where all elements of X are minimal is trivial. Suppose, therefore, that there is $\overline{x} \in X$ that is not minimal respect to $\succ$. Define $G_1 = \{a \in X : \overline{x} \succ\!\!\succ a\}$, $G_{-1} = \varnothing$. Take $G_q = qG_1$ for each $q > 0$ rational, and $G_q = \varnothing$ for each $q \leqslant 0$ rational. These are open (by semicontinuity of $\succ\!\!\succ$) and lower (by homotheticity of $\succ$ and therefore of $\succ\!\!\succ$) subsets. Let us first check that the collection $\{G_q\}_{q \in \mathbb{Q}}$ satisfies conditions (a), (b′), (c) and (d) of Theorem 1 (see Remark 1) in order to ensure the existence of a homogeneous of degree one, upper semicontinuous weak utility u for $\succ$ under (i) and (ii).

In order to verify that condition (a) of Theorem 1 holds, fix any $x \in X$. Because there is $q > 0$ rational such that $q\overline{x} \succ\!\!\succ x$ by (i), and $\succ\!\!\succ$ is homothetic too, $\overline{x} \succ\!\!\succ \frac{x}{q}$, which means $\frac{x}{q} \in G_1$ or, equivalently, $x \in qG_1 = G_q$. Hence, we have that $\bigcup\limits_{r \in \mathbb{Q}} G_r = X$. Furthermore, $\bigcap\limits_{r \in \mathbb{Q}} G_r = \varnothing$ by construction.

Now let us prove that condition (b′) of Theorem 1 is verified (see Remark 1). Let us fix $q < r \in \mathbb{Q}$. Unless $q, r > 0$, assumption (b) holds trivially. Suppose, therefore, that $q, r > 0$, and take an arbitrary element in G_q, that is, an element with the form qa where $a \in G_1$. Because $qa = r(\frac{q}{r}a)$, in order to check $qa \in G_r = rG_1$ we only need to justify that $\frac{q}{r}a \in G_1$. This fact derives from $\overline{x} \succ\!\!\succ a$ plus (ii) above easily.

Assumption (c) of Theorem 1 holds by construction.

Finally, let us show that condition (d) of Theorem 1 holds. Consider any two elements $x, y \in X$ such that $x \succ\!\!\succ y$. Then either $x = z \succ y$ or there is $z \in X$ with $x \succ\!\!\succ z \succ y$. Clearly, we are done if we justify the existence of $r_1, r_2 \in \mathbb{Q}$ such that $r_1 > r_2$, $z \notin G_{r_1}$, $y \in G_{r_2}$, whatever the case holds. Let us define the set $A(z) = \{t > 0 : \overline{x} \succ\!\!\succ tz\}$. It is non-empty by (i). And it is bounded above: since z is not minimal, there is $t_0 > 0$ such that $t_0 z \succ\succ \overline{x}$ by (i), and now acyclicity of $\succ$ prevents the existence of t arbitrarily large with $\overline{x} \succ\!\!\succ tz$ because of condition (ii). Denote $\lambda_0 = \sup A(z)$. Since $\succ$ is tc-upper semicontinuous, we have that $\lambda_0 \notin A(z)$ due to the continuity of scalar multiplication. We have thus shown $\lambda_0 z \notin G_1$, i.e., $z \notin \frac{1}{\lambda_0}G_1$. Let us check that $\lambda_0 y \in G_1$, i.e., $y \in \frac{1}{\lambda_0}G_1$. Note that $\lambda_0 z \succ \lambda_0 y$ by homotheticity. Upper semicontinuity of $\succ\!\!\succ$ grants the existence of $\epsilon_0 > 1$ with $\lambda_0 z \succ\!\!\succ (\lambda_0 \epsilon_0)y$, therefore $\frac{\lambda_0}{\epsilon_0}z \succ\!\!\succ \lambda_0 y$. By definition of λ_0, $\overline{x} \succ\!\!\succ \frac{\lambda_0}{\epsilon_0}z \succ\!\!\succ \lambda_0 y$ and so $\lambda_0 y \in G_1$. Now fix any rational number $\bar{r}_2$ such that $0 < \lambda_0 < \bar{r}_2$, $y \in \frac{1}{\bar{r}_2}G_1$; such a number exists by upper semicontinuity of $\succ\!\!\succ$ and continuity of scalar multiplication, since we have that $\bar{x} \succ\!\!\succ \lambda_0 y$. Now let $\bar{r}_1$ be any rational number such that $\lambda_0 < \bar{r}_1 < \bar{r}_2$. Then $\bar{r}_1 z \notin G_1$ from the definition of λ_0. Finally, set $r_1 = \frac{1}{\bar{r}_1}$ and $r_2 = \frac{1}{\bar{r}_2}$. Hence, we have that $r_1 > r_2$, $z \notin G_{r_1} = r_1 G_1$, $y \in G_{r_2} = r_2 G_1$, and therefore condition (d) of Theorem 1 holds.

The final statement is immediate now, because in fact the stronger condition (b) is satisfied if we select the element in (iii) and proceed as before. ☐

The next result builds on Property 3 and is interesting for abstract choice theory:

PROPOSITION 1 *Given a multivalued choice structure $(\mathcal{B}, c)$ on a real vector space X such that all subsets of X with 3 or fewer elements belong to $\mathcal{B}$: it is rationalizable by a homothetic preference if and only if WHA holds true.*

Proof: The necessity of the condition is plain (in fact, we have justified that SHA must hold in Property 3).

Conversely: provided that WHA is fulfilled, then F solves the problem. Indeed,

i) F is complete: given $x, y \in X$ then $x \in c(x, y)$ without loss of generality. Then $x\,V\,y$ and so $x\,F\,y$.

ii) F is transitive: given $x, y, z \in X$ such that $x\,F\,y\,F\,z$, denote $A = \{x, y, z\} \in \mathcal{B}$. Should we have $x \in c(A)$ then $x\,V\,z$ and so $x\,F\,z$. Otherwise, two options remain:

(a) $y \in c(A)$; then $y\,V\,x$ and so $y\,F\,x$. By WHA, $x \in c(A)$ and the reasoning above applies.

(b) $z \in c(A)$; then $z\,V\,x$ and so $z\,F\,x$. Again, WHA yields $x \in c(A)$ and we proceed as above.

iii) F is homothetic by construction.

iv) F rationalizes $(\mathcal{B}, c)$; that is, $c(B) = \{x \in B : xFy \text{ for all } y \in B\}$. Indeed, for any $x \in c(B)$ one has $x\,V\,y$ and so $x\,F\,y$, whenever $y \in B$. Conversely: let $x \in B$ with $x\,F\,y$ for all $y \in B$. Because there is $z \in c(B)$ and then $x\,F\,z$, the WHA axiom implies $x \in c(B)$. $\square$

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